Lecture Notes on Minimal Surfaces

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Chapter 1

Introduction

Minimal surface has zero curvature at every point on the surface. Since a surface surrounded by a boundary is minimal if it is an area minimizer, the study of minimal surface has arised many interesting applications in other fields in science, such as soap films.

In this book, we have included the lecture notes of a seminar course about minimal surfaces between September and December, 2004. The course consists of a review on multi-variable calculus (Lectures 2 - 4), some basic differential geometry (Lectures 5 - 10), and representation and properties of minimal surface (Lectures 11 - 16). The authors took turns in giving lectures. The target audience of the course was any advanced undergraduate student who had basic analysis and algebra knowledge.

Chapter 2

A Review on Differentiation

Reading: Spivak pp. 15-34, or Rudin 211-220

2.1 Differentiation

Recall from 18.01 that

Definition 2.1.1. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is* **differentiable** *at $a \in \mathbb{R}^n$ if there exists a linear transformation $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0 \tag{2.1}$$

The norm in Equation 2.1 is essential since $f(a+h) - f(a) - \lambda(h)$ is in $\mathbb{R}^m$ and h is in $\mathbb{R}^n$.

Theorem 2.1.2. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then there is a* **unique** *linear transformation $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies Equation* (2.1)*. We denote λ to be $Df(a)$ and call it the* **derivative** *of f at a*

Proof. Let $\mu : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \mu(h)|}{|h|} = 0 \tag{2.2}$$

and $d(h) = f(a+h) - f(a)$, then

$$\lim_{h \to 0} \frac{|\lambda(h) - \mu(h)|}{|h|} = \lim_{h \to 0} \frac{|\lambda(h) - d(h) + d(h) - \mu(h)|}{|h|} \tag{2.3}$$
$$\leq \lim_{h \to 0} \frac{|\lambda(h) - d(h)|}{|h|} + \lim_{h \to 0} \frac{|d(h) - \mu(h)|}{|h|} \tag{2.4}$$
$$= 0. \tag{2.5}$$

Now let $h = tx$ where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, then as $t \to 0$, $tx \to 0$. Thus, for $x \neq 0$, we have

$$\lim_{t \to 0} \frac{|\lambda(tx) - \mu(tx)|}{|tx|} = \frac{|\lambda(x) - \mu(x)|}{|x|} \tag{2.6}$$
$$= 0 \tag{2.7}$$

Thus $\mu(x) = \lambda(x)$. □

Although we proved in Theorem 2.1.2 that if $Df(a)$ exists, then it is unique. However, we still have not discovered a way to find it. All we can do at this moment is just by guessing, which will be illustrated in Example 1.

Example 1. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by*

$$g(x, y) = \ln x \tag{2.8}$$

Proposition 2.1.3. *$Dg(a, b) = \lambda$ where λ satisfies*

$$\lambda(x, y) = \frac{1}{a} \cdot x \tag{2.9}$$

Proof.

$$\lim_{(h,k) \to 0} \frac{|g(a+h, b+k) - g(a, b) - \lambda(h, k)|}{|(h, k)|} = \lim_{(h,k) \to 0} \frac{|\ln(a+h) - \ln(a) - \frac{1}{a} \cdot h|}{|(h, k)|} \tag{2.10}$$

Since $\ln'(a) = \frac{1}{a}$, we have

$$\lim_{h \to 0} \frac{|\ln(a+h) - \ln(a) - \frac{1}{a} \cdot h|}{|h|} = 0 \tag{2.11}$$

Since $|(h,k)| \geq |h|$, we have

$$\lim_{(h,k) \to 0} \frac{|\ln(a+h) - \ln(a) - \frac{1}{a} \cdot h|}{|(h,k)|} = 0 \tag{2.12}$$

$\square$

Definition 2.1.4. *The* **Jacobian matrix** *of f at a is the $m \times n$ matrix of $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the usual bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, and denoted $f'(a)$.*

Example 2. *Let g be the same as in Example 1, then*

$$g'(a,b) = (\frac{1}{a}, 0) \tag{2.13}$$

Definition 2.1.5. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is* **differentiable on** *$A \subset \mathbb{R}^n$ if f is diffrentiable at a for all $a \in A$. On the other hand, if $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$, then f is called* **differentiable** *if f can be extended to a differentiable function on some open set containing A.*

2.2 Properties of Derivatives

Theorem 2.2.1. *1. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a constant function, then $\forall a \in \mathbb{R}^n$,*

$$Df(a) = 0. \tag{2.14}$$

2. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $\forall a \in \mathbb{R}^n$

$$Df(a) = f. \tag{2.15}$$

Proof. The proofs are left to the readers $\square$

Theorem 2.2.2. *If $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by $g(x, y) = xy$, then*

$$Dg(a, b)(x, y) = bx + ay \tag{2.16}$$

In other words, $g'(a, b) = (b, a)$

Proof. Substitute p and Dp into L.H.S. of Equation 2.1, we have

$$\begin{align}
\lim_{(h,k)\to 0} \frac{|g(a+h, b+k) - g(a, b) - Dg(a, b)(h, k)|}{|(h, k)|} &= \lim_{(h,k)\to 0} \frac{|hk|}{|(h, k)|} \tag{2.17} \\
&\leq \lim_{(h,k)\to 0} \frac{\max(|h|^2, |k|^2)}{\sqrt{h^2 + k^2}} \tag{2.18} \\
&\leq \sqrt{h^2 + k^2} \tag{2.19} \\
&= 0 \tag{2.20}
\end{align}$$

$\square$

Theorem 2.2.3. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, and $g : \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $f(a)$, then the composition $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at a, and*

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a) \tag{2.21}$$

Proof. Put $b = f(a)$, $\lambda = f'(a)$, $\mu = g'(b)$, and

$$\begin{align}
u(h) &= f(a+h) - f(a) - \lambda(h) \tag{2.22} \\
v(k) &= g(b+k) - g(b) - \mu(k) \tag{2.23}
\end{align}$$

for all $h \in \mathbb{R}^n$ and $k \in \mathbb{R}^m$. Then we have

$$\begin{align}
|u(h)| &= \epsilon(h)|h| \tag{2.24} \\
|v(k)| &= \eta(k)|k| \tag{2.25}
\end{align}$$

where

$$\lim_{h\to 0} \epsilon(h) = 0 \tag{2.26}$$

$$\lim_{k\to 0} \eta(k) = 0 \tag{2.27}$$

Given h, we can put k such that $k = f(a+h) - f(a)$. Then we have

$$|k| = |\lambda(h) + u(h)| \leq [\|\lambda\| + \epsilon(h)]|h| \tag{2.28}$$

Thus,

$$\begin{aligned} g\circ f(a+h) - g\circ f(a) - \mu(\lambda(h)) &= g(b+k) - g(b) - \mu(\lambda(h)) && (2.29)\\ &= \mu(k - \lambda(h)) + v(k) && (2.30)\\ &= \mu(u(h)) + v(k) && (2.31) \end{aligned}$$

Thus

$$\frac{|g\circ f(a+h) - g\circ f(a) - \mu(\lambda(h))|}{|h|} \leq \|\mu\|\epsilon(h) + [\|\lambda\| + \epsilon(h)]\eta(h) \tag{2.32}$$

which equals 0 according to Equation 2.26 and 2.27. □

Exercise 1. *(Spivak 2-8) Let $f : \mathbb{R} \to \mathbb{R}^2$. Prove that f is differentiable at $a \in \mathbb{R}$ if and only if f^1 and f^2 are, and that in this case*

$$f'(a) = \begin{pmatrix} (f^1)'(a) \\ (f^2)'(a) \end{pmatrix} \tag{2.33}$$

Corollary 2.2.4. *If $f : \mathbb{R}^n \to \mathbb{R}^m$, then f is differentiable at $a \in \mathbb{R}^n$ if and* *only if each f^i is, and*

$$\lambda'(a) = \begin{pmatrix} (f^1)'(a) \\ (f^2)'(a) \\ . \\ . \\ . \\ (f^m)'(a) \end{pmatrix}. \tag{2.34}$$

Thus, $f'(a)$ is the $m \times n$ matrix whose ith row is $(f^i)'(a)$

Corollary 2.2.5. *If $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at a, then*

$$D(f+g)(a) = Df(a) + Dg(a) \tag{2.35}$$

$$D(fg)(a) = g(a)Df(a) + f(a)Dg(a) \tag{2.36}$$

$$D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{[g(a)]^2}, \ g(a) \neq 0 \tag{2.37}$$

Proof. The proofs are left to the readers. □

2.3 Partial Derivatives

Definition 2.3.1. *If $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$, then the limit*

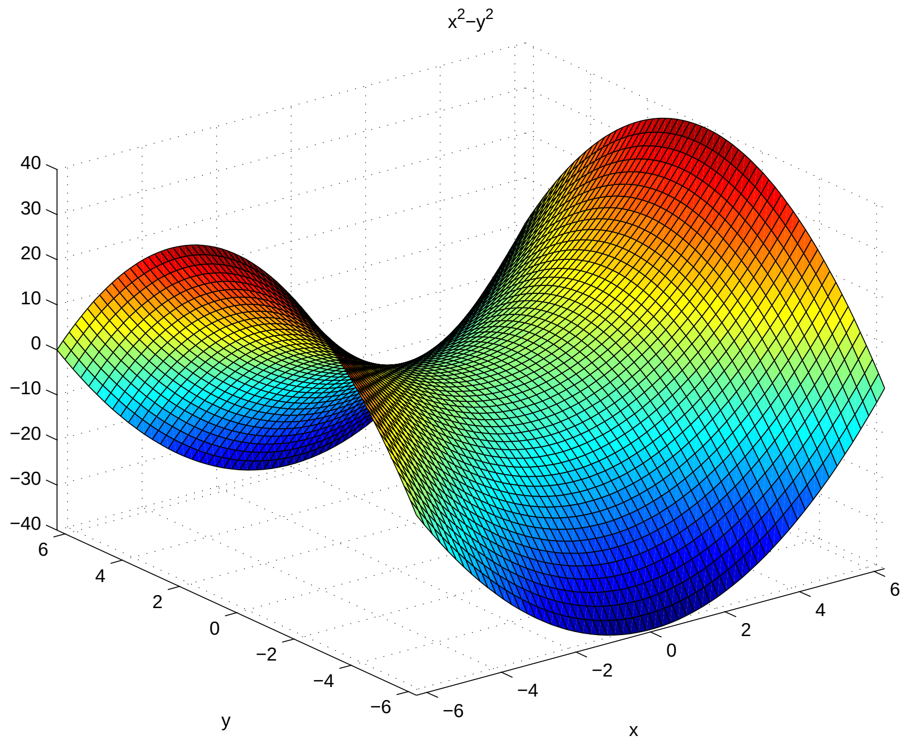
$$D_i f(a) = \lim_{h \to 0} \frac{f(a^1, ..., a^i + h, ..., a^n) - f(a^1, ..., a^n)}{h} \tag{2.38}$$

is called the ith **partial derivative** *of f at a if the limit exists.*

If we denote $D_j(D_i f)(x)$ to be $D_{i,j}(x)$, then we have the following theorem which is stated without proof. (The proof can be found in Problem 3-28 of Spivak)

Theorem 2.3.2. *If $D_{i,j}f$ and $D_{j,i}f$ are continuous in an open set containing a, then*

$$D_{i,j}f(a) = D_{j,i}f(a) \tag{2.39}$$



Partial derivatives are useful in finding the extrema of functions.

Theorem 2.3.3. *Let $A \subset \mathbb{R}^n$. If the maximum (or minimum) of $f : A \to \mathbb{R}$ occurs at a point a in the interior of A and $D_i f(a)$ exists, then $D_i f(a) = 0$.*

Proof. The proof is left to the readers. $\square$

However the converse of Theorem 2.3.3 may not be true in all cases. (Consider $f(x, y) = x^2 - y^2$).

2.4 Derivatives

Theorem 2.4.1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then $D_j f^i(a)$ exists for $1 \leq i \leq m, 1 \leq j \leq n$ and $f'(a)$ is the $m \times n$ matrix $(D_j f^i(a))$.*

Chapter 3

Inverse Function Theorem

(This lecture was given Thursday, September 16, 2004.)

3.1 Partial Derivatives

Definition 3.1.1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}^n$, then the limit*

$$D_i f(a) = \lim_{h \to 0} \frac{f(a^1, \ldots, a^i + h, \ldots, a^n) - f(a^1, \ldots, a^n)}{h} \tag{3.1}$$

is called the i^{th} ***partial derivative*** *of f at a, if the limit exists.*

Denote $D_j(D_i f(x))$ by $D_{i,j}(x)$. This is called a **second-order (mixed) partial derivative**. Then we have the following theorem (**equality of mixed partials**) which is given without proof. The proof is given later in Spivak, Problem 3-28.

Theorem 3.1.2. *If $D_{i,j} f$ and $D_{j,i} f$ are continuous in an open set containing a, then*

$$D_{i,j} f(a) = D_{j,i} f(a) \tag{3.2}$$

We also have the following theorem about partial derivatives and maxima and minima which follows directly from 1-variable calculus:

Theorem 3.1.3. *Let $A \subset \mathbb{R}^n$. If the maximum (or minimum) of $f : A \to \mathbb{R}$ occurs at a point a in the interior of A and $D_i f(a)$ exists, then $D_i f(a) = 0$.*

Proof: Let $g_i(x) = f(a^1, \ldots, x, \ldots, a^n)$. g_i has a maximum (or minimum) at a^i, and g_i is defined in an open interval containing a^i. Hence $0 = g_i'(a^i) = 0$.

The converse is not true: consider $f(x, y) = x^2 - y^2$. Then f has a minimum along the x-axis at 0, and a maximum along the y-axis at 0, but $(0, 0)$ is neither a relative minimum nor a relative maximum.

3.2 Derivatives

Theorem 3.2.1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then $D_j f^i(a)$ exists for $1 \le i \le m, 1 \le j \le n$ and $f'(a)$ is the m x n matrix $(D_j f^i(a))$.*

Proof: First consider $m = 1$, so $f : \mathbb{R}^n \to \mathbb{R}$. Define $h : \mathbb{R} \to \mathbb{R}^n$ by $h(x) = (a^1, \ldots, x, \ldots, a^n)$, with x in the j^{th} slot. Then $D_j f(a) = (f \circ h)'(a^j)$. Applying the chain rule,

$$\begin{aligned} (f \circ h)'(a^j) &= f'(a) \cdot h'(a^j) \\ &= f'(a) \cdot \begin{pmatrix} 0 \\ . \\ . \\ . \\ . \\ 1 \\ . \\ . \\ . \\ 0 \end{pmatrix} \end{aligned} \tag{3.3}$$

Thus $D_j f(a)$ exists and is the jth entry of the $1 \times n$ matrix $f'(a)$.

Spivak 2-3 (3) states that f is differentiable if and only if each f^i is. So the theorem holds for arbitrary m, since each f^i is differentiable and the ith row of $f'(a)$ is $(f^i)'(a)$.

The converse of this theorem – that if the partials exists, then the full derivative does – only holds if the partials are continuous.

Theorem 3.2.2. *If $f : \mathbb{R}^n \to \mathbb{R}^m$, then $Df(a)$ exists if all $D_j f(i)$ exist in an open set containing a and if each function $D_j f(i)$ is continuous at a. (In this case f is called **continuously differentiable**.)*

Proof.: As in the prior proof, it is sufficient to consider m = 1 (i.e., $f : \mathbb{R}^n \to \mathbb{R}$.)

$$\begin{aligned} f(a+h) - f(a) = \; & f(a^1 + h^1, a^2, \ldots, a^n) - f(a^1, \ldots, a^n) \\ & + f(a^1 + h^1, a^2 + h^2, a^3, \ldots, a^n) - f(a^1 + h^1, a^2, \ldots, a^n) \\ & + \ldots + f(a^1 + h^1, \ldots, a^n + h^n) \\ & - f(a^1 + h^1, \ldots, a^{n-1} + h^{n-1}, a^n). \end{aligned} \tag{3.4}$$

$D_1 f$ is the derivative of the function $g(x) = f(x, a^2, \ldots, a^n)$. Apply the mean-value theorem to g :

$$f(a^1 + h^1, a^2, \ldots, a^n) - f(a^1, \ldots, a^n) = h^1 \cdot D_1 f(b_1, a^2, \ldots, a^n). \tag{3.5}$$

for some b^1 between a^1 and $a^1 + h^1$. Similarly,

$$h^i \cdot D_i f(a^1 + h^1, \ldots, a^{i-1} + h^{i-1}, b_i, \ldots, a^n) = h^i D_i f(c_i) \tag{3.6}$$

for some c_i. Then

$$\begin{aligned}
&\lim_{h\to 0} \tfrac{|f(a+h)-f(a)-\sum_i D_i f(a)\cdot h^i|}{|h|} \\
&= \lim_{h\to 0} \tfrac{\sum_i [D_i f(c_i) - D_i f(a)\ \cdot h^i]}{|h|} \\
&\leq \lim_{h\to 0} \textstyle\sum_i |D_i f(c_i) - D_i f(a)| \cdot \tfrac{|h^i|}{|h|} \\
&\leq \lim_{h\to 0} \textstyle\sum_i |D_i f(c_i) - D_i f(a)| \\
&= 0
\end{aligned} \tag{3.7}$$

since $D_i f$ is continuous at 0.

Example 3. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be the function* $f(x,y) = xy/(\sqrt{x^2+y^2}$ *if* $(x,y) \neq (0,0)$ *and 0 otherwise (when* $(x,y) = (0,0)$*). Find the partial derivatives at* $(0,0)$ *and check if the function is differentiable there.*

3.3 The Inverse Function Theorem

(A sketch of the proof was given in class.)

