Chapter 18

Weierstrass-Enneper Representations

18.1 Weierstrass-Enneper Representations of Minimal Surfaces

Let M be a minimal surface defined by an isothermal parameterization $x(u,v)$. Let $z = u + iv$ be the corresponding complex coordinate, and recall that

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}), \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$$

Since $u = 1/2(z + \overline{z})$ and $v = -i/2(z - \overline{z})$ we may write

$$x(z,\overline{z}) = (x^1(z,\overline{z}), x^2(z,\overline{z}), x^3(z,\overline{z}))$$

Let $\phi = \frac{\partial x}{\partial z}, \phi^i = \frac{\partial x^i}{\partial z}$. Since M is minimal we know that ϕ^is are complex analytic functions. Since x is isothermal we have

$$(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0 \tag{18.1}$$

$$(\phi^1 + i\phi^2)(\phi^1 - i\phi^2) = -(\phi^3)^2 \tag{18.2}$$

Now if we let $f = \phi^1 - i\phi^2$ and $g = \phi^3/(\phi^1 - i\phi^2)$ we have

$$\phi^1 = 1/2f(1-g^2), \phi^2 = i/2f(1+g^2), \phi^3 = fg$$

Note that f is analytic and g is meromorphic. Furthermore fg^2 is analytic since $fg^2 = -(\phi^1 + i\phi^2)$. It is easy to verify that any ϕ satisfying the above equations and the conditions of the preceding sentence determines a minimal surface. (Note that the only condition that needs to be checked is isothermality.) Therefore we obtain:

Weierstrass-Enneper Representation I If f is analytic on a domain D, g is meromorphic on D and fg^2 is analytic on D, then a minimal surface is defined by the parameterization $x(z,\overline{z}) = (x^1(z,\overline{z}), x^2(z,\overline{z}), x^3(z,\overline{z})$, where

$$x^1(z,\overline{z}) = Re\int f(1-g^2)dz \tag{18.3}$$

$$x^2(z,\overline{z}) = Re\int if(1+g^2)dz \tag{18.4}$$

$$x^3(z,\overline{z}) = Re\int fgdz \tag{18.5}$$

Suppose in WERI g is analytic and has an inverse function g^{-1}. Then we consider g as a new complex variable $\tau = g$ with $d\tau = g'dz$ Define $F(\tau) = f/g'$ and obtain $F(\tau)d\tau = fdz$. Therefore, if we replace g with τ and fdz with $F(\tau)d\tau$ we get

Weierstrass-Enneper Representation II For any analytic function $F(\tau)$, a minimal surface is defined by the parameterization $x(z,\overline{z}) = (x^1(z, overlinez), x^2(z,\overline{z}), x^3(z,\overline{z}))$, where

$$x^1(z,\overline{z}) = Re\int F(\tau)(1-\tau^2)dz \tag{18.6}$$

$$x^2(z,\overline{z}) = Re\int iF(\tau)(1+\tau^2)dz \tag{18.7}$$

$$x^3(z,\overline{z}) = Re\int F(\tau)\tau dz \tag{18.8}$$

This representation tells us that any analytic function $F(\tau)$ defines a minimal surface.

class exercise Find the WERI of the helicoid given in isothermal coordinates (u, v)

$$x(u,v) = (sinhusinv, -sinhucosv, -v)$$

Find the associated WERII. (answer: $i/2\tau^2$) Show that $F(\tau) = 1/2\tau^2$ gives rise to catenoid. Show moreover that $\tilde{\phi} = -i\phi$ for conjugate minimal surfaces x and $\tilde{x}$.

Notational convention We have two Fs here: The F of the first fundamental form and the F in WERII. In order to avoid confusion well denote the latter by T and hope that Oprea will not introduce a parameter using the same symbol. Now given a surface $x(u,v)$ in R^3 with $F = 0$ we make the following observations:

i. x_u, x_v and $N(u,v)$ constitute an orthogonal basis of R^3.

ii. N_u and N_v can be written in this basis coefficients being the coefficients of matrix dNp

iii. $x_u u, x_v v$ and $x_u v$ can be written in this basis. One should just compute the dot products $\langle x_{uu}, x_u\rangle, \langle x_{uu}, x_v\rangle, \langle x_{uu}, N\rangle$ in order to represent x_{uu} in this basis. The same holds for x_{uv} and x_{vv}. Using the above ideas one gets the following equations:

$$x_{uu} = \frac{E_u}{2E} x_u - \frac{E_v}{2G} + eN \tag{18.9}$$
$$x_{uv} = \frac{E_v}{2E} x_u + \frac{G_v}{2G} + fN \tag{18.10}$$
$$x_{vv} = \frac{-G_u}{2E} x_u + \frac{G_v}{2G} + gN \tag{18.11}$$
$$N_u = -\frac{e}{E} x_u - \frac{f}{G} x_v \tag{18.12}$$
$$N_v = -\frac{f}{E} x_u - \frac{g}{G} x_v \tag{18.13}$$

