SYMPLECTIC GEOMETRY, LECTURE 22

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1. Symplectic Sum

Let $(M_1^{2n}, \omega_1), (M_2^{2n}, \omega_2)$ be symplectic manifolds, (Q^{2n-2}, ω_Q) a compact symplectic manifold with symplectic embeddings $\iota_1 : Q \to M_1, \iota_2 : Q \to M_2$ and trivial normal bundles. Then $v(Q_i)$ is symplectomorphic to $(Q \times D^2(\epsilon), \omega_Q \oplus \omega_0)$. Let

$$M = M_1 \#_Q M_2 = \left(M_1 \setminus \left(Q_1 \times D^2\left(\frac{\epsilon}{2}\right)\right)\right) \cup_\phi \left(M_2 \setminus \left(Q_2 \times D^2\left(\frac{\epsilon}{2}\right)\right)\right) \tag{1}$$

where ϕ is given in local coordinates by

$$Q \times \left(D^2(\epsilon) - D^2\left(\frac{\epsilon}{2}\right)\right) \to Q \times \left(D^2(\epsilon) - D^2\left(\frac{\epsilon}{2}\right)\right),\ (q, z) \mapsto (q, \psi(z)) \tag{2}$$

and ψ is an orientation- and area-preserving diffeomorphism that exchanges the boundaries. Then $\psi^*\omega_0 = \omega_0, \phi^*\omega_2 = \omega_1 \implies$ we get a natural symplectic structure on $M_1 \#_Q M_2$.

Remark. In this gluing, we "lost" an amount of volume depending on ϵ. If one instead forms the manifold as

$$(M_1 \setminus (Q_1 \times D^2(\frac{\epsilon}{2}))) \cup (Q \times \text{ cylinder}) \cup (M_2 \setminus (Q_2 \times D^2(\frac{\epsilon}{2}))) \tag{3}$$

one can force $\text{vol}(M) = \text{vol}M_1 + \text{vol}M_2$. Moreover, $M_1 \#_Q M_2$ depends on the isotopy class of $i_2 \circ i_1^{-1} : Q_1 \to Q \to Q_2$.

Remark. In dimension 4, it is enough to have $\Sigma_1 \subset M_1^4, \Sigma_2 \subset M_2^4$ symplectic submanifolds with the self-intersection 0 and identical genus and symplectic area.

We can generalize this construction to the case when the normal bundles are no longer trivial, but dual to each other, i.e. $c_1(NQ_1)+c_1(NQ_2) = 0$: this implies that we can do the gluing fiberwise since $(NQ_2) \cong (NQ_1)^*$. Letting $L = NQ_1$, we consider a manifold X which is the total space of $L \oplus L^* \to Q$, on which we can put a symplectic structure compatible with the symplectic structures on L, L^*. By local Moser, $\exists$ a local description

$$M_1 \cong \{(g, s_1, 0) \in Q \times L_q \times L_q^*\}, M_2 \cong \{(g, 0, s_2) \in Q \times L_q \times L_q^*\} \tag{4}$$

M_1, M_2 intersect along the zero section, and

$$M_1 \cup_Q M_2 = \{(q, s_1, s_2) | s_1 s_2 = 0\} \tag{5}$$

Let $M = \{(q, s_1, s_2) | s_1 s_2 = \delta\chi(|s_1|, |s_2|)\}$ for $\delta \neq 0$ small (can consider it to be a complex number fixing $L \otimes L^* \cong \underline{\mathbb{C}}$ or a nonvanishing section of $L \otimes L^*$) and χ a cutoff function which makes M look like M_1 or M_2 away from Q. We claim without proof that we can choose δ small enough that we get a symplectic structure on M.

Remark. In dimension 4, the above assumption implies that $[\Sigma_1] \cdot [\Sigma_1] + [\Sigma_2] \cdot [\Sigma_2] = 0$. We can do the same construction assuming only that $[\Sigma_1] \cdot [\Sigma_1] + [\Sigma_2] \cdot [\Sigma_2] \geq 0$. Consider, $L_1 = N\Sigma_1, L_2 = N\Sigma_2, X = L_1 \oplus L_2 \to \Sigma$, and set

$$M = \{(q, s_1, s_2) | s_1 s_2 = \delta\sigma(q)\chi(|s_1|, |s_2|)\} \tag{6}$$

where σ is a section of $L_1 \otimes L_2$. To ensure that M is smooth, we need σ to vanish transversally, i.e. its zeroes $\sim \sigma(z) = z$ or $\sigma(z) = \bar{z}$. To ensure that M is symplectic, we require all the zeros of σ to have complex orientation, which requires $\sim \sigma(z) = z$ and deg $(L_1 \otimes L_2) \geq 0$.