Chapter 4

Implicit Function Theorem

4.1 Implicit Functions

Theorem 4.1.1. ***Implicit Function Theorem*** *Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is continuously differentiable in an open set containing (a, b) and $f(a, b) = 0$. Let M be the $m \times m$ matrix $D_{n+j}f^i(a, b), 1 \leq i, j \leq m$ If $det(M) \neq 0$, there is an open set $A \subset \mathbb{R}^n$ containing a and an open set $B \subset \mathbb{R}^m$ containing b, with the following property: for each $x \in A$ there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$. The function g is differentiable.*

proof Define $F : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$ by $F(x, y) = (x, f(x, y))$. Then $det(dF(a, b)) = det(M) \neq 0$. By inverse function theorem there is an open set $W \subset \mathbb{R}^n \times \mathbb{R}^m$ containing $F(a, b) = (a, 0)$ and an open set in $\mathbb{R}^n \times \mathbb{R}^m$ containing (a, b), which we may take to be of the form $A \times B$, such that $F : A \times B \longrightarrow W$ has a differentiable inverse $h : W \longrightarrow A \times B$. Clearly h is the form $h(x, y) = (x, k(x, y))$ for some differentiable function k (since f is of this form)Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be defined by $\pi(x, y) = y$; then $\pi \circ F = f$. Therefore $f(x, k(x, y)) = f \circ h(x, y) = (\pi \circ F) \circ h(x, y) = \pi(x, y) = y$ Thus $f(x, k(x, 0)) = 0$ in other words we can define $g(x) = k(x, 0)$

As one might expect the position of the m columns that form M is immaterial. The same proof will work for any $f'(a, b)$ provided that the rank of the matrix is m.

Example $f : \mathbb{R}^2 \longrightarrow \mathbb{R}, f(x,y) = x^2 + y^2 - 1$. $Df = (2x2y)$ Let $(a,b) = (3/5, 4/5)$ M will be $(8/5)$. Now implicit function theorem guarantees the existence and teh uniqueness of g and open intervals $I, J \subset \mathbb{R}, 3/5 \in I, 4/5 in J$ so that $g : I \longrightarrow J$ is differentiable and $x^2 + g(x)^2 - 1 = 0$. One can easily verify this by choosing $I = (-1,1), J = (0,1)$ and $g(x) = \sqrt{1-x^2}$. Note that the uniqueness of $g(x)$ would fail to be true if we did not choose J appropriately.

example Let A be an $m \times (m+n)$ matrix. Consider the function $f : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m, f(x) = Ax$ Assume that last m columns $C_{n+1}, C_{n+2}, ..., C_{m+n}$ are linearly independent. Break A into blocks $A = [A'|M]$ so that M is the $m \times m$ matrix formed by the last m columns of A. Now the equation $AX = 0$ is a system of m linear equations in $m+n$ unknowns so it has a nontrivial solution. Moreover it can be solved as follows: Let $X = [X_1|X_2]$ where $X_1 \in \mathbb{R}^{n\times 1}$ and $X_2 \in \mathbb{R}^{m\times 1}$ $AX = 0$ implies $A'X_1 + MX_2 = 0 \Rightarrow X_2 = M^{-1}A'X_1$. Now treat f as a function mapping $\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ by setting $f(X_1, X_2) = AX$. Let $f(a,b) = 0$. Implicit function theorem asserts that there exist open sets $I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ and a function $g : I \longrightarrow J$ so that $f(x, g(x)) = 0$. By what we did above $g = M^{-1}A'$ is the desired function. So the theorem is true for linear transformations and actually I and J can be chosen $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

4.2 Parametric Surfaces

(Following the notation of *Osserman* E^n denotes the Euclidean n-space.) Let D be a domain in the u-plane, $u = (u_1, u_2)$. A parametric surface is simply the image of some differentiable transformation $u : D \longrightarrow E^n$.(A non-empty open set in $\mathbb{R}^2$ is called a domain.)
Let us denote the Jacobian matrix of the mapping $x(u)$ by

$$M = (m_{ij}); m_{ij} = \frac{\partial x_i}{\partial u_j}, i = 1, 2, .., n; j = 1, 2.$$

We introduce the exterior product

$$v \wedge w; w \wedge v \in E^{n(n-1)/2}$$

where the components of $v \wedge w$ are the determinants $\det \begin{pmatrix} v_i & v_j \\ u_i & u_j \end{pmatrix}$ arranged in some fixed order. Finally let

$$G = (g_{ij}) = M^T M; g_{ij} = \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}$$

Note that G is a 2×2 matrix. To compute $det(G)$ we recall Lagrange's identity:

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{1 \leq i,j \leq n} (a_i b_j - a_j b_i)^2$$

Proof of Lagrange's identity is left as an exercise. Using Langrange's identity one can deduce

$$det(G) = \left|\frac{\partial x}{\partial u_1} \wedge xu_2\right|^2 = \sum_{1 \leq i,j \leq n} \left(\frac{\partial(x_i, x_j)}{\partial(u_1, u_2)}\right)^2$$

Chapter 5

First Fundamental Form

5.1 Tangent Planes

One important tool for studying surfaces is the *tangent plane*. Given a given regular parametrized surface S embedded in $\mathbb{R}^n$ and a point $p \in S$, a *tangent vector* to S at p is a vector in $\mathbb{R}^n$ that is the tangent vector $\alpha'(0)$ of a differential parametrized curve $\alpha\colon (-\epsilon, \epsilon) \to S$ with $\alpha(0) = p$. Then the tangent plane $T_p(S)$ to S at p is the set of all tangent vectors to S at p. This is a set of $\mathbb{R}^3$-vectors that end up being a plane.

An equivalent way of thinking of the tangent plane is that it is the image of $\mathbb{R}^2$ under the linear transformation $Dx(q)$, where x is the map from a domain $D \to S$ that defines the surface, and q is the point of the domain that is mapped onto p. Why is this equivalent? We can show that x is invertible. So given any tangent vector $\alpha'(0)$, we can look at $\gamma = x^{-1} \circ \alpha$, which is a curve in D. Then $\alpha'(0) = (x \circ \gamma)'(0) = (Dx(\gamma(0)) \circ \gamma')(0) = Dx(q)(\gamma'(0))$. Now, γ can be chosen so that $\gamma'(0)$ is *any* vector in $\mathbb{R}^2$. So the tangent plane is the image of $\mathbb{R}^2$ under the linear transformation $Dx(q)$.

Certainly, though, the image of $\mathbb{R}^2$ under an invertible linear transformation (it's invertible since the surface is regular) is going to be a plane including the origin, which is what we'd want a tangent plane to be. (When

I say that the tangent plane includes the origin, I mean that the plane itself consists of all the vectors of a plane through the origin, even though usually you'd draw it with all the vectors emanating from p instead of the origin.)

This way of thinking about the tangent plane is like considering it as a "linearization" of the surface, in the same way that a tangent line to a function from $\mathbb{R} \to \mathbb{R}$ is a linear function that is locally similar to the function. Then we can understand why $Dx(q)(\mathbb{R}^2)$ makes sense: in the same way we can "replace" a function with its tangent line which is the image of $\mathbb{R}$ under the map $t \mapsto f'(p)t + C$, we can replace our surface with the image of $\mathbb{R}^2$ under the map $Dx(q)$.

The interesting part of seeing the tangent plane this way is that you can then consider it as having a basis consisting of the images of $(1, 0)$ and $(0, 1)$ under the map $Dx(q)$. These images are actually just (if the domain in $\mathbb{R}^2$ uses u_1 and u_2 as variables) $\frac{\partial x}{\partial u_1}$ and $\frac{\partial x}{\partial u_2}$ (which are n-vectors).

5.2 The First Fundamental Form

Nizam mentioned the First Fundamental Form. Basically, the FFF is a way of finding the length of a tangent vector (in a tangent plane). If w is a tangent vector, then $|w|^2 = w \cdot w$. Why is this interesting? Well, it becomes more interesting if you're considering w not just as its $\mathbb{R}^3$ coordinates, but as a linear combination of the two basis vectors $\frac{\partial x}{\partial u_1}$ and $\frac{\partial x}{\partial u_2}$. Say $w = a\frac{\partial x}{\partial u_1} + b\frac{\partial x}{\partial u_2}$; then

$$\begin{aligned} |w|^2 &= \left(a\frac{\partial x}{\partial u_1} + b\frac{\partial x}{\partial u_2}\right) \cdot \left(a\frac{\partial x}{\partial u_1} + b\frac{\partial x}{\partial u_2}\right) \\ &= a^2\frac{\partial x}{\partial u_1} \cdot \frac{\partial x}{\partial u_1} + 2ab\frac{\partial x}{\partial u_1} \cdot \frac{\partial x}{\partial u_2} + b^2\frac{\partial x}{\partial u_2} \cdot \frac{\partial x}{\partial u_2}. \end{aligned} \tag{5.1}$$

Let's deal with notational differences between do Carmo and Osserman. do Carmo writes this as $Ea^2 + 2Fab + Gb^2$, and refers to the whole thing as $I_p \colon T_p(S) \to \mathbb{R}$.[1] Osserman lets $g_{11} = E$, $g_{12} = g_{21} = F$ (though he never

[1]Well, actully he's using u' and v' instead of a and b at this point, which is because these coordinates come from a tangent vector, which is to say they are the $u'(q)$ and $v'(q)$

makes it too clear that these two are equal), and $g_{22} = G$, and then lets the matrix that these make up be G, which he also uses to refer to the whole form. I am using Osserman's notation.

Now we'll calculate the FFF on the cylinder over the unit circle; the parametrized surface here is $x\colon (0, 2\pi) \times \mathbb{R} \to S \subset \mathbb{R}^3$ defined by $x(u, v) = (\cos u, \sin u, v)$. (Yes, this misses a vertical line of the cylinder; we'll fix this once we get away from *parametrized* surfaces.) First we find that $\frac{\partial x}{\partial u} = (-\sin u, \cos u, 0)$ and $\frac{\partial x}{\partial v} = (0, 0, 1)$. Thus $g_{11} = \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial u} = \sin^2 u + \cos^2 u = 1$, $g_{21} = g_{12} = 0$, and $g_{22} = 1$. So then $|w|^2 = a^2 + b^2$, which basically means that the length of a vector in the tangent plane to the cylinder is the same as it is in the $(0, 2\pi) \times \mathbb{R}$ that it's coming from.

As an exercise, calculate the first fundamental form for the sphere S^2 parametrized by $x\colon (0, \pi) \times (0, 2\pi) \to S^2$ with

$$x(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{5.2}$$

We first calculate that $\frac{\partial x}{\partial \theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$ and $\frac{\partial x}{\partial \varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$. So we find eventually that $|w|^2 = a^2 + b^2\sin^2\theta$. This makes sense — movement in the φ direction (latitudinally) should be "worth more" closer to the equator, which is where $\sin^2\theta$ is maximal.

5.3 Area

If we recall the exterior product from last time, we can see that $\left|\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v}\right|$ is the area of the parallelogram determined by $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$. This is analogous to the fact that in 18.02 the magnitude of the cross product of two vectors is the area of the parallelogram they determine. Then $\int_Q \left|\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v}\right| \mathrm{d}u\mathrm{d}v$ is the area of the bounded region Q in the surface. But Nizam showed yesterday

of some curve in the domain D.

that Lagrange's Identity implies that

$$\left|\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v}\right|^2 = \left|\frac{\partial x}{\partial u}\right|^2 \left|\frac{\partial x}{\partial v}\right|^2 - \left(\frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v}\right)^2 \tag{5.3}$$

Thus $\left|\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v}\right| = \sqrt{g_{11}g_{22} - g_{12}^2}$. Thus, the area of a bounded region Q in the surface is $\int_Q \sqrt{g_{11}g_{22} - g_{12}^2}\mathrm{d}u\mathrm{d}v$.

For example, let us compute the surface area of a torus; let's let the radius of a meridian be r and the longitudinal radius be a. Then the torus (minus some tiny strip) is the image of $x\colon (0, 2\pi) \times (0, 2\pi) \to S^1 \times S^1$ where $x(u, v) = ((a + r\cos u)\cos v, (a + r\cos u)\sin v), r\sin u)$. Then $\frac{\partial x}{\partial u} = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$, and $\frac{\partial x}{\partial v} = (-(a+r\cos u)\sin v, (a+r\cos u)\cos v, 0)$. So $g_{11} = r^2$, $g_{12} = 0$, and $g_{22} = (r\cos u + a)^2$. Then $\sqrt{g_{11}g_{22} - g_{12}^2} = r(r\cos u + a)$. Integrating this over the whole square, we get

$$\begin{aligned}
A &= \int_0^{2\pi}\int_0^{2\pi} (r^2\cos u + ra)\mathrm{d}u\mathrm{d}v \\
&= \left(\int_0^{2\pi} (r^2\cos u + ra)\mathrm{d}u\right)\left(\int_0^{2\pi} \mathrm{d}v\right) \\
&= (r^2\sin 2\pi + ra2\pi)(2\pi) = 4\pi^2 ra
\end{aligned}$$

And this is the surface area of a torus!

(This lecture was given Wednesday, September 29, 2004.)

Chapter 6

Curves

6.1 Curves as a map from $\mathbb{R}$ to $\mathbb{R}^n$

As we've seen, we can say that a **parameterized differentiable curve** is a differentiable map α from an open interval $I = (-\epsilon, \epsilon)$ to $\mathbb{R}^n$. Differentiability here means that each of our coordinate functions are differentiable: if α is a map from some I to $\mathbb{R}^3$, and $\alpha = (x(t), y(t), z(t))$, then α being differentiable is saying that $x(t), y(t)$, and $z(t)$ are all differentiable. The vector $\alpha'(t) = (x'(t), y'(t), z'(t))$ is called the tangent vector of α at t.

One thing to note is that, as with our notation for surfaces, our curve is a differentiable map, and not a subset of $\mathbb{R}^n$. do Carmo calls the image set $\alpha(I) \subset \mathbb{R}^3$ the **trace** of α. But multiple "curves," i.e., differentiable maps, can have the same image or trace. For example,

Example 4. *Let*

$$\alpha(t) = (\cos(t), \sin(t)) \tag{6.1}$$

$$\beta(t) = (\cos(2t), \sin(2t)) \tag{6.2}$$

$$\gamma(t) = (\cos(-t), \sin(-t)) \tag{6.3}$$

with t in the interval $(-\epsilon, 2\pi + \epsilon)$.

Then the image of α , β, and γ are all the same, namely, the unit circle centered at the origin. But the velocity vector of β is twice that of α's, and γ runs as fast as α does, but in the opposite direction – the orientation of the curve is reversed. (In general, when we have a curve α defined on (a, b), and define $\gamma(-t) = \alpha(t)$ on $(-b, -a)$, we say α and γ differ by a change of orientation.*)*

6.2 Arc Length and Curvature

Now we want to describe properties of curves. A natural one to start with is **arc length.** We define the arc length s of a curve α from a time t_0 to a time t as follows

$$s(t) = \int_{t_0}^{t} ds \tag{6.4}$$
$$= \int_{t_0}^{t} \frac{ds}{dt} dt \tag{6.5}$$
$$= \int_{t_0}^{t} |\alpha'(t)| dt \tag{6.6}$$

The next property that we would like to define is **curvature.** We want curvature to reflect properties of the image of our curve – i.e., the actual subset C in $\mathbb{R}^n$ – as opposed to our map. Letting our curvature vector be defined as $\alpha''(t)$ has the problem that, while it measures how fast the tangent vector is changing along the curve, it also measures how large our velocity vector is, which can vary for maps with the same image. Looking back at Example 1, while both α and β's images were the unit circle, the velocity and acceleration vectors were different:

$$\alpha'(t) = (-\sin t, \cos t) \tag{6.7}$$

$$\alpha''(t) = (-\cos t, -\sin t), |\alpha''(t)| = 1 \tag{6.8}$$

$$\beta'(t) = (-2\sin(2t), 2\cos(2t)) \tag{6.9}$$

$$\beta''(t) = (-4\cos(2t), -4\sin(2t)), |\beta''(t)| = 4 \tag{6.10}$$

A way to correct this problem is to scale our velocity vector down to unit length – i.e., let ds/dt = 1. Then the magnitude of the velocity vector won't skew our curvature vector, and we'll just be able to look at how much the angle between neighboring tangent vectors is changing when we look at $\alpha''(t)$ (how curved our curve is!)

To do this we **parameterize by arc length.** First, we look at the arc length function $s(t) = \int_{t_0}^{t} |\alpha'(t)|dt$. If $\alpha'(t) \neq 0$ for all t, then our function is always increasing and thus has an inverse s^{-1}. So instead of mapping from time, along an interval I, into length, we can map from length into an interval I.

Definition 6.2.1. *: A paramaterized differentiable curve $\alpha : I \to \mathbb{R}^3$ is **regular** if $\alpha'(t) \neq 0$ for all $t \in I$.*

Thus to parameterize by arc length, we require our curve to be regular. Then, given a fixed starting point t_0 going to a fixed end point t_1, and the length of our arc from t_0 to t_1 being L, we can reparameterize as follows:

$$(0, L) \to (t_0, t_1) \to \mathbb{R}^3 \text{ (or } \mathbb{R}^n.)$$

where the first arrow is given by s^{-1}, and the second arrow is just our curve α mapping into real-space. If $s \in (0, L)$, and our reparameterized curve is β, then β and α have the same image,and also $|\beta'(s)|$ is $|dt/ds \cdot \alpha'(t)| = 1$. So after reparamterizing by arc length, we have fixed the length of our velocity vector to be 1.

Now we can properly define curvature.

Definition 6.2.2. *Let α be a curve paramaterized by arc length. Then we say that $\alpha''(s)$ (where s denotes length) is the* ***curvature vector of*** α ***at*** *s, and the* ***curvature at s*** *is the norm of the curvature vector, $|\alpha''(s)|$.*

Example 5. *Let's go back to the example of a circle, in this case with radius r.*

$\alpha(t) = (r\sin(t), r\cos(t))$, and $\alpha'(t) = (r\cos t, -r\sin t)$. So $|\alpha'(t)| = r$, and not 1. In order to correct for this, set

$\beta(s) = (r\sin(s/r), r\cos(s/r))$. Then $\beta'(s) = (\cos(s/r), -\sin(s/r))$ and $|\beta'(s)| = 1$. Our circle is now parameterized by arc length.

The curvature vector at a given length s is then

$$\beta''(t) = (-(1/r)\sin(s/r), -(1/r)\cos(s/r)) \tag{6.11}$$

and $|\beta''(s)| = 1/r$. Appropriately, the bigger our circle is, the smaller the curvature.