Now we state the Gausss theorem egregium:

Gausss Theorem Egregium The Gauss curvature K depends only on the metric $E, F = 0$and G:

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right)$$

This is an important theorem showing that the isometries do not change the Gaussian curvature.

proof If one works out the coefficient of x_v in the representation of $x_{uuv} - x_{uvu}$ one gets:

$$x_{uuv} = []x_u + [\frac{E_u G_u}{4EG} - (\frac{E_v}{2G})_v - \frac{E_v G_v}{4G^2} - \frac{eg}{G}]x_v + []N \tag{18.14}$$
$$x_{uvu} = []x_u + \frac{E_v}{2E} x_{uu} + (\frac{G_u}{2G})_u x_u v + f_u N + f N_u \tag{18.15}$$
$$x_{uvu} = []x_u + [-\frac{E_v E_v}{4EG} + (\frac{G_u}{2G})_u + \frac{G_u G_u}{4G^2} - \frac{f^2}{G}]x_v + []U \tag{18.16}$$

Because the x_v coefficient of $x_{uuv} - x_{uvu}$ is zero we get:

$$0 = \frac{E_u G_u}{4EG} - (\frac{E_v}{2G})_v - \frac{E_v G_v}{4G^2} + \frac{E_v E_v}{4EG} - (\frac{G_u}{2G})_u - \frac{GuGu}{4G^2} - \frac{eg - f^2}{G}$$

dividing by E, we have

$$\frac{eg-f^2}{EG} = \frac{E_uG_u}{4E^2G} - \frac{1}{E}(\frac{E_v}{2G})_v - \frac{E_vG_v}{4EG^2} + \frac{E_vE_v}{4E^2G} - \frac{1}{E}(\frac{G_u}{2G})_u - \frac{G_uG_u}{4EG^2}$$

Thus we have a formula for K which does not make explicit use of N:

$$K = -\frac{1}{2\sqrt{EG}}(\frac{\partial}{\partial v}(\frac{\partial E_v}{\partial\sqrt{EG}}) + \frac{\partial}{\partial u}(\frac{G_u}{\sqrt{EG}}))$$

Now we use Gausss theorem egregium to find an expression for K in terms of T of WERII

$$K = -\frac{1}{2\sqrt{EG}}(\frac{\partial}{\partial v}(\frac{E_v}{\sqrt{EG}}) + \frac{\partial}{\partial u}(\frac{G_u}{\sqrt{EG}})) \quad (18.17)$$

$$= -\frac{1}{2E}(\frac{\partial}{\partial v}(\frac{E_v}{E}) + \frac{\partial}{\partial u}(\frac{E_u}{E})) \quad (18.18)$$

$$= -\frac{1}{2E}\Delta(lnE) \quad (18.19)$$

Theorem The Gauss curvature of the minimal surface determined by the WER II is

$$K = \frac{-4}{|T|^2(1+u2+v2)^4}$$

where $\tau = u + iv$. That of a minimal surface determined by WER I is:

$$K = \frac{4|g'|^2}{|f|^2(1+|g|^2)^4}$$

In order to prove this thm one just sees that $E = 2|\phi|^2$ and makes use of the equation (20). Now we prove a proposition that will show WERs importance later.

Proposition Let M be a minimal surface with isothermal parameterization $x(u,v)$. Then the Gauss map of M is a conformal map.

proof In order to show N to be conformal we only need to show $|dNp(x_u)| =$

$\rho(u,v)|x_u|, |dNp(x_v)| = \rho(u,v)|x_v|$ and $dNp(x_u).dNp(x_v) = \rho^2 x_u.x_v$ Latter is trivial because of the isothermal coordinates. We have the following eqns for $dNp(x_u)$ and $dNp(x_v)$

$$dNp(x_u) = N_u = -\frac{e}{E}x_u - \frac{f}{G}x_v \tag{18.20}$$

$$dNp(x_v) = N_v = -\frac{f}{E}x_u - \frac{g}{G}x_v \tag{18.21}$$

By minimality we have $e + g = 0$. Using above eqns the Gauss map is conformal with scaling factor $\frac{\sqrt{e^2+f^2}}{E} = \sqrt{|K|}$ It turns out that having a conformal Gauss map almost characterizes minimal surfaces:

Proposition Let M be a surface parameterized by $x(u,v)$ whose Gauss map $N : M \longrightarrow S^2$ is conformal. Then either M is (part of) sphere or M is a minimal surface.

