10 Lecture 10 (Notes: K. Venkatram)

Last time, we defined an almost Dirac structure on any Lie group G with a bi-invariant metric B by

$$L_C = \langle a^L - a^R + B(a^L + a^R) | a \in \mathfrak{g} \rangle \tag{13}$$

10.1 Integrability

Lemma 2. *$d(B(a^L))(x^L, y^L) = x^L B(a^L, y^L) - y^L B(a^L, x^L) - B(a^L, [x^L, y^L]) = -i_{a^L} H(x^L, y^L)$, where $H(a,b,c) = B(a^L, [b^L, c^L])$.*

Problem. Show that $B(\theta^L, [\theta^L, \theta^L])(a^L, b^L, c^L) = 6B(a^L, [b^L, c^L])$.

Note also that

$$dB(a^R)(x^R, y^R) = -B(a^R, [x^R, y^R]) = i_{a^R} H(x^R, y^R) \tag{14}$$

Now,

$$\begin{aligned}[a^L - a^R + B(a^L + a^R), b^L - b^R + B(b^L + b^R)]_0 &= [a,b]^L - [a,b]^R - i_{b^L - b^R} dB(a^L + a^R) + L_{a^L - a^R} B(b^L + b^R) \\ &= [a,b]^L - [a,b]^R + i_{b^L - b^R} i_{a^L - a^R} H + B([a,b]^L + [a,b]^R)\end{aligned} \tag{15}$$

Corollary 3. *L_C is involutive under $[,]_H$.*

Comments about the Cartan-Dirac structure:

1. $a^L - a^R$ generates the adjoint action so generalized, and $\pi L_C = \Delta$ is a foliation by the conjugacy classes.
2. T^* component is $B(a^L + a^R)$, which spans T^* whenever $\mathfrak{g} \to T^*_g, a \mapsto a^L + a^R$ is surjective $\Leftrightarrow$ ($\mathrm{ad}_g + 1$ is invertible. This is true, in particular, for an open set containing $e \in G$.

In this region, $L_c = \Gamma_\beta$ for an H-twisted Poisson structure.

1. Determine explicitly the bivector β when it is defined.
2. For $G = SU(2) = S^3$, describe the conjugacy classes and the locus where $\mathrm{ad}_g + 1$ is invertible, rank 2, rank 1, and rank 0.
3. Determine the Lie algebroid cohomology $H^*(L_c)$. Hint: $\mathfrak{g} \to L_c, a \mapsto a^L - a^R + B(a^L + a^R)$ is bracket-preserving.

10.2 Dirac Maps

A linear map $f : V \to W$ of vector spaces induces a map $f_* : \mathrm{Dir}(V) \to \mathrm{Dir}(W)$ (the forward Dirac map) given by $f_* L_V = \{f_* v + \eta \in W \oplus W^* | v + f^* \eta \in L_V\}$ and a map $f^* : \mathrm{Dir}(W) \to \mathrm{Dir}(V)$ (the backward Dirac map) given by $f^* L_W = \{v + f^* \eta \in V \oplus V^* | f_* v + \eta \in L_W\}$.

Example.
$\beta \in \bigwedge^2 V$. Then

$$\begin{aligned} f_*\Gamma_\beta &= \{f_*v + \eta | v + f^*\eta = \beta(\xi) + \xi \forall \xi \in V^*\} = \{f_*\beta f^*\eta + \eta | \eta \in W^*\} \\ &= \{(f_*\beta)(\eta) + \eta\} = \Gamma_{f_*\beta} \end{aligned} \tag{16}$$

so f_* coincides with the usual pushforward.
$L = L(E, \epsilon), f : E \hookrightarrow V, \epsilon \in \bigwedge^2 E^*$. Then L is precisely $f_*\Gamma_\epsilon$ via the pushforward $E \oplus E^* \to V \oplus V^*$.
In general, $L = L(F, \gamma), F \subset V^*, \gamma \in \bigwedge^2 F^*$ is equivalent to specifying
$(C = \text{Ann } F = L \cap V, \gamma \iota \bigwedge^2 F^* = \bigwedge^2(V/L \cap V) = \bigwedge^2(V/C))$. Note that $(f_*L_V) \cap W = f_*(L_V \cap V)$.

Problem. $f_*L(C, \gamma) = L(f_*C, f_*\gamma)$.

This proves that pushforward commutes properly with composition.

10.3 Manifolds with Courant Structure

Let $(M, H_M), (N, H_N)$ be manifolds equipped with $H \in \Omega^3)cl$-structure.