Exercise 2. *Now we take a catenary, the curve we get if we hang a string from two poles.*

Let $\alpha(t) = (t, \cosh t)$, where $\cosh(t) = (1/2)(e^t + e^{-t})$. Parameterize by arc length and check that it works. The identities $\sinh(t) = (1/2)(e^t - e^{-t})$ and $\sinh^{-1}(t) = \ln(t + (t^2+1)^{1/2})$ will be of use.

Chapter 7

Tangent Planes

Reading: Do Carmo sections 2.4 and 3.2

Today I am discussing

1. Differentials of maps between surfaces

2. Geometry of Gauss map

7.1 Tangent Planes; Differentials of Maps Between Surfaces

7.1.1 Tangent Planes

Recall from previous lectures the definition of *tangent plane.*

(Proposition 2-4-1). *Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parameterization of a regular surface S and let $q \in U$. The vector subspace of dimension 2,*

$$d\mathbf{x}_q(\mathbb{R}^2) \subset \mathbb{R}^3 \tag{7.1}$$

coincides with the set of tangent vectors to S at $\mathbf{x}(q)$. We call the plane $d\mathbf{x}_q(\mathbb{R}^2)$ the **Tangent Plane** *to S at p, denoted by $T_p(S)$.*

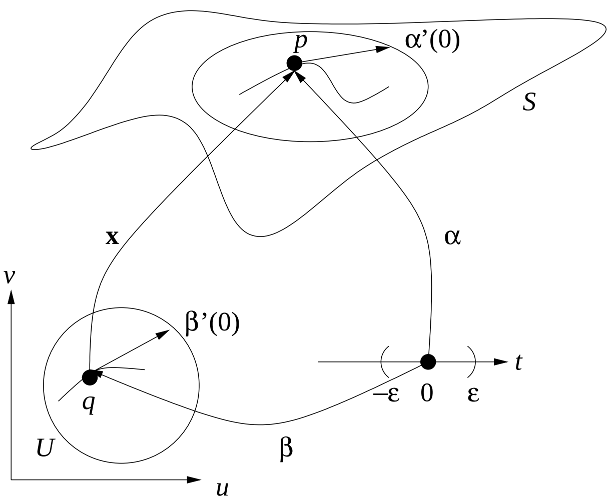


Figure 7.1: Graphical representation of the map dx_q that sends $\beta'(0) \in T_q(\mathbb{R}^2$ to $\alpha'(0) \in T_p(S)$.

Note that the plane $dx_q(\mathbb{R}^2)$ does not depend on the parameterization $\mathbf{x}$. However, the choice of the parameterization determines the basis on $T_p(S)$, namely $\{(\frac{\partial \mathbf{x}}{\partial u})(q), (\frac{\partial \mathbf{x}}{\partial v})(q)\}$, or $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$.

7.1.2 Coordinates of $w \in T_p(S)$ in the Basis Associated to Parameterization $\mathbf{x}$

Let w be the velocity vector $\alpha'(0)$, where $\alpha = \mathbf{x} \circ \beta$ is a curve in the surface S, and the map $\beta : (-\epsilon, \epsilon) \to U$, $\beta(t) = (u(t), v(t))$. Then in the basis of $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$, we have

$$w = (u'(0), v'(0)) \tag{7.2}$$

7.1.3 Differential of a (Differentiable) Map Between Surfaces

It is natural to extend the idea of differential map from $T(\mathbb{R}^2) \to T(S)$ to $T(S_1) \to T(S_2)$.

Let S_1, S_2 be two regular surfaces, and a differential mapping $\varphi \subset S_1 \to S_2$ where V is open. Let $p \in V$, then all the vectors $w \in T_p(S_1)$ are velocity vectors $\alpha'(0)$ of some differentiable parameterized curve $\alpha : (-\epsilon, \epsilon) \to V$ with $\alpha(0) = p$.

Define $\beta = \varphi \circ \alpha$ with $\beta(0) = \varphi(p)$, then $\beta'(0)$ is a vector of $T_{\varphi(p)}(S_2)$.

(Proposition 2-4-2). *Given w, the velocity vector $\beta'(0)$ does not depend on the choice of α. Moreover, the map*

$$d\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2) \tag{7.3}$$

$$d\varphi_p(w) = \beta'(0) \tag{7.4}$$

is linear. We call the linear map $d\varphi_p$ to be the **differential** *of φ at $p \in S_1$.*

Proof. Suppose φ is expressed in $\varphi(u, v) = (\varphi_1(u, v), \varphi_2(u, v))$, and $\alpha(t) = (u(t), v(t)), t \in (-\epsilon, \epsilon)$ is a regular curve on the surface S_1. Then

$$\beta(t) = (\varphi_1(u(t), v(t)), \varphi_2(u(t), v(t)). \tag{7.5}$$

Differentiating β w.r.t. t, we have

$$\beta'(0) = \left(\frac{\partial \varphi_1}{\partial u} u'(0) + \frac{\varphi_1}{\partial v} v'(0), \frac{\partial \varphi_2}{\partial u} u'(0) + \frac{\varphi_2}{\partial v} v'(0) \right) \tag{7.6}$$

in the basis of $(\bar{\mathbf{x}}_u, \bar{\mathbf{x}}_v)$.

As shown above, $\beta'(0)$ depends on the map φ and the coordinates of $(u'(0), v'(0)$ in the basis of $\{\mathbf{x}_u, \mathbf{x}_v\}$. Therefore, it is independent on the choice of α.

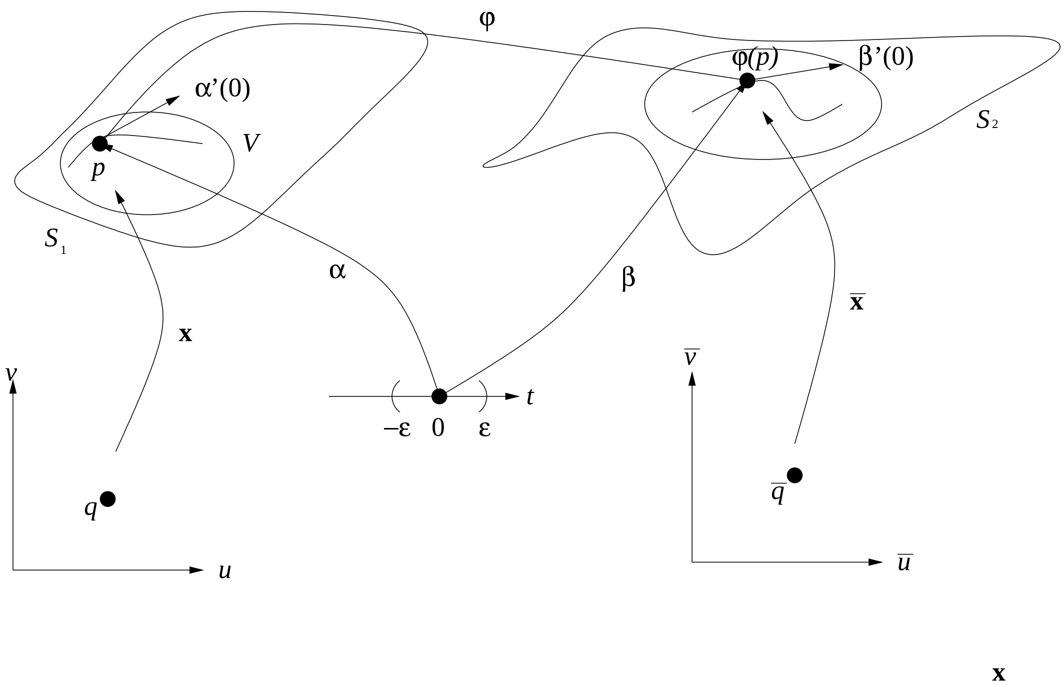


Figure 7.2: Graphical representation of the map $d\varphi_p$ that sends $\alpha'(0) \in T_q(S_1)$ to $\beta'(0) \in T_p(S_2)$.

Moreover, Equation 7.6 can be expressed as

$$\beta'(0) = d\varphi_p(w) = \begin{pmatrix} \dfrac{\partial \varphi_1}{\partial u} & \dfrac{\partial \varphi_1}{\partial v} \\ \dfrac{\partial \varphi_2}{\partial u} & \dfrac{\partial \varphi_2}{\partial v} \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix} \tag{7.7}$$

which shows that the map $d\varphi_p$ is a mapping from $T_p(S_1)$ to $T_{\varphi(p)}(S_2)$. Note that the 2×2 matrix is respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p(S_1)$ and $\{\bar{\mathbf{x}}_u, \bar{\mathbf{x}}_v\}$ of $T_{\varphi(p)}(S_2)$ respectively. $\square$

We can define the differential of a (differentiable) function $f : U \subset S \to R$ at $p \in U$ as a linear map $df_p : T_p(S) \to \mathbb{R}$.

Example 2-4-1: Differential of the height function Let $v \in \mathbb{R}^3$. Consider the map

$$h : S \subset \mathbb{R}^3 \to \mathbb{R} \tag{7.8}$$

$$h(p) = v \cdot p, p \in S \tag{7.9}$$

We want to compute the differential $dh_p(w), w \in T_p(S)$. We can choose a differential curve $\alpha : (-\epsilon, \epsilon)) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = w$. We are able to choose such α since the differential $dh_p(w)$ is independent on the choice of α. Thus

$$h(\alpha(t)) = \alpha(t) \cdot v \tag{7.10}$$

Taking derivatives, we have

$$dh_p(w) = \frac{d}{dt} h(\alpha(t))|_{t=0} = \alpha'(0) \cdot v = w \cdot v \tag{7.11}$$

Example 2-4-2: Differential of the rotation Let $S^2 \subset \mathbb{R}^3$ be the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\} \tag{7.12}$$

Consider the map

$$R_{z,\theta} : \mathbb{R}^3 \to \mathbb{R}^3 \tag{7.13}$$

be the rotation of angle θ about the z axis. When $R_{z,\theta}$ is restricted to S^2, it becomes a differential map that maps S^2 into itself. For simplicity, we denote the restriction map $R_{z,\theta}$. We want to compute the differential $(dR_{z,\theta})_p(w), p \in S^2, w \in T_p(S^2)$. Let $\alpha : (-\epsilon, \epsilon) \to S^2$ be a curve on S^2 such that $\alpha(0) = p, \alpha'(0) = w$. Now

$$(dR_{z,\theta})(w) = \frac{d}{dt}(R_{z,\theta} \circ \alpha(t))_{t=0} = R_{z,\theta}(\alpha'(0)) = R_{z,\theta}(w) \tag{7.14}$$

7.1.4 Inverse Function Theorem

All we have done is extending differential calculus in $\mathbb{R}^2$ to regular surfaces. Thus, it is natural to have the Inverse Function Theorem extended to the regular surfaces.

A mapping $\varphi : U \subset S_1 \to S_2$ is a **local diffeomorphism** at $p \in U$ if there exists a neighborhood $V \subset U$ of p, such that φ restricted to V is a diffeomorphism onto the open set $\varphi(V) \subset S_2$.

(Proposition 2-4-3). *Let S_1, S_2 be regular surfaces and $\varphi : U \subset S_1 \to S_2$ a differentiable mapping. If $d\varphi_p : T_p(S_1) \to T_{\varphi(p)}(S_2)$ at $p \in U$ is an isomorphism, then φ is a local diffeomorphism at p.*

The proof is a direct application of the inverse function theorem in $\mathbb{R}^2$.

7.2 The Geometry of Gauss Map

In this section we will extend the idea of curvature in curves to regular surfaces. Thus, we want to study how rapidly a surface S pulls away from the tangent plane $T_p(S)$ in a neighborhood of $p \in S$. This is equivalent to measuring the rate of change of a unit normal vector field N on a neighborhood of p. We will show that this rate of change is a linear map on $T_p(S)$ which is self adjoint.

7.2.1 Orientation of Surfaces

Given a parameterization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ of a regular surface S at a point $p \in S$, we choose a unit normal vector at each point $\mathbf{x}(U)$ by

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(q), q \in \mathbf{x}(U) \tag{7.15}$$

We can think of N to be a map $N : \mathbf{x}(U) \to \mathbb{R}^3$. Thus, each point $q \in \mathbf{x}(U)$ has a normal vector associated to it. We say that N is a **differential field**

of unit normal vectors on U.

We say that a regular surface is **orientable** if it has a differentiable field of unit normal vectors defined on the whole surface. The choice of such a field N is called an **orientation** of S. An example of non-orientable surface is Möbius strip (see Figure 3).

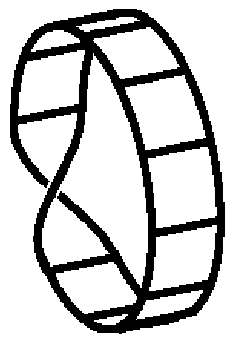


Figure 7.3: Möbius strip, an example of non-orientable surface.

In this section (and probably for the rest of the course), we will only study regular orientable surface. We will denote S to be such a surface with an orientation N which has been chosen.

7.2.2 Gauss Map

(Definition 3-2-1). *Let $S \subset \mathbb{R}^3$ be a surface with an orientation N and $S^2 \subset \mathbb{R}^3$ be the unit sphere*

$$S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}. \tag{7.16}$$

The map $N : S \to S^2$ is called the **Gauss map**.

The map N is differentiable since the differential,

$$dN_p : T_p(S) \to T_{N(p)}(S^2) \tag{7.17}$$

at $p \in S$ is a linear map.

For a point $p \in S$, we look at each curve $\alpha(t)$ with $\alpha(0) = p$ and compute $N \circ \alpha(t) = N(t)$ where we define that map $N : (-\epsilon, \epsilon) \to S^2$ with the same notation as the normal field. By this method, we restrict the normal vector N to the curve $\alpha(t)$. The tangent vector $N'(0) \in T_p(S^2)$ thus measures the rate of change of the normal vector N restrict to the curve $\alpha(t)$ at $t = 0$. In other words, dN_p measure how N pulls away from $N(p)$ in a neighborhood of p. In the case of the surfaces, this measure is given by a linear map.

Example 3-2-1 (Trivial) Consider S to be the plane $ax + by + cz + d = 0$, the tangent vector at any point $p \in S$ is given by

$$N = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} \tag{7.18}$$

Since N is a constant throughout S, $dN = 0$.

Example 3-2-2 (Gauss map on the Unit Sphere)

Consider $S = S^2 \subset \mathbb{R}^3$, the unit sphere in the space $\mathbb{R}^3$. Let $\alpha(t) = (x(t), y(t), z(t))$ be a curve on S, then we have

$$2xx' + 2yy' + 2zz' = 0 \tag{7.19}$$

which means that the vector (x, y, z) is normal to the surface at the point (x,y,z). We will choose $N = (-x, -y, -z)$ to be the normal field of S. Restricting to the curve $\alpha(t)$, we have

$$N(t) = (-x(t), -y(t), -z(t)) \tag{7.20}$$

and therefore

$$dN(x'(t), y'(t), z'(t)) = (-x'(t), -y'(t), -z'(t)) \tag{7.21}$$

or $dN_p(v) = -v$ for all $p \in S$ and $v \in T_p(S^2)$.

Example 3-2-4 (Exercise: Gauss map on a hyperbolic paraboloid) Find the differential $dN_{p=(0,0,0)}$ of the normal field of the paraboloid $S \subset \mathbb{R}^3$ defined by

$$\mathbf{x}(u,v) = (u, v, v^2 - u^2) \tag{7.22}$$

under the parameterization $\mathbf{x} : U \subset \mathbb{R}^2 \to S$.

7.2.3 Self-Adjoint Linear Maps and Quadratic Forms

Let V now be a vector space of dimension 2 endowed with an inner product $\langle \ , \ \rangle$.

Let $A : V \to V$ be a linear map. If $\langle Av, w\rangle = \langle v, Aw\rangle$ for all $v, w \in V$, then we call A to be a **self-adjoint** linear map.

Let $\{e_1, e_2\}$ be a orthonormal basis for V and $(\alpha_{ij}), i, j = 1, 2$ be the matrix elements of A in this basis. Then, according to the axiom of self-adjoint, we have

$$\langle Ae_i, e_j\rangle = \alpha_{ij} = \langle e_i, Ae_j\rangle = \langle Ae_j, e_i\rangle = \alpha_{ji} \tag{7.23}$$

There A is **symmetric**.

To each self-adjoint linear map, there is a **bilinear map** $B : V \times V \to \mathbb{R}$ given by

$$B(v, w) = \langle Av, w\rangle \tag{7.24}$$

It is easy to prove that B is a bilinear symmetric form in V.

For each bilinear form B in V, there is a **quadratic form** $Q : V \to \mathbb{R}$ given by

$$Q(v) = B(v, v), v \in V. \tag{7.25}$$

Exercise (Trivial): Show that

$$B(u, v) = \frac{1}{2}\left[Q(u + v) - Q(v) - Q(u)\right] \tag{7.26}$$

Therefore, there is a 1-1 correspondence between quadratic form and self-adjoint linear maps of V.

Goal for the rest of this section: Show that given a self-adjoint linear map $A : V \to V$, there exist a orthonormal basis for V such that, relative to this basis, the matrix A is diagonal matrix. Moreover, the elements of the diagonal are the maximum and minimum of the corresponding quadratic form restricted to the unit circle of V.

(Lemma (Exercise)). *If $Q(x, y) = ax^2 = 2bxy+cy^2$ restricted to $\{(x, y); x^2+y^2 = 1\}$ has a maximum at $(1, 0)$, then $b = 0$*

Hint: Reparametrize (x, y) using $x = \cos t, y = \cos t, t \in (-\epsilon, 2\pi + \epsilon)$ and set $\frac{dQ}{dt}|_{t=0} = 0$.

(Proposition 3A-1). *Given a quadratic form Q in V, there exists an orthonormal basis $\{e_1, e_2\}$ of V such that if $v \in V$ is given by $v = xe_1 + ye_2$, then*

$$Q(v) = \lambda_1 x^2 + \lambda_2 y^2 \tag{7.27}$$

where $\lambda_i, i = 1, 2$ are the maximum and minimum of the map Q on $|v| = 1$ respectively.