proof We assume that the surface is given by an orthogonal parameterization (F = 0) Since $x_u.x_v = 0$ by conformality of N $N_u.N_v = 0$ using the formulas (13) (14) one gets $f(Ge + Eg) = 0$ therefore either $e = 0$ (at every point) or $Ge + eG = 0$(everywhere). The latter is minimal surface equality. If the surface is not minimal then $f = 0$. Now use $f = 0$, confomality and (13), (14) to get

$$\frac{e^2}{E} = N_u.N_u = \rho^2 E, \frac{g^2}{G} = N_v.N_v = \rho^2 G$$

Multiplying across each equation produces

$$\frac{e^2}{E^2} = \frac{g^2}{G^2} \Rightarrow \frac{e}{G} = \pm\frac{g}{G}$$

The last equation with minus sign on LHS is minimal surface equation so we may just consider the case $e/E = g/G = k$. Together with $f = 0$ we have $N_u = -kx_u$ and $N_v = -kx_v$ this shows that x_u and x_v are eigenvectors of the differential of the Gauss map with the same eigenvalue. Therefore any point on M is an umbilical point. The only surface satisfying this property is sphere so were done.

Steographic Projection: $St : S^2 - N \longrightarrow R^2$ is given by $St(x, y, z) = (x/(1-z), y/(1-z), 0)$ We can consider the Gauss map as a mapping from the surface to $C \cup \infty$ by taking its composite with steographic projection.Note that the resulting map is still conformal since both of Gauss map and Steographic are conformal. Now we state a thm which shows that WER can actually be attained naturally:

Theorem Let M be a minimal surface with isothermal parameterization $x(u, v)$ and WER (f, g). Then the Gauss map of M, $G : M \longrightarrow C \cup \infty$ can be identified with the meromorphic function g.

proof Recall that

$$\phi^1 = \frac{1}{2} f(1 - g^2), \phi^2 = i2f(1 + g^2), \phi^3 = fg$$

We will describe the Gauss map in terms of ϕ^1, ϕ^2 and ϕ^3.

$$x_u \times x_v = ((x_u \times x_v)^1, (x_u \times x_v)^2, (x_u \times x_v)^3) \tag{18.22}$$

$$= (x_u^2 x_v^3 - x_u^3 x_v^2, x_u^3 x_v^1 - x_u^1 x_v^3, x_u^1 x_v^2 - x_u^2 x_v^1) \tag{18.23}$$

and consider the first component $x_u^2 x_v^3 - x_u^3 x_v^2$ we have

$$x_u^2 x_v^3 - x_u^3 x_v^2 = 4Im(\phi^2 \overline{\phi}^3)$$

Similarly $(x_u \times x_v)^2 = 4Im(\phi^2 \overline{\phi}^1)$ and $(x_u \times x_v)^3 = 4Im(\phi^1 \overline{\phi^2})$ Hence we obtain

$$x_u \times x_v = 4Im(\phi^2 \overline{\phi^3}, \phi^3 \overline{\phi^1}, \phi^1 \overline{\phi^2}) = 2Im(\phi \times \overline{\phi})$$

Now since $x(u, v)$ is isothermal $|x_u \times x_v| = |x_u||x_v| = E = 2|\phi|^2$. Therefore we have

$$N = \frac{x_u \times x_v}{|x_u \times x_v|} = \frac{\phi \times \overline{\phi}}{|\phi|^2}$$

Now

$$\begin{aligned}
G(u,v) &= St(N(u,v)) && (18.24)\\
&= St(\frac{x_u \times x_v}{|x_u \times x_v|}) && (18.25)\\
&= St(\frac{\phi \times \overline{\phi}}{|\phi|^2}) && (18.26)\\
&= St(2Im(\phi^2\overline{\phi^3}, \phi^3\overline{\phi^1}, \phi^1\overline{\phi^2})|\phi|^2) && (18.27)\\
&= (\frac{2Im(\phi^2\overline{\phi^3})}{|\phi|^2 - 2Im(\phi^1\overline{\phi^2})}, \frac{2Im(\phi^3\overline{\phi^1})}{|\phi|^2 - 2Im(\phi^1\overline{\phi^2})}, 0) && (18.28)
\end{aligned}$$

Identifying (x, y) in R^2 with $x + iy \in C$ allows us to write

$$G(u,v) = \frac{2Im(\phi^2\overline{\phi^3}) + 2iIm(\phi^3\overline{\phi^1})}{|\phi|^2 - 2Im(\phi^1\overline{\phi^2})}$$

Now its simple algebra to show that

$$G(u,v) = \frac{\phi^3}{\phi^1 - i\phi^2}$$

But that equals to g so were done.