An application of the symplectic sum construction is the following result:

Theorem 1 (Gompf 1994)**.** *Every finitely presented group G is π_1 of a compact symplectic 4 manifold.*

Write $G = \langle g_1, \dots, g_k | r_1, \dots, r_k \rangle$ where g_i are generators and r_i are relations. Let F be a compact genus k surface with standard generators $(\alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$ of π_1 s.t.

$$\pi_1(F) = \langle \alpha_1, \beta_1, \dots, \alpha_k, \beta_k | \prod_{i=1}^{k} [\alpha_i, \beta_i] = 1 \rangle \tag{7}$$

That is, $F = F^0 \cup D^2$, where $F^0 = \bigvee^{2g} S^1$ is the 1-skeleton and D^2 is attached along $\prod \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$.

Now, for $i = 1, \dots, \ell$, choose γ_i an immersed closed curve in F representing $\sigma_i(\alpha_1 \cdots \alpha_k)$. Let $\gamma_{\ell+j} = \beta_j$ for $j = 1, \dots, k$. Then

$$G = \pi_1(F) / \langle \gamma_1 \cdots \gamma_{k+\ell} \rangle \tag{8}$$

Assume $\exists \rho \in \Omega^1(F)$ a closed 1-form s.t. $\rho|_{\gamma_i}$ is a positive form at every point of every γ_i (there exists a procedure to do this, at the expense of increasing the genus and the number of γ_i's). Set $X = F \times T^2, \omega = \omega_1 + \omega_2$. From before we have $\gamma_i \subset F, \rho \in \Omega^1(F)$ closed s.t. $\rho|_{\gamma_i} > 0$, and we can similarly find $\alpha_i \subset T^2$ disjoint nontrivial simple closed curves and $\theta \in \Omega^1(T^2)$ closed with $\theta|_{\alpha_i} > 0$ (for instance, $\theta = dx$ for $\alpha_i = S^1 \times \{p_i\}$). Then $T_i = \gamma_i \times \alpha_i$ are Lagrangian w.r.t ω, symplectic w.r.t. $\omega' = \omega + \rho \wedge \theta$. Now do a symplectic sum construction, attaching $(\mathbb{CP}^2, E = \{(x_0 : x_1 : x_2) | x_0^3 + x_1^3 + x_2^3 = 0\})$.

Remark (Adjunction Formula). For a connected, embedded compact symplectic $\Sigma^2 \subset (M^4, \omega)$, $TM|_\Sigma = T\Sigma \oplus N\Sigma$ as symplectic vector bundles, so

$$\begin{gathered} c_1(TM|_\Sigma) = c_1(T\Sigma) + c_1(N\Sigma) \in H^2(\Sigma) = \mathbb{Z} \\ c_1(TM) \cdot [\Sigma] = 2 - 2g(\Sigma) + [\Sigma] \cdot [\Sigma] \end{gathered} \tag{9}$$

In our case, this implies that the genus is 1, i.e. we have a torus on both sides that can be glued. The tori T_i are disjoint, and $[T_i] \cdot [T_i] = 0$: since $[E] \cdot [E] = 9$, we can do the symplectic sum. Doing the sums along the T_i as well as $\{z\} \times T^2, z \in F \setminus (\bigcup \gamma_i)$, we kill γ_i and the generators of the T^2. Indeed, using Van Kampen, we can show that $\#_E \mathbb{CP}^2$ just kills $\mathrm{Im}(\pi_1(\Sigma) \to \pi_1(M))$, giving us the desired manifold.

Now we will study further the topology of 4-manifolds.

Problem. Let M be the connected sum of 9 copies of $\mathbb{CP}^2$ and 44 copies of $\overline{\mathbb{C}}\mathbb{P}^2$. Then M is homeomorphic but not diffeomorphic to $\{x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0\} \subset \mathbb{CP}^3$.