Proof. Let λ_1 be the maximum of Q on the circle $|v| = 1$, and e_1 to be the unit vector with $Q(e_1) = \lambda_1$. Such e_1 exists by continuity of Q on the compact set $|v| = 1$.

Now let e_2 to be the unit vector orthonormal to e_1, and let $\lambda_2 = Q(e_2)$. We will show that this set of basis satisfy the proposition.

Let B be a bilinear form associated to Q. If $v = xe_1 + ye_2$, then

$$Q(v) = B(v, v) = \lambda_1 x^2 + 2bxy + \lambda_2 y^2 \tag{7.28}$$

where $b = B(e_1, e_2)$. From previous lemma, we know that $b = 0$. So now it suffices to show that λ_2 is the minimum of Q on $|v| = 1$. This is trivial since we know that $x^2 + y^2 = 1$ and

$$Q(v) = \lambda_1 x^2 + \lambda_2 y^2 \geq \lambda_2(x^2 + y^2) = \lambda_2 \tag{7.29}$$

as $\lambda 2 \leq \lambda 1$. $\square$

If $v \neq 0$, then v is called the **eigenvector** of $A : V \to V$ if $Av = \lambda v$ for some real λ. We call the λ the corresponding **eigenvalue**.

(Theorem 3A-1). *Let $A : V \to V$ be a self-adjoint linear map, then there exist an orthonormal basis $\{e_1, e_2\}$ of V such that*

$$A(e_1) = \lambda_1 e_1, \quad A(e_2) = \lambda_2 e_2. \tag{7.30}$$

Thus, A is diagonal in this basis and $\lambda_i, i = 1, 2$ are the maximum and minimum of $Q(v) = \langle Av, v\rangle$ on the unit circle of V.

Proof. Consider $Q(v) = \langle Av, v\rangle$ where $v = (x, y)$ in the basis of $e_i, i = 1, 2$. Recall from the previous lemma that $Q(x, y) = ax^2 + cy^2$ for some $a, c \in \mathbb{R}$. We have $Q(e_1) = Q(1, 0) = a, Q(e_2) = Q(0, 1) = c$, therefore $Q(e_1 + e_2) = Q(1, 1) = a + c$ and

$$B(e_1, e_2) = \frac{1}{2}[Q(e_1 + e_2) - Q(e_1) - Q(e_2)] = 0 \tag{7.31}$$

Thus, Ae_1 is either parallel to e_1 or equal to 0. In any case, we have $Ae_1 = \lambda_1 e_1$. Using $B(e_1, e_2) = \langle Ae_2, e_1\rangle = 0$ and $\langle Ae_2, e_2\rangle = \lambda_2$, we have $Ae_2 = \lambda_2 e_2$. $\square$

Now let us go back to the discussion of Gauss map.

(Proposition 3-2-1). *The differential map $dN_p : T_p(S) \to T_p(S)$ of the Gauss map is a self-adjoint linear map.*

Proof. It suffices to show that

$$\langle dN_p(w_1), w_2\rangle = \langle w_1, dN_p(w_2)\rangle \tag{7.32}$$

for the basis $\{w_1, w_2\}$ of $T_p(S)$.

Let $\mathbf{x}(u, v)$ be a parameterization of S at p, then $\mathbf{x}_u, \mathbf{x}_v$ is a basis of $T_p(S)$. Let $\alpha(t) = \mathbf{x}(u(t), v(t))$ be a parameterized curve in S with $\alpha(0) = p$, we have

$$dN_p(\alpha'(0)) = dN_p(x_u u'(0) + x_v v'(0)) \tag{7.33}$$

$$= \frac{d}{dt} N(u(t), v(t))|_{t=0} \tag{7.34}$$

$$= N_u u'(0) + N_v v'(0) \tag{7.35}$$

with $dN_p(\mathbf{x}_u) = N_u$ and $dN_p(\mathbf{x}_v) = N_v$. So now it suffices to show that

$$\langle N_u, \mathbf{x}_v \rangle = \langle \mathbf{x}_u, N_v \rangle \tag{7.36}$$

If we take the derivative of $\langle N, \mathbf{x}_u \rangle = 0$ and $\langle N, \mathbf{x}_v \rangle = 0$, we have

$$\langle N_v, \mathbf{x}_u \rangle + \langle N, \mathbf{x}_u v \rangle = 0 \tag{7.37}$$

$$\langle N_u, \mathbf{x}_v \rangle + \langle N, \mathbf{x}_v u \rangle = 0 \tag{7.38}$$

Therefore

$$\langle N_u, \mathbf{x}_v \rangle = - \langle N, \mathbf{x}_u v \rangle = \langle N_v, \mathbf{x}_u \rangle \tag{7.39}$$

□

Chapter 8

Gauss Map I

8.1 "Curvature" of a Surface

We've already discussed the curvature of a curve. We'd like to come up with an analogous concept for the "curvature" of a regular parametrized surface S parametrized by $x \colon U \to \mathbb{R}^n$. This can't be just a number — we need at the very least to talk about the "curvature of S at p in the direction $v \in T_p(S)$".

So given $v \in T_p(S)$, we can take a curve $\alpha \colon I \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = v$. (This exists by the definition of the tangent plane.) The curvature of α itself as a curve in $\mathbb{R}^n$ is $\frac{d^2\alpha}{ds^2}$ (note that this is with respect to arc length). However, this depends on the choice of α — for example, if you have the cylinder over the unit circle, and let v be in the tangential direction, both a curve that just goes around the cylinder and a curve that looks more like a parabola that happens to be going purely tangentially at p have the same α', but they do not have the same curvature. But if we choose a field of normal vectors N on the surface, then $\frac{d^2\alpha}{ds^2} \cdot N_p$ is independent of the choice of α (as long as $\alpha(0) = p$ and $\alpha'(0) = v$). It's even independent of the magnitude of v — it only depends on its direction $\hat{v}$. We call this curvature $k_p(N, \hat{v})$. For the example, we can see that the first curve's α'' is 0, and that the second one's α'' points in the negative $\hat{z}$ direction, whereas N points in the radial direction, so $k_p(N, \hat{v})$ is zero no matter which α you choose.

(In 3-space with a parametrized surface, we can always choose N to be $N = \frac{x_u \wedge x_v}{|x_u \wedge x_v|}$.)

To prove this, we see that $\alpha(s) = x(u_1(s), u_2(s))$, so that $\frac{d\alpha}{ds} = \frac{du_1}{ds} x_{u_1} + \frac{du_2}{ds} x_{u_2}$ and $\frac{d^2\alpha}{ds^2} = \frac{d^2u_1}{ds} x_{u_1} + \frac{du_1}{ds}\left(\frac{du_1}{ds} x_{u_1u_1} + \frac{du_2}{s} x_{u_1u_2}\right) + \frac{d^2u_2}{ds} x_{u_2} + \frac{du_2}{ds}\left(\frac{du_1}{ds} x_{u_1u_2} + \frac{du_2}{s} x_{u_2u_2}\right)$. But by normality, $N \cdot x_{u_1} = N \cdot x_{u_2} = 0$, so $\frac{d^2\alpha}{ds^2} \cdot N = \sum_{i,j=1}^{2} b_{ij}(N) \frac{du_i}{ds}\frac{du_j}{ds}$, where $b_{ij}(N) = x_{u_iu_j} \cdot N$.

We can put the values b_{ij} into a matrix $B(N) = [b_{ij}(N)]$. It is symmetric, and so it defines a symmetric quadratic form $B = II \colon T_p(S) \to \mathbb{R}$. If we use $\{x_{u_1}, x_{u_2}\}$ as a basis for $T_p(S)$, then $II(cx_{u_1} + dx_{u_2}) = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} b_{11}(N) & b_{12}(N) \\ b_{21}(N) & b_{22}(N) \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$. We call II the *Second Fundamental Form.*

II is independent of α, since it depends only on the surface (not on α). To show that $k_p(N, \hat{v})$ is independent of choice of α, we see that

$$k_p(N, \hat{V}) = \frac{d^2\alpha}{ds^2} \cdot N = \sum_{ij} b_{ij}(N) \frac{du_i}{ds}\frac{du_j}{ds} = \frac{\sum_{i,j} b_{ij}(N) \frac{du_i}{dt}\frac{du_j}{dt}}{\left(\frac{ds}{dt}\right)^2}$$

Now, $s(t) = \int_{t_0}^{t} |\alpha'(t)|\, dt$, so that $\left(\frac{ds}{dt}\right)^2 = |\alpha'(t)|^2 = |\frac{du_1}{dt} x_{u_1} + \frac{du_2}{dt} x_{u_2}|^2 = \sum_{i,j} \left(\frac{du_i}{dt}\right)\left(\frac{du_j}{dt}\right) g_{ij}$, where g_{ij} comes from the *First* Fundamental Form. So

$$k_p(N, \hat{v}) = \frac{\sum_{i,j} b_{ij}(N) \frac{du_i}{dt}\frac{du_j}{dt}}{\sum_{i,j} g_{ij} \frac{du_i}{dt}\frac{du_j}{dt}}$$

The numerator is just the First Fundamental Form of v, which is to say its length. So the only property of α that this depends on are the derivatives of its components at p, which are just the components of the given vector v. And in fact if we multiply v by a scalar λ, we multiply both the numerator and the denominator by λ^2, so that $k_p(N, \hat{v})$ doesn't change. So $k_p(N, \hat{v})$ depends only on the *direction* of v, not its magnitude.

If we now let $k_1(N)_p$ be the maximum value of $k_p(N, \hat{v})$. This exists because $\hat{v}$ is chosen from the compact set $S^1 \subset T_p(S)$. Similarly, we let

$k_2(N)_p$ be the minimal value of $k_p(N,\hat{v})$. These are called the *principle curvatures* of S at p with regards to N. The directions e_1 and e_2 yielding these curvatures are called the *principal directions*. It will turn out that these are the eigenvectors and eigenvalues of a linear operator defined by the Gauss map.

8.2 Gauss Map

Recall that for a surface $x\colon U \to S$ in $\mathbb{R}^3$, we can define the Gauss map $N\colon S \to S^2$ which sends p to $N_p = \frac{x_{u_1}\wedge x_{u_2}}{|x_{u_1}\wedge x_{u_2}|}$, the unit normal vector at p. Then $dN_p\colon T_p(S) \to T_{N_p}(S^2)$; but $T_p(S)$ and $T_{N_p}(S^2)$ are the same plane (they have the same normal vector), so we can see this as a linear operator $T_p(S) \to T_p(S)$.

For example, if $S = S^2$, then $N_p = p$, so N_p is a linear transform so it is its own derivative, so dN_p is also the identity.

For example, if S is a plane, then N_p is constant, so its derivative is the zero map.

For example, if S is a right cylinder defined by $(\theta, z) \mapsto (\cos\theta, \sin\theta, z)$, then $N(x,y,z) = (x,y,0)$. (We can see this because the cylinder is defined by $x^2+y^2 = 1$, so $2xx'+2yy' = 0$, which means that $(x,y,0)\cdot(x',y',z') = 0$, so that $(x,y,0)$ is normal to the velocity of *any* vector through (x,y,z).) Let us consider the curve α with $\alpha(t) = (\cos t, \sin t, z(t))$, then $\alpha'(t) = (-\sin t, \cos t, z'(t))$. So $(N\circ\alpha)(t) = (x(t), y(t), 0)$, and so $(N\circ\alpha)'(t) = (-\sin t, \cos t, 0)$. So $dN_p(x_\theta) = x_\theta$. So in the basis $\{x_\theta, x_z\}$, the matrix is $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. It has determinant 0 and $\frac{1}{2}$ trace equal to $\frac{1}{2}$. It turns out that the determinant of this matrix only depends on the First Fundamental Form, and not how it sits in space — this is why the determinant is the same for the cylinder and the plane. A zero eigenvalue can't go away no matter how you curve the surface, as long as you don't stretch it.

Chapter 9

Gauss Map II

9.1 Mean and Gaussian Curvatures of Surfaces in $\mathbb{R}^3$

We'll assume that the curves are in $\mathbb{R}^3$ unless otherwise noted. We start off by quoting the following useful theorem about self adjoint linear maps over $\mathbb{R}^2$:

Theorem 9.1.1 (Do Carmo pp. 216). *: Let V denote a two dimensional vector space over $\mathbb{R}$. Let $A : V \to V$ be a self adjoint linear map. Then there exists an orthonormal basis e_1, e_2 of V such that $A(e_1) = \lambda_1 e_1$, and $A(e_2) = \lambda_2 e_2$ (that is, e_1 and e_2 are eigenvectors, and λ_1 and λ_2 are eigenvalues of A). In the basis e_1, e_2, the matrix of A is clearly diagonal and the elements λ_1, λ_2, $\lambda_1 \geq \lambda_2$, on the diagonal are the maximum and minimum, respectively, of the quadratic form $Q(v) = \langle Av, v\rangle$ on the unit circle of V.*

Proposition 9.1.2. *: The differential $dN_p : T_p(S) \to T_p(S)$ of the Gauss map is a self-adjoint linear map.*

Proof. Since dN_p is linear, it suffices to verify that $\langle dN_p(w_1), w_2\rangle = \langle w_1, dN_p(w_2)\rangle$ for a basis w_1, w_2 of $T_p(S)$. Let $x(u, v)$ be a parametrization of S at P

and x_u, x_v be the associated basis of $T_p(S)$. If $\alpha(t) = x(u(t), v(t))$ is a parametrized curve in S with $\alpha(0) = p$, we have

$$dN_p(\alpha'(0)) = dN_p(x_u u'(0) + x_v v'(0)) \tag{9.1}$$
$$= \frac{d}{dt} N(u(t), v(t))|_{t=0} \tag{9.2}$$
$$= N_u u'(0) + N_v v'(0) \tag{9.3}$$

in particular, $dN_p(x_u) = N_u$ and $dN_p(x_v) = N_v$. Therefore to prove that dN_p is self adjoint, it suffices to show that

$$\langle N_u, x_v \rangle = \langle x_u, N_v \rangle. \tag{9.4}$$

To see this, take the derivatives of $\langle N, x_u \rangle = 0$ and $\langle N, x_v \rangle = 0$, relative to v and u respectively, and obtain

$$\langle N_v, x_u \rangle + \langle N, x_{uv} \rangle = 0, \tag{9.5}$$
$$\langle N_u, x_v \rangle + \langle N, x_{uv} \rangle = 0, \tag{9.6}$$

Thus,

$$\langle N_u, x_v \rangle = -\langle N, x_{uv} \rangle = \langle N_v, x_u \rangle \tag{9.7}$$

□

Now given that dN_p is self-adjoint one can think of the associated quadratic form.

Definition 9.1.3. *The quadratic form II_p defined in $T_p(S)$ by $II_p(v) = -\langle dN_p(v), v \rangle$ is called the second fundamental form of S at p.*

Now that we have two definitions for the second fundamental form we better show that they're equivalent. (Recall from the last lecture that $II_p(\alpha'(0)) = \langle N(0), \alpha''(0) \rangle$ where α is considered as a function of arc length.)

Let $N(s)$ denote the restriction of normal to the curve $\alpha(s)$. We have $\langle N(s), \alpha'(s)\rangle = 0$ Differentiating yields

$$\langle N(s), \alpha''(s)\rangle = -\langle N'(s), \alpha'(s)\rangle. \tag{9.8}$$

Therefore,

$$\begin{aligned} II_p(\alpha'(0)) &= -\langle dN_p(\alpha'(0)), \alpha'(0)\rangle \\ &= -\langle N'(0), \alpha'(0)\rangle \\ &= \langle N(0), \alpha''(0)\rangle \end{aligned} \tag{9.9}$$

which agrees with our previous definition.

Definition 9.1.4. *: The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the* principal curvatures *at p; and the corresponding eigenvectors are called principal directions at p.*

So for instance if we take cylinder $k_1 = 0$ and $k_2 = -1$ for all points p.

Definition 9.1.5. *: If a regular connected curve C on S is such that for all $p \in C$ the tangent line of C is a principal direction at p, then C is said to be a* line of curvature *of S.*

For cylinder a circle perpendicular to axis and the axis itself are lines of curvature of the cylinder.

Proposition 9.1.6. *A necessary and sufficient condition for a connected regular curve X on S to be a line of curvature of S is that*

$$N'(t) = \lambda(t)\alpha'(t)$$

for any parametrization $\alpha(t)$ of C, where $N(t) = N(\alpha(t))$ and λ is a differentiable function of t. In this case, $-\lambda(t)$ is the principal curvature along $\alpha'(t)$

Proof. : Obvious since principal curvature is an eigenvalue of the linear transformation N'. $\square$

A nice application of the principal directions is computing the normal curvature along a given direction of $T_p(s)$. If e_1 and e_2 are two orthogonal eigenvectors of unit length then one can represent any unit tangent vector as

$$v = e_1 \cos\theta + e_2 \sin\theta \tag{9.10}$$

The normal curvature along v is given by

$$\begin{aligned} II_p(v) &= -\langle dN_p(v), v\rangle \\ &= k_1 cos^2\theta + k_2 sin^2\theta \end{aligned} \tag{9.11}$$

Definition 9.1.7. *Let $p \in S$ and let $dN_p : T_p(S) \to T_p(S)$ be the differential of the Gauss map. The determinant of dN_p is the* Gaussian curvature *K at p. The negative of half of the trace of dN_p is called the mean curvature H of S at p.*

In terms of principal curvatures we can write

$$K = k_1 k_2, H = \frac{k_1 + k_2}{2} \tag{9.12}$$

Definition 9.1.8. *: A point of a surface S is called*

1. *Elliptic if $K > 0$,*
2. *Hyperbolic if $K < 0$,*
3. *Parabolic if $K = 0$, with $dN_p \neq 0$*
4. *Planar if $dN_p = 0$*

Note that above definitions are independent of the choice of the orientation.