Definition 17. *A morphism $\Phi : (M, H_M) \to (N, H_N)$ is a pair (ϕ, B) for $\phi : M \to N$ a smooth map and $B \in \Omega^2(M)$ s.t. $\phi^*H_N - H_M = dB$, i.e. B gives an isomorphism $\phi^*G_N \to G_M$.*

Now, suppose that $L_M \subset TM \oplus T^*M, L_N \subset TN \oplus T^*N$ are Dirac structures.

Definition 18. Φ *is a* Dirac morphism $\Leftrightarrow \phi_*e^BL_M = L_N$.

If L_M is transverse to T^*M, then a Dirac morphism to (N, H_N, L_N) is called a *Dirac brane* for N: this object is important because ϕ^*G_N is trivial.

Example. Let L_N be a Dirac structure, and let $M \subset N$ be a leaf of $\Delta = \pi L_N$. Then $L_N = L(\Delta, \epsilon \in \bigwedge^2 \Delta^*)$ and so $\epsilon \in \Omega^2(M)$. Furthermore, integrability means that $d\epsilon = H|_M$, hence $(M, \epsilon) \to (N, H, L)$ is a Dirac brane. So any Dirac manifold is foliated by Dirac branes, and for G, is foliated by conjugacy classes C and 2-forms $\epsilon \in \Omega^2(C)$ called *GHJW (Guruprasad-Huebschmann-Jeffrey-Weinstein) 2-forms.*

Theorem 7. $(m, \tau) : (G \times G, p_1^*H + p_2^*H) \to (G, H)$ *is a Dirac morphism from $L_C \times L_C \to L_C$, i.e. $m_*e^\tau(L_C \times L_C) = L_C$.*

Proof. Set $\rho(a) = a^L - a^R, \sigma(a) = B(a^L + a^R)$, so $[\rho(a), \rho(b)] = \rho([a, b])$, $[\rho(a), \sigma(b)] = \sigma([a, b])$, and $d\sigma(a) = -i_{\rho(a)}H$. Then

$$e^\tau(L_C \times L_C) = \langle(\rho(a), \rho(b)), (\sigma(a), \sigma(b)) + i_{\rho(a),\rho(b)}\tau\rangle \tag{17}$$

We want to show that this object contains L_C, so choose $(X, \xi) \in L_C|_{gh}, X = \rho(x), \xi = \sigma(x)$. Want to find a, b s.t. $X = m_*(\rho(a), \rho(b))$ and $m^*\sigma(x) = (\sigma(a), \sigma(b)) + i_{\rho(a),\rho(b)}\tau$.

I $m_*|_{(g,h))} = [R_{h*}, L_{g*}]$ and

$$\begin{aligned} m_* \begin{pmatrix} \rho(x)_g \\ \rho(x)_h \end{pmatrix} &= \begin{pmatrix} R_{h^*} & L_{g^*} \end{pmatrix} \begin{pmatrix} (L_{g*} - R_{g^*})x \\ (L_{h*} - R_{h^*})x \end{pmatrix} \\ &= (R_{h^*}(L_{g*} - R_{g^*}) + L_{g*}(L_{h*} - R_{h^*}))x = \rho(x)_{gh} \end{aligned} \tag{18}$$

II Want to show $m^*\sigma(x)_{gh} = (\sigma(a)_g, \sigma(b)_h) + i_{\rho(a)_g,\rho(b)_h}\tau$. At gh, we have that

$$m^*\sigma(x)\begin{pmatrix} a^R \\ b^L \end{pmatrix} = \sigma(x)(R_{h*}a^R + L_{g*}b^L) = \sigma(x)(a^R + b^L) = B(x^L - x^R, a^R + b^L) \tag{19}$$

Then

$$(\sigma(x), \sigma(x))\begin{pmatrix} a^R \\ b^L \end{pmatrix} = \sigma(x)_g(a^R) + \sigma(x)_h(b^L) \tag{20}$$

and the rest follows.

□

This leads to a *fusion* operation on Dirac morphisms: given $\Phi_1 : M_1 \to G, \Phi_2 : M_2 \to G$, composing the product with (m, τ) gives $\Phi_1 \circledast \Phi_2 : M_1 \times M_2 \to G$.

Example. Given two copies of the map $m : G \times G \to G$, obtain $m \circledast m : G^4 \to G$: more generally, get Dirac morphisms $M^{\circledast h} : G^{2h} \to G$. This is used by AMM to get a symplectic structure on the moduli space of flat G-connections on a genus h Riemann surface.

By Freed-Hopkins, fusion on branes implies a form of fusion on $K^\tau_G(G)$.