Definition 9.1.9. *Let p be a point in S. An* asymptotic direction *of S at p is a direction of $T_p(S)$ for which the normal curvature is zero. An* asymptotic curve *of S is a regular connected curve $C \subset S$ such that for each $p \in C$ the tangent line of C at p is an asymptotic direction.*

9.2 Gauss Map in Local Coordinates

Let $x(u,v)$ be a parametrization at a point $p \in S$ of a surface S, and let $\alpha(t) = x(u(t), v(t))$ be a parametrized curve on S, with $\alpha(0) = p$ To simplify the notation, we shall make the convention that all functions to appear below denote their values at the point p.

The tangent vector to $\alpha(t)$ at p is $\alpha' = x_u u + x_v v$ and

$$dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v' \tag{9.13}$$

Since N_u and N_v belong to $T_p(S)$, we may write

$$\begin{aligned} N_u &= a_{11}x_u + a_{21}x_v \\ N_v &= a_{12}x_u + a_{22}x_v \end{aligned} \tag{9.14}$$

Therefore,

$$dN = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with respect to basis $\{x_u, x_v\}$.

On the other hand, the expression of the second fundamental form in the basis $\{x_u, x_v\}$ is given by

$$\begin{aligned} II_p(\alpha') &= -\langle dN(\alpha'), \alpha' \rangle \\ &= e(u')^2 + 2fu'v' + g(v')^2 \end{aligned} \tag{9.15}$$

where, since $\langle N, x_u \rangle = \langle N, x_v \rangle = 0$

$$e = -\langle N_u, x_u \rangle = \langle N, x_{uu} \rangle, \tag{9.16}$$

$$f = -\langle N_v, x_u \rangle = \langle N, x_{uv} \rangle = \langle N, x_{vu} \rangle = -\langle N_u, x_v \rangle \tag{9.17}$$

$$g = -\langle N_v, x_v \rangle = \langle N, x_{vv} \rangle \tag{9.18}$$

From eqns. (11), (12) we have

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

From the above equation we immediately obtain

$$K = \det(a_{ij}) = \frac{eg - f^2}{EG - F^2} \tag{9.19}$$

Formula for the mean curvature:

$$H = \frac{1}{2} \frac{sG - 2fF + gE}{EG - F^2} \tag{9.20}$$

Exercise 3. *Compute H and K for sphere and plane.*

Example 6. *Determine the asymptotic curves and the lines of curvature of the helicoid $x = v\cos u, y = v\sin u, z = cu$ and show that its mean curvature is zero.*

Chapter 10

Introduction to Minimal Surfaces I

10.1 Calculating the Gauss Map using Coordinates

Last time, we used the differential of the Gauss map to define several interesting features of a surface — mean curvature H, Gauss curvature K, and principal curvatures k_1 and k_2. We did this using relatively general statements. Now we will calculate these quantities in terms of the entries g_{ij} and b_{ij} of the two fundamental form matrices. (Note that do Carmo still uses E, F, G, and e, f, g here respectively.) Don't forget that the terms g_{ij} and $b_{ij}(N)$ can be calculated with just a bunch of partial derivatives, dot products, and a wedge product — the algebra might be messy but there's no creativity required.

Let $dN_p = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in terms of the basis $\{\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\}$ of $T_p(S)$. Now, $\frac{\partial N}{\partial u} = a_{11}\frac{\partial x}{\partial u} + a_{21}\frac{\partial x}{\partial v}$; so $\left\langle \frac{\partial N}{\partial u}, \frac{\partial x}{\partial u}, = \right\rangle a_{11}g_{11} + a_{21}g_{12}$. But by a proof from last time, $\left\langle \frac{\partial N}{\partial u}, \frac{\partial x}{\partial u}, = \right\rangle - \left\langle N, \frac{\partial^2 x}{\partial u^2}, = \right\rangle - b_{11}(N)$. So $-b_{11}(N) = a_{11}g_{11} + a_{21}g_{12}$.

Three more similar calculations will show us that

$$-\big[b_{ij}(N)\big] = \big[a_{ij}\big]\big[g_{ij}\big]$$

If we recall that the Gaussian curvature $K = k_1 k_2$ is the determinant of $dN_p = \big(a_{ij}\big)$, then we can see that $\det\big[b_{ij}(N)\big] = K \det\big[g_{ij}\big]^{-1}$, so that $K = \frac{b_{11}(N)b_{22}(N)-b_{12}(N)^2}{g_{11}g_{22}-g_{12}^2}$.

If we solve the matrix equality for the matrix of a_{ij}, we get that

$$\big[a_{ij}\big] = \frac{1}{\det G}\begin{bmatrix} g_{12}b_{12}(N) - g_{22}b_{11}(N) & g_{12}b_{22}(N) - g_{22}b_{12}(N) \\ g_{12}b_{11}(N) - g_{11}b_{12}(N) & g_{12}b_{12}(N) - g_{11}b_{22}(N) \end{bmatrix}$$

We recall that $-k_1$ and $-k_2$ are the eigenvalues of dN_p. Thus, for some nonzero vector v_i, we have that $dN_p(v_i) = -k_i v_i = -k_i I v_i$. Thus $\begin{bmatrix} a_{11}+k_i & a_{12} \\ a_{21} & a_{22}+k_i \end{bmatrix}$ maps some nonzero vector to zero, so its determinant must be zero. That is, $k_i^2 + k_i(a_{11}+a_{22}) + a_{11}a_{22} - a_{21}a_{12} = 0$; both k_1 and k_2 are roots of this polynomial. Now, for any quadratic, the coefficient of the linear term is the opposite of the sum of the roots. So $H = \frac{1}{2}(k_1+k_2) = -\frac{1}{2}(a_{11}+a_{22}) = \frac{1}{2}\frac{b_{11}(N)g_{22}-2b_{12}(N)g_{12}+b_{22}(N)g_{11}}{g_{11}g_{22}-g_{12}^2}$. (This is going to be the Really Important Equation.)

Last, we find the actual values k_1 and k_2. Remembering that the constant term of a quadratic is the product of its roots and thus K, which we've already calculated, we see that the quadratic we have is just $k_i^2 - 2Hk_i + K = 0$; this has solutions $k_i = H \pm \sqrt{H^2 - K}$.

As an exercise, calculate the mean curvature H of the helicoid $x(uv) = (v\cos u, v\sin u, cu)$. (This was in fact a homework problem for today, but work it out again anyway.)

10.2 Minimal Surfaces

Since the $b_{ij}(N)$ are defined as the dot product of N and something independent of N, they are each linear in N. So $H(N) = \frac{1}{2}\frac{b_{11}(N)g_{22}-2b_{12}(N)g_{12}+b_{22}(N)g_{11}}{g_{11}g_{22}-g_{12}^2}$ is also linear in N. We can actually consider mean curvature as a vector H instead of as a function from N to a scalar, by finding the unique vector H such that $H(N) = H \cdot N$. I'm pretty sure that this is more interesting when we're embedded in something higher than $\mathbb{R}^3$.

We define a surface where $H = 0$ everywhere to be a *minimal surface*. Michael Nagle will explain this choice of name next time. You just calculated that the helicoid is a minimal surface. So a surface is minimal iff $g_{22}b_{11}(N) + g_{11}b_{22}(N) - 2g_{12}b_{12}(N) = 0$.

Another example of a minimal surface is the catenoid: $x(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$. (We've looked at this in at least one homework exercise.) We calculate $\frac{\partial x}{\partial u} = (-\cosh v \sin u, \cosh v \cos u, 0)$ and $\frac{\partial x}{\partial v} = (\sinh v \cos u, \sinh v \sin u, 1)$, so that $\begin{bmatrix} g_{ij} \end{bmatrix} = \begin{bmatrix} \cosh^2 v & 0 \\ 0 & \cosh^2 v \end{bmatrix}$. Next, $\frac{\partial x}{\partial u} \wedge \frac{\partial x}{\partial v} = (\cosh v \cos u, \cos v \sin u, -\cosh v \sinh v)$, with norm $\cosh^2 v$. So $N_p = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, -\tanh v\right)$.

Taking the second partials, $\frac{\partial^2 x}{\partial u^2} = (-\cosh v \cos u, -\cosh v \sin u, 0)$, $\frac{\partial^2 x}{\partial v^2} = (\cosh v \cos u, \cosh v \sin u, 0)$, and $\frac{\partial^2 x}{\partial u \partial v} = (-\sinh v \sin u, \sinh v \cos u, 0)$. So $\begin{bmatrix} b_{ij}(N) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Finally, the numerator of H is $g_{22}b_{11}(N) + g_{11}b_{22}(N) - 2g_{12}b_{12}(N) = -\cosh^2 v + \cosh^2 v - 0 = 0$. So the catenoid is a minimal surface. In fact, it's the *only* surface of rotation that's a minimal surface. (Note: there are formulas in do Carmo for the second fundamental form of a surface of rotation on page 161, but they assume that the rotating curve is parametrized by arc length, so they'll give the wrong answers for this particular question.)

Why is it the only one? Say we have a curve $y(x) = f(x)$ in the xy-plane Let S be the surface of rotation around the x-axis from this. We can show that the lines of curvature of the surface are the circles in the yz-plane and the lines of fixed θ. We can show that the first have curvature $\frac{1}{y}\frac{1}{(1+(y')^2)^{\frac{1}{2}}}$, and the second have the same curvature as the graph y, which is $\frac{y''}{(1+(y')^2)^{\frac{3}{2}}}$. So H is the sum of these: $\frac{1+(y')^2-yy''}{2y(1+(y')^2)^{\frac{3}{2}}}$. So this is 0 if $1+\left(\frac{dy}{x}\right)^2 - y\frac{d^2y}{dx^2} = 0$. If we let $p = \frac{dy}{dx}$, then $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$. So our equation becomes $1+p^2 - yp\frac{dp}{dy} = 0$, or $\frac{p}{1+p^2}dp = \frac{1}{y}dy$. Integrating, we get $\frac{1}{2}\log(1+p^2) = \log y + C$, so that $y = C_0\sqrt{1+p^2}$. Then $p = \frac{dy}{dx} = \sqrt{cy^2-1}$, so that $\frac{dy}{\sqrt{cy^2-1}} = dx$. Integrating (if you knew this!), you get $\frac{\cosh^{-1} cy}{c} = x+k$, which is to say that $y = c\cosh\frac{x+l}{c}$. Whew!

Chapter 11

Introduction to Minimal Surface II

11.1 Why a Minimal Surface is Minimal (or Critical)

We want to show why a regular surface $im(x) = S$ with mean curvature $H = 0$ everywhere is called a minimal surface – i.e., that this is the surface of least area among all surfaces with the same boundary γ (and conversely, that a surface that minimizes area (for a given boundary γ) has $H = 0$ everywhere.) To do this we first use *normal variations* to derive a formula for the change in area in terms of mean curvature, and then as an application of our formula we find that a surface has minimal area if and only if it has mean curvature 0.

Let D be the domain on which x is defined, and let γ be a closed curve in D which bounds a subdomain $\triangle$. (This is the notation used in Osserman, p. 20 - 23.) We choose a differentiable function $N(u)$ (here $u = (u_1, u_2)$ is a point in our domain D) normal to S at u, i.e.,

$$N(u) \cdot \frac{\delta x}{\delta u_i} = 0. \tag{11.1}$$

Differentiating yields

$$\frac{\delta N}{\delta u_j} \cdot \frac{\delta x}{\delta u_i} = -N \cdot \frac{\delta^2}{\delta u_i \delta u_j} = -b_{ij}(N). \tag{11.2}$$

Now let $h(u)$ be an arbitrary C^2 function in D, and for every real number λ let the surface S_λ be given by

$$S_\lambda : y(u) = x(u) + \lambda h(u) N(u) \tag{11.3}$$

y is called a **normal variation** of x – since we are varying the surface x via the parameter λ along the direction of our normal N. Letting $A(\lambda)$ denote the area of the surface S_λ, we will show that:

Theorem 11.1.1. *$A'(0) = -2\int\int_S H(N)h(u)dA$,*

where the integral of f with respect to surface area on S is defined as

$$\int\int_S f(u)dA = \int\int_\triangle f(u)\sqrt{\det g_{ij}}du_1 du_2 \tag{11.4}$$

($A'(0)$ denotes the derivative with respect to λ.)

Proof. Differentiating y with repsect to the domain coordinates u_i, we get

$$\frac{\delta y}{\delta u_i} = \frac{\delta x}{\delta u_i} + \lambda(h\frac{\delta N}{\delta u_i} + \frac{\delta h}{\delta u_i}N) \tag{11.5}$$

If we let g^λ_{ij} denote the entries of the first fundamental form for the surface S_λ, we get

$$g^\lambda_{ij} = \frac{\delta y}{\delta u_i} \cdot \frac{\delta y}{\delta u_j} = g_{ij} - 2\lambda h b_{ij}(N) + \lambda^2 c_{ij} \tag{11.6}$$

where c_{ij} is a continuous function of u in D.

Then we have

$$\det(g_{ij}^{\lambda}) = a_o + a_1\lambda + a_2\lambda^2 \tag{11.7}$$

with $a_0 = \det\, g_{ij}$, $a_1 = -2h(g_{11}b_{22}(N) + g_{22}b_{11}(N) - 2g_{12}b_{12}(N))$, and a_2 is a continuous function in u_1, u_2, and λ.

Because S is regular, and the determinant function is continuous, we know that a_0 has a positive minimum on $cl(\triangle)$ (the closure of $\triangle$.) Then we can find an ϵ such that $|\lambda| < \epsilon$ means that $\det(g_{ij}^{\lambda}) > 0$ on $cl(\triangle)$. Thus, for a small enough ϵ, all surfaces S_λ restricted to $\triangle$ are regular surfaces.

Now, looking at the Taylor series expansion of the determinant function, we get, for some positive constant M,

$$|\sqrt{(\det(g_{ij}^{\lambda})} - (\sqrt{a_0} + \frac{a_1}{2\sqrt{a_1}})\lambda| < M\lambda^2 \tag{11.8}$$

Then, using the formula for the area of a surface, we have that the area of our original surface $S, A(0) = \int\int_{\triangle} \sqrt{a_0}du_1du_2$.

Integrating the equation with an M in it, we get

$$|A(\lambda) - A(0) - \lambda\int\int_{\triangle}\frac{a_1}{2\sqrt{a_0}}du_1du_2| < M_1\lambda^2 \tag{11.9}$$

$$|\frac{A(\lambda) - A(0)}{\lambda} - \int\int_{\triangle}\frac{a_1}{2\sqrt{a_0}}du_1du_2| < M_1\lambda. \tag{11.10}$$

Letting λ go to 0, and using $H(N) = \frac{g_{22}b_{11}(N)+g_{11}b_{22}(N)-2g_{12}b_{12}(N)}{2\det(g_{ij})}$, we get

$$A'(0) = -2\int\int_{\triangle} H(N)h(u)\sqrt{\det g_{ij}}du_1du_2(*) \tag{11.11}$$

or when integrating with respect to surface area

$$A'(0) = -2\int\int_{\triangle} H(N)h(u)dA \tag{11.12}$$

□

From here it is clear that if $H(N)$ is zero everywhere, then $A'(0)$ is zero, and thus we have a critical point (hence minimal surfaces being misnamed: we can only ensure that A has a critical point by setting $H(N)$ to zero everywhere.) Now we show the converse:

Corollary 11.1.2. *If S minimizes area, then its mean curvature vanishes everywhere.*

Proof. : Suppose the mean curvature doesn't vanish. Then there's some point a and a normal $N(a)$ where $H(N) \neq 0$ (we can assume $H(N) > 0$ by choosing an appropriately oriented normal.) Then, with Lemma 2.2 from Osserman, we can find a neighborhood V_1 of a where N is normal to S. This implies that there's a smaller neighborhood V_2 contained in V_1 where the mean curvature $H(N)$ is positive. Now choose a function h which is positive on V_2 and 0 elsewhere. Then the integral in (*) is strictly positive, and thus $A'(0)$ is strictly negative.

If V_2 is small enough (contained in $\triangle$), then on the boundary γ, $x(u) = y(u)$ for the original surface S and a surface S_λ respectively. Assuming that S minimizes area says that for all λ, $A(\lambda) \geq A(0)$, which implies $A'(0) = 0$ which is a contradiction since $A'(0)$ was just shown to be strictly negative. $\square$

11.2 Complex Functions

Working in $\mathbb{C}$ is way different than working in just $\mathbb{R}^2$. For example: a complex function of a complex variable (i.e., $f : \mathbb{C} \to \mathbb{C}$) is called analytic if it is differentiable, and it can be shown that any analytic function is infinitely differentiable! It's pretty crazy.

I don't think we're going to show that in this class, though. But let's talk about derivatives of complex functions. They're defined in the same way as for real functions, i.e. the derivative of a function f (with either a real or complex variable x) at a point a is

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{11.13}$$

These derivatives work like we expect them to (things like the product rule, quotient rule, and chain rule are still valid.) But, there is a fundamental difference between considering a real variable and a complex variable.

Exercise 4. *Let $f(z)$ be a real function of a complex variable ($f : \mathbb{C} \to \mathbb{R}$.) What can we say about $f'(a)$ for any point a?*

11.3 Analytic Functions and the Cauchy-Riemann Equations

So we defined earlier what an analytic function was, but I'll restate it here:

Definition 11.3.1. *A function $f : \mathbb{C} \to \mathbb{C}$ is called* analytic *(or* holomorphic*, equivalently) if its first derivative exists where f is defined.*

We can also represent a complex function f by writing $f(z) = u(z) + iv(z)$, where u and v are real-valued.

When we look at the derivative of f at a :

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{11.14}$$

we know that the limit as h approaches 0 must agree from all directions. So if we look at $f'(a)$ as h approaches 0 along the real line (keeping the imaginary part of h constant), our derivative is a partial with respect to x and we get:

$$f'(z) = \frac{\delta f}{\delta x} = \frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \tag{11.15}$$

Similarly, taking purely imaginary values for h, we get that

$$f'(z) = \lim_{k \to 0} \frac{f(z+ik) - f(z)}{k} = -i\frac{\delta f}{\delta y} = -i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \tag{11.16}$$

So we get that

$$\frac{\delta f}{\delta x} = -i\frac{\delta f}{\delta y} \tag{11.17}$$

and comparing real parts and imaginary parts,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}, \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \tag{11.18}$$

These are the **Cauchy-Riemann equations**, and any analytic function must satisfy them.

11.4 Harmonic Functions

We assume that the functions u and v (given some analytic $f = u + iv$) have continuous partial derivatives of all orders, and that the mixed partials are equal (this follows from knowing the derivative of an analytic function is itself analytic, as raved about earlier.) Then, using equality of mixed partials and the Cauchy-Riemann equations we can show that:

$$\Delta u = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \tag{11.19}$$

and

$$\Delta v = \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0 \tag{11.20}$$

Defining any function f which satisfies *Laplace's equation* $\Delta f = 0$ to be **harmonic**, we get that the real and imaginary parts of an analytic function are harmonic.

Conversely, say we have two harmonic funcions u and v, and that they satisfy the Cauchy-Riemann equations (here v is called the **conjugate har-**

monic function of u.) We want to show that $f = u + iv$ is analytic. This is done in the next lecture (Kai's Monday 10/18 lecture!)

Chapter 12

Review on Complex Analysis I

Reading: Alfors [1]:

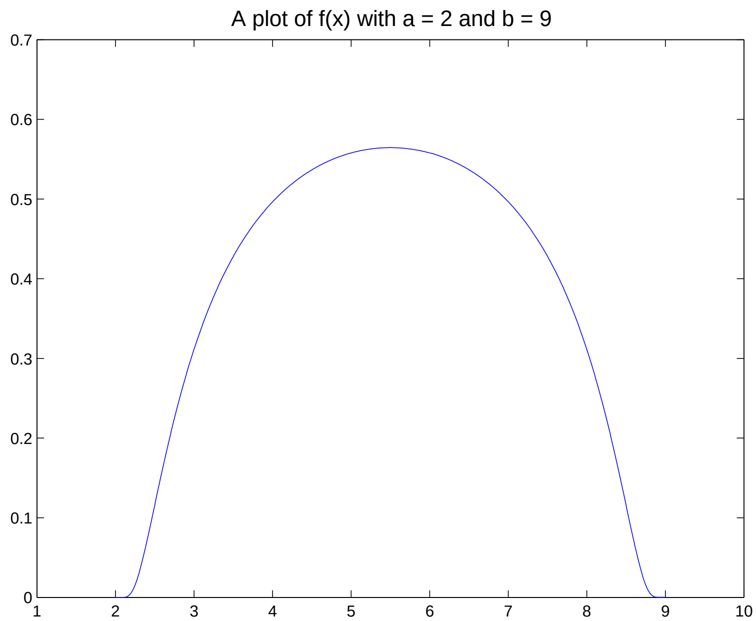
- Chapter 2, 2.4, 3.1-3.4
- Chapter 3, 2.2, 2.3
- Chapter 4, 3.2
- Chapter 5, 1.2

12.1 Cutoff Function

Last time we talked about cutoff function. Here is the way to construct one on $\mathbb{R}^n$ [5].

Proposition 12.1.1. *Let A and B be two disjoint subsets in $\mathbb{R}^m$, A compact and B closed. There exists a differentiable function φ which is identically 1 on A and identically 0 on B*

Proof. We will complete the proof by constructing such a function.



Let $0 < a < b$ and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \left\{ \begin{array}{cc} \exp\left(\frac{1}{x-b} - \frac{1}{x-a}\right) & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{array} \right\} \tag{12.1}$$

It is easy to check that f and the function

$$F(x) = \frac{\int_x^b f(t)\ dt}{\int_a^b f(t)\ dt} \tag{12.2}$$

are differentiable. Note that the function F has value 1 for $x \leq a$ and 0 for $x \geq b$. Thus, the function

$$\psi(x_1, \ldots, x_m) = F(x_1^2 + \ldots + x_m^2) \tag{12.3}$$

is differentiable and has values 1 for $x_1^2 + \ldots + x_m^2 \leq a$ and 0 for $x_1^2 + \ldots + x_m^2 \geq b$.

Let S and S' be two concentric spheres in $\mathbb{R}^m$, $S' \subset S$. By using ψ and linear transformation, we can construct a differentiable function that has value 1 in the interior of S' and value 0 outside S.

Now, since A is compact, we can find finitely many spheres $S_i (1 \leq i \leq n)$ and the corresponding open balls V_i such that

$$A \subset \bigcup_{i=1}^{n} V_i \tag{12.4}$$

and such that the closed balls $\bar{V}_i$ do not intersect B. □

We can shrink each S_i to a concentric sphere S_i' such that the corresponding open balls V_i' still form a covering of A. Let ψ_i be a differentiable function on $\mathbb{R}^m$ which is identically 1 on B_i' and identically 0 in the complement of V_i', then the function

$$\varphi = 1 - (1 - \psi_1)(1 - \psi_2) \ldots (1 - \psi_n) \tag{12.5}$$

is the desired cutoff function.

12.2 Power Series in Complex Plane

In this notes, z and a_i's are complex numbers, $i \in \mathbb{Z}$.

Definition 12.2.1. *Any series in the form*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots + a_n(z - z_0)^n + \ldots \tag{12.6}$$

is called power series.

Without loss of generality, we can take z_0 to be 0.

Theorem 12.2.2. *For every power series 12.6 there exists a number R, $0 \leq R \leq \infty$, called the* radius of convergence*, with the following properties:*

1. *The series converges absolutely for every z with $|z| < R$. If $0 \le \rho \le R$ the convergence is uniform for $|z| \le \rho$.*

2. *If $|z| > R$ the terms of the series are unbounded, and the series is consequently divergent.*

3. *In $|z| < R$ the sum of the series is an analytic function. The derivative can be obtained by termwise differentiation, and the derived series has the same radius of convergence.*

Proof. The assertions in the theorem is true if we choose R according to the *Hadamard's formula*

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}. \tag{12.7}$$

The proof of the above formula, along with assertion (1) and (2), can be found in page 39 of Alfors.

For assertion (3), first I will prove that the derived series $\sum_1^\infty n a_n z^{n-1}$ has the same radius of convergence. It suffices to show that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1 \tag{12.8}$$

Let $\sqrt[n]{n} = 1 + \delta_n$. We want to show that $\lim_{n\to\infty} \delta_n = 0$. By the binomial theorem,

$$n = (1+\delta_n)^n > 1 + \frac{1}{2}n(n-1)\delta_n^2 \tag{12.9}$$

which gives

$$\delta_n^2 < \frac{2}{n} \tag{12.10}$$

and thus

$$\lim_{n\to\infty} \delta_n = 0 \tag{12.11}$$

Let us write

$$f(z) = \sum_0^\infty a_n z^n = s_n(z) + R_n(z) \tag{12.12}$$

where

$$s_n(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} \tag{12.13}$$

is the partial sum of the series, and

$$R_n(z) = \sum_{k=n}^{\infty} a_k z^k \tag{12.14}$$

is the remainder of the series. Also let

$$f_1(z) = \sum_{1}^{\infty} n a_n z^{n-1} = \lim_{n\to\infty} s_n'(z). \tag{12.15}$$

If we can show that $f'(z) = f_1(z)$, then we can prove that the sum of the series is an analytic function, and the derivative can be obtained by termwise differentiation.

Consider the identity

$$\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0) = \left(\frac{s_n(z)-s_n(z_0)}{z-z_0} - s_n'(z_0)\right) + (s_n'(z_0) - f_1(z_0)) + \left(\frac{R_n(z)-R_n(z_0)}{z-z_0}\right) \tag{12.16}$$

and assume $z \neq z_0$ and $|z|, |z_0| < \rho < R$. The last term can be rewritten as

$$\sum_{k=n}^{\infty} a_k (z^{k-1} + z^{k-2} z_0 + \ldots + z z_o^{k-2}) = z_0^{k-1}, \tag{12.17}$$

and thus

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| \leq \sum_{k=n}^{\infty} k|a_k|\rho^{k-1} \tag{12.18}$$

Since the left hand side of the inequality is a convergent sequence, we can find n_0 such that for $n \geq n_0$,

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| < \frac{\epsilon}{3}. \tag{12.19}$$

From Eq. 12.15, we know that there is also an n_1 such that for $n \geq n_1$,

$$|s'_n(z_0) - f_1(z_0)| < \frac{\epsilon}{3}. \tag{12.20}$$

Now if we choose $n \geq n_0, n_1$, from the definition of derivative we can find $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies

$$\left| \frac{s_n(z) - s_n(z_0)}{z - z_0} - s'_n(z_0) \right| < \frac{\epsilon}{3}. \tag{12.21}$$

Combining Eq. 12.19, 12.20 and 12.21, we have for $0 < |z - z_0| < \delta$

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) \right| < \epsilon \tag{12.22}$$

Thus, we have proved that $f'(z_0)$ exists and equals $f_1(z_0)$. $\square$

12.3 Taylor Series

Note that we have proved that a power series with positive radius of convergence has derivatives of all orders. Explicitly,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots \tag{12.23}$$

$$f'(z) = z_1 + 2a_2 z + 3a_3 z^2 + \dots \tag{12.24}$$

$$f''(z) = 2a_2 + 6a_3 z + 12a_4 z^2 + \dots \tag{12.25}$$

$$\vdots \tag{12.26}$$

$$f^{(k)}(z) = k! a_k + \frac{(k+1)!}{1!} a_{k+1} z + \frac{(k+2)!}{2!} a_{k+2} z^2 + \dots \tag{12.27}$$

Since $a_k = f^{(k)}(0)/k!$, we have the Taylor-Maclaurin series:

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!} z^2 + \dots + \frac{f^{(n)}(0)}{n!} z^n + \dots \tag{12.28}$$

Note that we have proved this only under the assumption that $f(z)$ has a power series development. We did not prove that every analytic function has a Taylor development, but this is what we are going to state without proof. The proof can be found in Chapter 4, Sec. 3.1 of [1].

Theorem 12.3.1. *If $f(z)$ is analytic in the region Ω, containing z_0, then the representation*

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \ldots \tag{12.29}$$

is valid in the largest open disk of center z_0 contained in Ω.

12.3.1 The Exponential Functions

We define the *exponential function* as the solution to the following differential equation:

$$f'(z) = f(z) \tag{12.30}$$

with the initial value $f(0) = 1$. The solution is denoted by e^z and is given by

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots \tag{12.31}$$

Since $R = \limsup_{n\to\infty} \sqrt[n]{n!}$, we can prove that the above series converges if

$$\lim_{n\to\infty} \sqrt[n]{n!} = \infty \tag{12.32}$$

Proposition 12.3.2.

$$e^{a+b} = e^a e^b \tag{12.33}$$

Proof. Since $D(e^z \cdot e^{c-z}) = e^z \cdot e^{c-z} + e^z \cdot (-e^{c-z}) = 0$, we know that $e^z \cdot e^{c-z}$ is a constant. The value can be found by putting $z = 0$, and thus $e^z \cdot e^{c-z} = e^c$. Putting $z = a$ and $c = a + b$, we have the desired result. $\square$

Corollary 12.3.3. $e^z \cdot e^{-z} = 1$, *and thus e^z is never 0.*

Moreover, if $z = x + iy$, we have

$$|e^{iy}|^2 = e^{iy}e^{-iy} = 1 \tag{12.34}$$

and

$$|e^{x+iy}| = |e^x|. \tag{12.35}$$

12.3.2 The Trigonometric Functions

We define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{12.36}$$

In other words,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots, \tag{12.37}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots. \tag{12.38}$$

From Eq. 12.36, we can obtain the Euler's equation,

$$e^{iz} = \cos z + i \sin z \tag{12.39}$$

and

$$\cos^2 z + \sin^2 z = 1 \tag{12.40}$$

From Eq. 12.39, we can directly find

$$D\cos z = -\sin z, \quad D\sin z = \cos z \tag{12.41}$$

and the additions formulas

$$\cos(a + b) = \cos a \cos b - \sin a \sin b \tag{12.42}$$

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \tag{12.43}$$

12.3.3 The Logarithm

The logarithm function is the inverse of the exponential function. Therefore, $z = \log w$ is a root of the equation $e^z = w$. Since e^z is never 0, we know that the number 0 has no logarithm. For $w \neq 0$ the equation $e^{x+iy} = w$ is equivalent to

$$e^x = |w|, \quad e^{iy} = \frac{w}{|w|} \tag{12.44}$$

The first equation has a unique solution $x = log|w|$, the real logarithm of $|w|$. The second equation is a complex number of absolute value 1. Therefore, one of the solution is in the interval $0 \leq y < 2\pi$. Also, all y that differ from this solution by an integral multiple of 2π satisfy the equation. Therefore, every complex number other that 0 has infinitely many logarithms which differ from each other by multiples of $2\pi i$.

If we denote $\arg w$ to be the imaginary part of $\log w$, then it is interpreted as the angle, measured in radians, between the positive real axis and the half line from 0 through the point w. And thus we can write

$$\log w = \log |w| + i \arg w \tag{12.45}$$

The addition formulas of the exponential function implies that

$$\log(z_1 z_2) = \log z_1 + \log z_2 \tag{12.46}$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \tag{12.47}$$

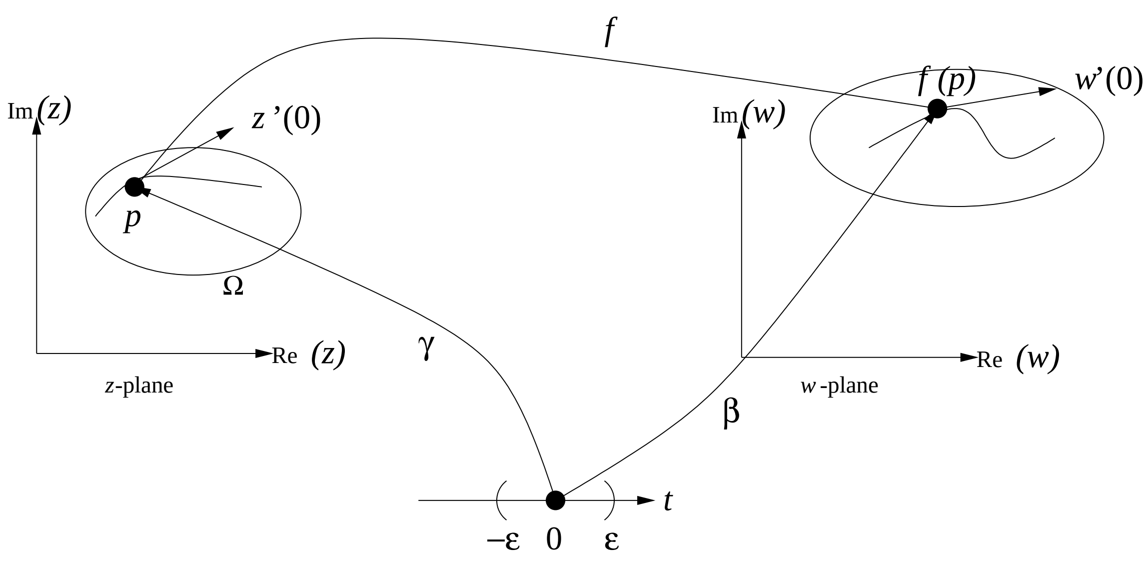
12.4 Analytic Functions in Regions

Definition 12.4.1. *A function $f(z)$ is analytic on an arbitrary point set A if it is the restriction to A of a function which is analytic in some open set containing A.*

Although the definition of analytic functions requires them to be single-valued, we can choose a definite region such that a multiple-valued function, such as $z^{1/2}$, is single-valued and analytic when restricted to the region. For example, for the function $f(z) = z^{1/2}$, we can choose the region Ω to be the complement of the negative real axis. With this choice of Ω, one and only one of the values of $z^{1/2}$ has a positive real part, and thus $f(z)$ is a single-valued function in Ω. The proof of continuity and differentiability of $f(z)$ is in [1] and thus omitted.

12.5 Conformal Mapping

Let γ be an arc with equation $z = z(t), t \in [-\epsilon, \epsilon]$ contained in region Ω with $z(0) = p$. Let $f(z)$ be a continuous function on Ω. The equation $w = w(t) = f(z(t))$ defines an arc β in the w-plane which we call it the image of γ.



We can find $w'(0)$ by

$$w'(0) = f'(p)z'(0). \tag{12.48}$$

The above equation implies that

$$\arg w'(0) = \arg f'(p) + \arg z'(0). \tag{12.49}$$

In words, it means that the angle between the directed tangents to γ at p and to β and $f(p)$ is equal to $\arg f'(p)$, and thus independent of γ. Consequently, curves through p which are tangent to each other are mapped onto curves with a common tangent at $f(p)$. Moreover, two curves which form an angle at p are mapped upon curves forming the same angle. In view of this, we call the mapping f to be *conformal* at all points with $f'(z) \neq 0$.

12.6 Zeros of Analytic Function

The goal of this section is to show that the zeros of analytic functions are isolated.

Proposition 12.6.1. *If f is an analytic function on a region Ω and it does not vanish identically in Ω, then the zeros of f are isolated.*

Proof. Remember that we have assumed in 12.3 that every function f that is analytic in the region Ω can be written as

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \ldots \tag{12.50}$$

Let E_1 be the set on which $f(z)$ and all derivatives vanish at $z_0 \in \mathbb{C}$ and E_2 the set on which the function or one of the derivatives evaluated at z_0 is different from zero. When $f(z)$ and all derivatives vanish at z_0, then $f(z) = 0$ inside the whole region Ω. Thus, E_1 is open. E_2 is open because the function and all derivatives are continuous. Since Ω is connected, we know that either E_1 or E_2 has to be empty. If E_2 is empty, then the function is identically zero. If E_1 is empty, $f(z)$ can never vanish together with all its derivatives.

Assume now that $f(z)$ is not identically zero, and $f(z_0) = 0$. Then there exists a first derivative $f^{(h)}(z_0)$ that is not zero. We say that a is a zero of f of order h. Moreover, it is possible to write

$$f(z) = (z - z_0)^h f_h(z) \tag{12.51}$$

where $f_h(z)$ is analytic and $f_h(z_0) \neq 0$.

Since $f_h(z)$ is continuous, $f_h(z) \neq 0$ in the neighbourhood of z_0 and $z = z_0$ is the unique zero of $f(z)$ in the neighborhood of z_0. □

Corollary 12.6.2. *If f(z) and g(z) are analytic in Ω, and if $f(z) = g(z)$ on a set which has an accumulation point in Ω, then $f(z)$ is identically equal to $g(z)$.*

Proof. Consider the difference $f(z) - g(z)$ and the result from Proposition 12.6.1. □

Chapter 13

Review on Complex Analysis II

(This lecture was given Friday, October 22, 2004.)

13.1 Poles and Singularities

(Following Ahlfors pages 127 - 129.)

We consider a function $f(z)$ analytic in a neighborhood of a, except perhaps at a itself. (So $f(z)$ is analytic on a region $0 < |z - a| < \delta$.)

Definition 13.1.1. *The number a, as above, is called an* isolated singularity *of f.*

We call a a *removable singularity* if we can simply define $f(a)$ appropriately so that $f(z)$ becomes analytic in the entire disk $|z - a| < \delta$. This is discussed in Ahlfors, page 124.

Definition 13.1.2. *If* $\lim_{z\to a} f(z) = \infty, a$ *is a* pole *of* $f(z)$.

With the case of a pole, we set $f(a) = \infty$. Then there exists a $\delta' \leq \delta$ such that $f(z) \neq 0$ on $0 < |z - a| < \delta'$. On this disk, we can look at $g(z) = 1/f(z)$, which is analytic on this disk, and more importantly has a removable singularity at a. So we can set $g(a) = 0$.

Now $g(z)$ doesn't vanish identically, so we know that the zero of g at a has finite order and we can write $g(z) = (z-a)^h g_h(z)$, where $g_h(z) \neq 0$ and is analytic (we can do this because an analytic function behaves locally like a polynomial. Since our function isn't identically zero, we can find a derivative $g^{(h)}(a)$ that doesn't vanish, and then look at the Taylor series expansion from that term on, factoring out $(z-a)^h$ from each term. See Kai's previous notes, Eq. 60.) We call h the *order* of the pole, and we can now write $f(z) = (z-a)^{-h} f_h(z)$, where $f_h(z) = 1/g_h(z)$ is analytic and non-zero in a neighborhood of a.

Definition 13.1.3. *: A function $f(z)$ analytic in a region Ω, except for at its poles, is called* meromorphic *in Ω.*

Equivalently, for every $a \in \Omega$, there is either a neighborhood $|z-a| < \delta$ where the function is analytic, or else $f(z)$ is analytic on $0 < |z-a| < \delta$ and the isolated singularity there is a pole. So that means that the poles of a meromorphic function are isolated by definition. (What would happen if the poles weren't isolated?)

Looking at the quotient $f(z)/g(z)$ of two analytic functions in Ω, and assuming that $g(z)$ isn't identically zero, we get a meromorphic function in Ω. The possible poles here are the zeroes of $g(z)$, but a common zero of $f(z)$ and $g(z)$ could be a removable singularity ($f(z) = (z^2-1)/(z+1)$, for example.) Similarly, the sum, product, and quotient of meromorphic functions are again meromorphic. When regarding the quotient of meromorphic functions, we exclude the possibility of the denominator being identically zero (otherwise, we'd have to consider $f(z) = \infty$ to be a meromorphic function.)

Let's now take a deeper look at isolated singularities. Consider the conditions

$$\lim_{z \to a} |z-a|^\alpha |f(z)| = 0 \tag{13.1}$$

$$\lim_{z \to a} |z-a|^\alpha |f(z)| = \infty \tag{13.2}$$

where α is a real number. If (1) holds for a given value of α, then it holds for all larger values of α, and thus we can find some integer m where it is true. This means that $g(z) = (z-a)^m f(z)$ has a removable singularity and vanishes on a. From here, we know that either $f(z)$ is identically zero (and then (1) holds for all α), or $g(z) = (z-a)^m f(z)$ has a zero of finite order k. In the latter case, we can write $g(z) = (z-a)^k (z-a)^{m-k} f(z)$, where $(z-a)^{m-k} f(z)$ is analytic. So if $\alpha > h = m-k$, (1) holds, and if $\alpha < h$ (2) holds.

Now we assume that condition (2) holds for some α. Then it holds for all smaller α, and likewise for some integer n. The function $g(z) = (z-a)^n f(z)$ has a pole of finite order l, and setting $h = n+l$ (since now we write $g(z) = (z-a)^{-l}(z-a)^{l+n} f(z)$, where $(z-a)^{l+n} f(z)$ is analytic) we find that condition (1) holds when $\alpha > h$ and condition (2) holds when $\alpha < h$.

This means that given an isolated singularity, we have three cases to examine:

i) $f(z)$ is identically zero

ii) there exists an integer h such that (1) holds for $h > \alpha$ and (2) holds for $h < \alpha$

iii) neither (1) nor (2) holds for any α.

Case i) is not interesting.

With case ii), we call h the *algebraic order* of $f(z)$ at a. For a pole this is positive, for a zero it is negative, and it is zero when $f(z)$ is analytic but not equal to zero at a. The algebraic order is always an integer – there is no single-valued analytic function which tends to 0 or ∞ like a fractional power of $|z-a|$

For case iii), a is called an *essential isolated singularity.* So in any neighborhood of an essential isolated singularity, $f(z)$ is both unbounded and comes arbitrarily close to zero. This is illustrated by:

Theorem 13.1.4 (Weierstrass). *An analytic function comes arbritrarily close to any complex value in every neighborhood of an essential singularity.*

Proof. : Suppose that isn't true. Then we can find a complex number A and a $\delta > 0$ such that $|f(z) - A| > \delta$ in a neighborhood of a (except at $z = a$.) For any $\alpha < 0$ we then have $\lim_{z\to a} |z - a|^{\alpha}|f(z) - A| = \infty$. So a would not be an essential singularity of $f(z) - A$.

Then we can find a β where $\lim_{z\to a} |z - a|^{\beta}|f(z) - A| = 0$ (since we're now looking at a case ii) singularity) and we're free to choose $\beta > 0$. Then since in that case $\lim_{z\to a} |z-a|^{\beta} = 0$, it follows that $\lim_{z\to a} |z-a|^{\beta}|f(z)| = 0$, contradicting the fact that a is an essential singularity of $f(z)$. $\square$

Chapter 14

Isotherman Parameters

Let $x : U \to S$ be a regular surface. Let

$$\phi_k(z) = \frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}, z = u_1 + iu_2. \tag{14.1}$$

Recall from last lecture that

a)ϕ is analytic in z $\Leftrightarrow$ x_k is harmonic in u_1 and u_2.

b)u_1 and u_2 are isothermal parameters $\Leftrightarrow$

$$\sum_{k=1}^{n} \phi_k^2(z) = 0 \tag{14.2}$$

c) If u_1, u_2 are isothermal parameters, then S is regular $\Leftrightarrow$

$$\sum_{k=1}^{n} |\phi_k(z)|^2 \neq 0 \tag{14.3}$$

We start by stating a lemma that summarizes what we did in the last lecture:

Lemma 4.3 in Osserman: Let x(u) define a minimal surface, with u_1, u_2 isothermal parameters. Then the functions $\phi_k(z)$ are analytic and they satisfy the eqns in b) and c). Conversely if $\phi_1, \phi_2, .., \phi_n$ are analytic functions satisfying the eqns in b) and c) in a simply connected domain D then there exists a regular minimal surface defined over domain D, such that the eqn on the top of the page is valid.

Now we take a surface in non-parametric form:

$$x_k = f_k(x_1, x_2), k = 3, ..., n \tag{14.4}$$

and we have the notation from the last time:

$$f = (f_3, f_4, ..., f_n), p = \frac{\partial f}{\partial x_1}, q = \frac{\partial f}{\partial x_2}, r = \frac{\partial^2 f}{\partial x_1^2}, s = \frac{\partial^2 f}{\partial x_1 \partial x_2}, t = \frac{\partial^2 f}{\partial x_2^2} \tag{14.5}$$

Then the minimal surface eqn may be written as:

$$(1 + |q|^2)\frac{\partial p}{\partial x_1} - (p.q)(\frac{\partial p}{\partial x_2} + \frac{\partial q}{\partial x_1}) + (1 + |p|^2)\frac{\partial q}{\partial x_2} = 0 \tag{14.6}$$

equivalently

$$(1 + |q|^2)r - 2(p.q)s + (1 + |p|^2)t = 0 \tag{14.7}$$

One also has the following:

$$det g_{ij} = 1 + |p|^2 + |q|^2 + |p|^2|q|^2 - (p.q)^2 \tag{14.8}$$

Define

$$W = \sqrt{det g_{ij}} \tag{14.9}$$

Below we'll do exactly the same things with what we did when we showed that the mean curvature equals 0 if the surface is minimizer for some curve. Now we make a variation in our surface just like the one that we did before (the only difference is that x_1 and x_2 are not varied.)

$$\tilde{f}_k = f_k + \lambda h_k, k = 3, ..., n, \tag{14.10}$$

where λ is a real number, and $h_k \in C^1$ in the domain of definition D of the

f_k We have

$$\tilde{f} = f + \lambda h, \tilde{p} = p + \lambda \frac{\partial h}{\partial x_1}, \tilde{q} = q + \lambda \frac{\partial h}{\partial x_2} \tag{14.11}$$

One has

$$\tilde{W}^2 = W^2 + 2\lambda X + \lambda^2 Y \tag{14.12}$$

where

$$X = [(1 + |q|^2)p - (p.q)q].\frac{\partial h}{\partial x_1} + [(1 + |p|^2)q - (p.q)p].\frac{\partial h}{\partial x_2} \tag{14.13}$$

and Y is a continuous function in x_1 and x_2. It follows that

$$\tilde{W} = W + \lambda \frac{X}{W} + O(\lambda^2) \tag{14.14}$$

as $|\lambda| \to 0$ Now we consider a closed curve Γ on our surface. Let Δ be the region bounded by Γ If our surface is a minimizer for Δ then for every choice of h such that $h = 0$ on Γ we have

$$\int \int_\Delta \tilde{W} dx_1 dx_2 \geq \int \int_\Delta W dx_1 dx_2 \tag{14.15}$$

which implies

$$\int \int_\Delta \frac{X}{W} = 0 \tag{14.16}$$

Substituting for X, integrating by parts, and using the fact that $h = 0$ on Γ , we find

$$\int \int_\Delta \left[\frac{\partial}{\partial x_1} \left[\frac{1 + |q|^2}{W} p - \frac{p.q}{W} q \right] + \frac{\partial}{\partial x_2} \left[\frac{1 + |p|^2}{W} q - \frac{p.q}{W} p \right] \right] h dx_1 dx_2 = 0 \tag{14.17}$$

must hold everywhere. By the same reasoning that we used when we found the condition for a minimal surface the above integrand should be zero.

$$\frac{\partial}{\partial x_1} \left[\frac{1 + |q|^2}{W} p - \frac{p.q}{W} q \right] + \frac{\partial}{\partial x_2} \left[\frac{1 + |p|^2}{W} q - \frac{p.q}{W} p \right] = 0 \tag{14.18}$$

Once we found this equation it makes sense to look for ways to derive it from the original equation since after all there should only be one constraint for a minimal surface. In fact the LHS of the above eqn can be written as the sum of three terms:

$$\left[\frac{1+|q|^2}{W}\frac{\partial p}{\partial x_1} - \frac{p.q}{W}(\frac{\partial q}{\partial x_1} + \frac{\partial p}{\partial x_2}) + \frac{1+|p|^2}{W}\frac{\partial q}{\partial x_2}\right] \tag{14.19}$$

$$+\left[\frac{\partial}{\partial x_1}(\frac{1+|q|^2}{W}) - \frac{\partial}{\partial x_2}(\frac{p.q}{W})\right] p \tag{14.20}$$

$$+\left[\frac{\partial}{\partial x_2}(\frac{1+|p|^2}{W}) - \frac{\partial}{\partial x_1}(\frac{p.q}{W})\right] q \tag{14.21}$$

The first term is the minimal surface eqn given on the top of the second page. If we expand out the coefficient of p in the second term we find the expression:

$$\frac{1}{W^3}[(p.q)q - (1+|q|^2)p].[(1+|q|^2)r - 2(p.q)s + (1+|p|^2)t] \tag{14.22}$$

which vanishes by the second version of the minimal surface eqns. Similarly the coefficient of q in third term vanishes so the while expression equals zero. In the process we've also shown that

$$\frac{\partial}{\partial x_1}\left(\frac{1+|q|^2}{W}\right) = \frac{\partial}{\partial x_2}\left(\frac{p.q}{W}\right) \tag{14.23}$$

$$\frac{\partial}{\partial x_2}\left(\frac{1+|p|^2}{W}\right) = \frac{\partial}{\partial x_1}\left(\frac{p.q}{W}\right) \tag{14.24}$$

Existence of isothermal parameters or Lemma 4.4 in Osserman Let S be a minimal surface. Every regular point of S has a neighborhood in which there exists a reparametrization of S in terms of isothermal parameters.

Proof: Since the surface is regular for any point there exists a neighborhood of that point in which S may be represented in non-parametric form. In particular we can find a disk around that point where the surface can be represented in non parametric form. Now the above eqns imply the existence of functions $F(x_1, x_2)$ $G(x_1, x_2)$ defined on this disk, satisfying

$$\frac{\partial F}{\partial x_1} = \frac{1+|p|^2}{W}, \frac{\partial F}{\partial x_2} = \frac{p.q}{W}; \tag{14.25}$$

$$\frac{\partial G}{\partial x_1} = \frac{p.q}{W}, \frac{\partial G}{\partial x_2} = \frac{1+|q|^2}{W} \tag{14.26}$$

If we set

$$\xi_1 = x_1 + F(x_1, x_2), \xi_2 = x_2 + G(x_1, x_2), \tag{14.27}$$

we find

$$J = \frac{\partial(x_1, x_2)}{\partial(x_1, x_2)} = 2 + \frac{2+|p|^2+|q|^2}{W} \geq 0 \tag{14.28}$$

Thus the transformation $(x_1, x_2) \to (\xi_1, \xi_2)$ has a local inverse $(\xi_1, \xi_2) \to (x_1, x_2)$. We find the derivative of x at point (ξ_1, ξ_2):

$$Dx = J^{-1}[x_1, x_2, f_3, ..., f_n] \tag{14.29}$$

It follows that with respect to the parameters ξ_1, ξ_2 we have

$$g_{11} = g_{22} = |\frac{\partial x}{\partial \xi_1}|^2 = |\frac{\partial x}{\partial \xi_2}|^2 \tag{14.30}$$

$$g_{12} = \frac{\partial x}{\partial \xi_1}.\frac{\partial x}{\partial \xi_2} = 0 \tag{14.31}$$

so that ξ_1, ξ_2 are isothermal coordinates.

Chapter 15

Bernstein's Theorem

15.1 Minimal Surfaces and isothermal parametrizations

Note: This section will not be gone over in class, but it will be referred to.

Lemma 15.1.1 (Osserman 4.4). *Let S be a minimal surface. Every regular point p of S has a neighborhood in which there exists of reparametrization of S in terms of isothermal parameters.*

Proof. By a previous theorem (not discussed in class) there exists a neighborhood of the regular point which may be represented in a non-parametric form. Then we have that $x(x_1, x_2) = (x_1, x_2, f_3(x_1, x_2), \ldots, f_n(x_1, x_2))$. Defining $f = (f_3, f_4, \ldots, f_n)$, we let $p = \frac{\partial f}{\partial x_1}$, $q = \frac{\partial f}{\partial x_2}$, $r = \frac{\partial^2 f}{\partial x_1^2}$, $s = \frac{\partial^2 f}{\partial x_1 \partial x_2}$, and $t = \frac{\partial^t f}{\partial x_2^2}$. Last, we let $W = \sqrt{\det g_{ij}} = \sqrt{1 + |p|^2 + |q|^2 + |p|^2|q|^2 - (p \cdot q)^2}$. We then have (from last lecture)

$$\frac{\partial}{\partial x_1}\left(\frac{1 + |q|^2}{W}\right) = \frac{\partial}{\partial x_2}\left(\frac{p \cdot q}{W}\right)$$

$$\frac{\partial}{\partial x_1}\left(\frac{p \cdot q}{W}\right) = \frac{\partial}{\partial x_2}\left(\frac{1 + |q|^2}{W}\right)$$

Then there exists a function $F(x_1, x_2)$ such that $\frac{\partial F}{\partial x_1} = \frac{1+|p|^2}{W}$ and $\frac{\partial F}{\partial x_2} = \frac{p \cdot q}{W}$. Why? Think back to 18.02 and let $V = (\frac{1+|p|^2}{W}, \frac{p \cdot q}{W}, 0)$ be a vector field in $\mathbb{R}^3$; then $|\bigtriangledown \times V| = \frac{\partial}{\partial x_2} \frac{1+|p|^2}{W} - \frac{\partial}{\partial x_1} \frac{p \cdot q}{W} = 0$, so there exists a function F such that $\bigtriangledown F = V$, which is the exact condition we wanted (once we get rid of the third dimension). Similarly there exists a function $G(x_1, x_2)$ with $\frac{\partial G}{\partial x_1} = \frac{p \cdot q}{W}$ and $\frac{\partial G}{\partial x_2} = \frac{\partial 1+|q|^2}{\partial W}$.

We now define $\xi(x_1, x_2)(x_1 + F(x_1, x_2), x_2 + G(x_1, x_2))$. We then find that $\frac{\partial \xi_1}{\partial x_1} = 1 + \frac{1+|p|^2}{W}$, $\frac{\partial \xi_2}{\partial x_2} = 1 + \frac{1+|q|^2}{W}$, and $\frac{\partial \xi_1}{\partial x_2} = \frac{\partial \xi_2}{\partial x_1} = \frac{p \cdot q}{W}$. Then (recalling the defintion of W^2) we can find that the magnitude of the Jacobian $\frac{\partial(\xi_1, \xi_2)}{\partial(x_1, x_)}$ is $2 + \frac{2+|p|^2+|q|^2}{W} > 0$. This implies that the transformation ξ has a local inverse $\hat{x}$ at p. Judicial use of the inverse function theorem will show that with respect to the parameters ξ_1 and ξ_2, $g_{11} = g_{22}$ and $g_{12} = 0$, so these are isothermal coordinates; see Osserman p 32 for details. $\square$

We also have the following result:

Lemma 15.1.2 (Osserman 4.5). *Let a surface S be defined by an isothermal parametrization $x(u)$, and let $\widetilde{S}$ be a reparametrization of S defined by a diffeomorphism with matrix U. Then $\tilde{u_1}$, $\tilde{u_2}$ are isothermal parameters if and only if the map U is either conformal or anti-conformal.*

Proof. For a map to be conformal or anti-conformal means that it preserves $|\theta|$, or alternatively that it preserves $\cos\theta$. (It also needs to be continuous enough that it isn't flipping the sign back and forth.) If U is a constant μ times an orthogonal matrix, then $\mu|v| = |Uv|$ for all v since $\mu^2 \langle v, v\rangle = \langle Uv, Uv\rangle$; thus if θ is the angle between vectors v and w and θ' is the angle between Uv and Uw, we have that $\cos\theta = \frac{\mu^2\langle v,w\rangle}{\mu^2|v||w|} = \frac{\langle Uv, Uw\rangle}{|Uv||Uw|} = \cos\theta'$. So for diffeomorphisms with matrix U, U being conformal or anti-conformal is equivalent to U being a constant multiple of an orthogonal matrix.

Now, since x is isothermal, we have that $g_{ij} = \lambda^2 \delta_{ij}$ (where δ_{ij} is the Kronecker delta). By a theorem on page 5 about change of coordinates, we know that $\widetilde{G} = U^T G U = \lambda^2 U^T U$. So $\tilde{u_1}$, $\tilde{u_2}$ is isothermal iff $\tilde{g}_{ij} = \tilde{\lambda}^2 \delta_{ij}$, which is to say that $I = \frac{\lambda^2}{\tilde{\lambda}^2} U^T U$, which is to say that $\frac{\tilde{\lambda}}{\lambda} U$ is orthogonal. But we have already shown that this is equivalent to U being conformal or anti-conformal. $\square$

15.2 Bernstein's Theorem: Some Preliminary Lemmas

The main goal of today is to prove Bernstein's Theorem, which has the nice corollary that in $\mathbb{R}^3$, the only minimal surface that is defined in non-parametric form on the entire x_1, x_2 plane is a plane. This makes sense: the catenoid and helicoid are not going to give you nonparametric forms since no projection of them is injective, and Scherk's surface may be nonparametric but it's only defined on a checkerboard. We have a bunch of lemmas to work through first.

Lemma 15.2.1 (Osserman 5.1). *Let $E \colon D \to \mathbb{R}$ be a C^2 function on a convex domain D, and suppose that the Hessian matrix $\left(\frac{\partial^2 E}{\partial x_i \partial x_j}\right)$ evaluated at any point is positive definite. (This means that the quadratic form it defines sends every nonzero vector to a positive number, or equivalently that it is symmetric with positive eigenvalues.) Define a mapping $\phi \colon D \to \mathbb{R}^2$ with $\phi(x_1, x_2) = (\frac{\partial E}{\partial x_1}(x_1, x_2), \frac{\partial E}{\partial x_2}(x_1, x_2))$ (since $\frac{\partial E}{\partial x_1} \colon D \to \mathbb{R}$). Let a and b be distinct points of D; then $(b - a) \cdot (\phi(b) - \phi(a)) > 0$.*

Proof. Let $G(t) = E(tb + (1-t)a) = E(tb_1 + (1-t)a_1, tb_2 + (1-t)b_2)$ for $t \in [0, 1]$. Then

$$G'(t) = \sum_{i=1}^{2} \left(\frac{\partial E}{\partial x_i}(tb + (1-t)a) \right) (b_i - a_i)$$

(note that the $tb + (1-t)a$ here is the *argument* of $\frac{\partial E}{\partial x_i}$, not a multiplied factor) and

$$G''(t) = \sum_{i,j=1}^{2} \left(\frac{\partial^2}{\partial x_i \partial x_j}(tb + (1-t)a) \right) (b_i - a_i)(b_j - a_j)$$

But this is just the quadratic form of $\left(\frac{\partial^2 E}{\partial x_i \partial x_j}\right)$ evaluated at the point $tb + (1-t)a$, applied to the nonzero vector $b - a$. By positive definiteness, we have that $G''(t) > 0$ for $t \in [0,1]$. So $G'(1) > G'(0)$, which is to say that $\sum \phi(b)_i(b_i - a_i) > \sum \phi(a)_i(b_i - a_i)$, which is to say that $(\phi(b) - \phi(a)) \cdot (b - a) > 0$. ☐

Lemma 15.2.2 (Osserman 5.2). *Assume the hypotheses of Osserman Lemma 5.1. Define the map $z \colon D \to \mathbb{R}^2$ by $z_i(x_1, x_2) = x_i + \phi_i(x_1, x_2)$. Then given distinct points $a, b \in D$, we have that $(z(b) - z(a)) \cdot (b - a) > |b - a|^2$, and $|z(b) - z(a)| > |b - a|$.*

Proof. Since $z(b) - z(a) = (b - a) + (\phi(b) - \phi(a))$, we have that $(z(b) - z(a)) \cdot (b - a) = |b - a|^2 + (\phi(b) - \phi(a)) \cdot (b - a) > |b - a|^2$ by the previous lemma.

Then $|b - a|^2 < |(z(b) - z(a)) \cdot (b - a)| \leq |z(b) - z(a)||b - a|$, where the second inequality holds by Cauchy-Schwarz; so $|b - a| < |z(b) - z(a)|$. ☐

Lemma 15.2.3 (Osserman 5.3). *Assume the hypotheses of Osserman Lemma 5.2. If D is the disk $x_1^2 + x_2^2 < R^2$, then the map z is a diffeomorphism of D onto a domain Δ which includes a disk of radius R around $z(0)$.*

Proof. We know that z is continuously differentiable, since $E \in C^2$. If $x(t)$ is any differentiable curve in D and $z(t)$ is its image under z, then it follows from the previous lemma that $|z'(t)| > |x'(t)|$; thus the determinant of the matrix dz (which is to say, the Jacobian) is greater than 1, since $z'(t) = (dz)x'(t)$ implies that $|z'(t)| = \det dz|x'(t)|$. So since the Jacobian is everywhere greater than 1, the map is a local diffeomorphism everywhere. It's also injective (because

$\phi(b) - \phi(a) = 0$ implies that $b - a = 0$ by the previous lemma), so it's in fact a (global) diffeomorphism onto a domain Δ.

We must show that Δ includes all points z such that $z - z(0) < R$. If Δ is the whole plane this is obvious; otherwise there is a point Z in the complement of Δ (which is closed) which minimizes the distance to $z(0)$. Let Z^k be a sequence of points in $\mathbb{R}^2 - \Delta$ which approach Z (if this didn't exist, we could find a point in $\mathbb{R}^2 - \Delta$ closer to $z(0)$ than Z), and since z is a diffeomorphism, we let x^k be the sequence of points mapped onto Z^k by z. The points x^k cannot have a point of accumulation in D, since that would be mapped by z onto a point of Δ, and we are assuming that $Z \notin \Delta$. But x^k must have an accumulation point in $\mathbb{R}^2$ in order for their image to; so $|x^k| \to R$ as $k \to \infty$; since $|\mathbb{Z}^k - z(0)| > |x^k - 0|$ by the previous lemma, we have that $|Z - z(0)| \geq R$, so every point within R of $z(0)$ is in Δ. $\square$

Lemma 15.2.4 (Osserman 5.4). *Let $f(x_1, x_2)$ be a non-parametric solution to the minimal surface equation in the disk of radius R around the origin. Then the map ξ defined earlier is a diffeomorphism onto a domain Δ which includes a disk of radius R around $\xi(0)$.*

Proof. It follows from the defining characteristics of F and G that there exists a function E satisfying $\frac{\partial E}{\partial x_1} = F$ and $\frac{\partial E}{\partial x_2} = G$, for the same reason that F and G exist. Then $E \in C^2$, and $\frac{\partial^2 E}{\partial x_1^2} = \frac{1+|p|^2}{W} > 0$, and det $\frac{\partial^2 E}{\partial x_i \partial x_j} = \frac{\partial(F,G)}{\partial(x_1,x_2)} = 1 > 0$ (by the definition of W, it's a simple check). Any matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $a > 0$ and $ac - b^2 > 0$ must have $c > 0$, so its trace and determinant are both positive, so the sum and product of its eigenvalues are both positive, so it is positive definite. So the Hessian of E is positive definite. We can see that the mapping z defined in (our version of) Osserman Lemma 5.2 is in this case the same map as ξ defined in (our version of) Osserman Lemma 4.4. So by Osserman Lemma 5.3, we have that ξ is a diffeomorphism onto a domain Δ which includes a disk of radius R around $\xi(0)$. $\square$

Lemma 15.2.5 (Osserman 5.5). *Let $f\colon D \to \mathbb{R}$ be a C^1 function. Then the surface S in $\mathbb{R}^3$ defined in non-parametric form by $x_3 = f(x_1, f_2)$ lies on a plane iff there exists a nonsingular linear transformation $\psi\colon U \to D$ from some domain U such that u_1, u_2 are isothermal parameters on S.*

Proof. Suppose such parameters u_1, u_2 exist. Letting $\phi_k(\zeta) = \frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}$, for $1 \le k \le 3$, we see that ϕ_1 and ϕ_2 are constant because x_1 and x_2 are linear in u_1 and u_2. We know from a previous lecture that u_1 and u_2 are isothermal parameters iff $\sum_{k=1}^{3} \phi_k^2(\zeta)$ is zero for all ζ, so ϕ_3 is constant too. (Well, it implies that ϕ_3^2 is constant, which constrains it to at most two values, and since ϕ_3 must be continuous, it must be constant.) This means that x_3 has a constant gradient with respect to u_1, u_2 and thus also with respect to x_1, x_2. This means that we must have $f(x_1, x_2) = Ax_1 + Bx_2 + C$; but this is the equation of a plane.

Conversely, if $f(x_1, x_2)$ is a part of a plane, then it equals Ax_1+Bx_2+C for some constants A, B, and C. Then the map $x(u_1, u_2) = (\lambda Au_1+Bu_2, \lambda Bu_1-Au_2)$ with $\lambda^2 = \frac{1}{1+A^2+B^2}$ is isothermal. To check this, we see that $\phi_1 = \lambda A - iB$, $\phi_2 = \lambda B + iA$, $\phi_1^2 = \lambda^2 A^2 - B^2 - 2\lambda ABi$, $\phi_2^2 = \lambda^2 B^2 - A^2 + 2\lambda ABi$. $x_3 = Ax_1 + Bx_2 + C = A(\lambda Au_1 + Bu_2) + B(\lambda Bu_1 - Au_2) + C$, so $\phi_3 = \lambda(A^2 + B^2)$ and $\phi^2 = \lambda^2(A^2+B^2)^2$. Then $\phi_1^2+\phi_2^2+\phi_3^2 = \lambda^2(A^2+B^2)-(A^2+B^2)+\lambda^2(A^2+B^2)^2 = (A^2+B^2)(\lambda^2 - 1 + \lambda^2(A^2+B^2)) = (A^2+B^2)(\lambda^2(1+A^2+B^2)-1) = (A^2+B^2)(1-1) = 0$, so this is isothermal. □

15.3 Bernstein's Theorem

Theorem 15.3.1 (Bernstein's Theorem, Osserman 5.1). *Let $f(x_1, x_2)$ be a solution of the non-parametric minimal surface equation defined in the entire x_1, x_2 plane. Then there exists a nonsingular linear transformation $x_1 = u_1$, $x_2 = au_1+bu_2$ with $b > 0$ such that u_1, u_2 are isothermal parameters on the entire u-plane for the minimal surface S defined by $x_k = f_k(x_1, x_2)$ ($3 \le k \le n$).*

Proof. Define the map ξ as in our version of Osserman Lemma 4.4. Osserman Lemma 5.4 shows that this is a diffeomorphism from the entire x-plane onto the entire ξ-plane. We know from Osserman Lemma 4.4 that ξ is a set of isothermal parameters on S. By Osserman Lemma 4.3 (which Nizam proved), the functions $\phi_k(\zeta) = \frac{\partial x_k}{\partial \xi_1} - i\frac{\partial x_k}{\partial \xi_2}$ $(1 \leq k \leq n)$ are analytic functions of ζ. We can see that $\Im(\bar{\phi_1}\phi_2) = -\frac{\partial(x_1,x_2)}{\partial(\xi_1,\xi_2)}$; since this Jacobian is always positive (see proof of Osserman Lemma 4.4), we can see that $\phi_1 \neq 0$, $\phi_2 \neq 0$, and that $\Im\frac{\phi_2}{\phi_1} = \frac{1}{|\phi_1|^2}\Im(\bar{\phi_1}\phi_2) < 0$. So the function $\frac{\phi_2}{\phi_1}$ is analytic on the whole ζ-plane and has negative imaginary part everywhere. By Picard's Theorem, an analytic function that misses more than one value is constant, so $\frac{\phi_2}{\phi_1} = C$ where $C = a - ib$. That is $\phi_2 = (a - ib)\phi_1$. The real part of this equation is $\frac{\partial x_2}{\partial \xi_1} = a\frac{\partial x_1}{\partial \xi_1} - b\frac{\partial x_1}{\partial \xi_2}$, and the imaginary part is $\frac{\partial x_2}{\partial \xi_2} = b\frac{\partial x_1}{\partial \xi_1} + a\frac{\partial x_1}{\partial \xi_2}$. If we then apply the linear transformation from the statement of the theorem, using the a and b that we have, this becomes $\frac{\partial u_1}{\partial \xi_1} = \frac{\partial u_2}{\partial \xi_2}$ and $\frac{\partial u_2}{\partial \xi_2} = -\frac{\partial u_1}{\partial \xi_2}$: the Cauchy-Reimann equations! So $u_1 + iu_2$ is an analytic function of $\xi_1 + i\xi_2$. But by Osserman Lemma 4.5, this implies that u_1, u_2 are also isothermal parameters, which proves the theorem. $\square$

This (with Osserman Lemma 5.5) has the immediate corollary that for $n = 3$, the only solution of the non-parametric minimal surface equation on the entire x-plane is surface that is a plane. This gives us a nice way to generate lots of weird minimal surfaces in dimensions 4 and up by starting with analytic functions; this is Osserman Corollary 3, but I do not have time to show this.

Chapter 16

Manifolds and Geodesics

Reading:

- Osserman [7] Pg. 43-52, 55, 63-65,
- Do Carmo [2] Pg. 238-247, 325-335.

16.1 Manifold Theory

Let us recall the definition of differentiable manifolds

Definition 16.1.1. *An* **n-manifold** *is a Hausdorff space, each point of which has a neighborhood homeomorphic to a domain in* $\mathbb{R}^n$.

Definition 16.1.2. *An* **atlas** *A for an n-manifold* M^n *is a collection of triples* $(U_\alpha, V_\alpha, \varphi_\alpha)$ *where* U_α *is a domain in* $\mathbf{R}^n$, V_α *is an open set on* M^n, *and* φ_α *is a homeomorphism of* U_α *onto* V_α, *and*

$$\bigcup_\alpha V_\alpha = M^n \tag{16.1}$$

Each triple is called a **map**.

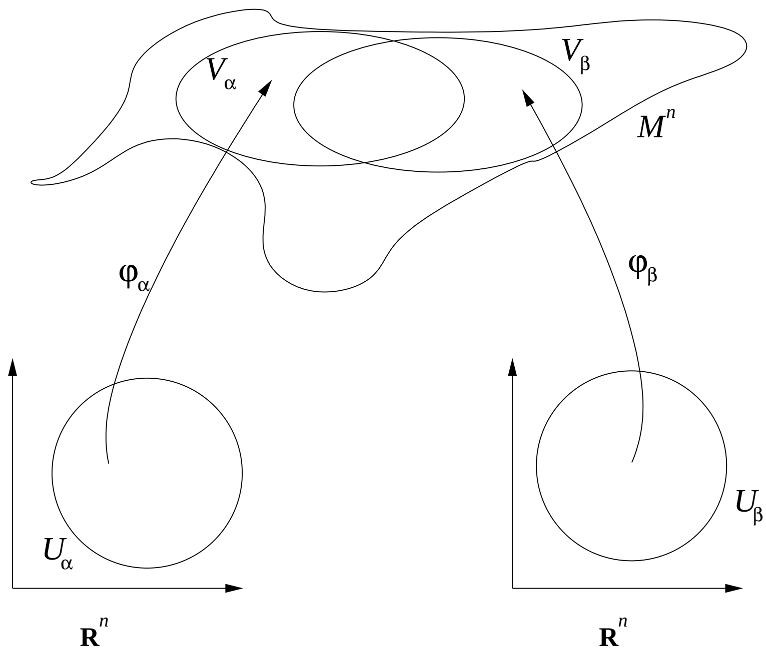


Figure 16.1: Definition of atlas

Definition 16.1.3. *A C^r- (resp. conformal) structure on M^n is at atlas for which each transformation $\varphi_\alpha^{-1} \circ \varphi_\beta \in C^r$ (resp. conformal) wherever it is defined.*

Corollary 16.1.4. *The space $\mathbf{R}^n$ has a canonical C^r-structure for all r, defined by letting A consists of the single triple $U_\alpha = V_\alpha = \mathbf{R}^n$, and φ_α the identity map.*

Let S be a C^r-surface in $\mathbf{R}^n$, and A the C^r-structure on the associated 2-manifold M. We discussed that all local properties of surfaces which are independent of parameters are well defined on a global surface S by the change of parameters. The global properties of S will be defined simply to be those of M, such as orientation, compactness, connectedness, simply connectedness, etc.

In the rest of the course, all surfaces will be connected and orientable.

Definition 16.1.5. *A regular C^2-surface S in $\mathbf{R}^n$ is a* **minimal surface** *if its mean curvature vector vanishes at each point.*

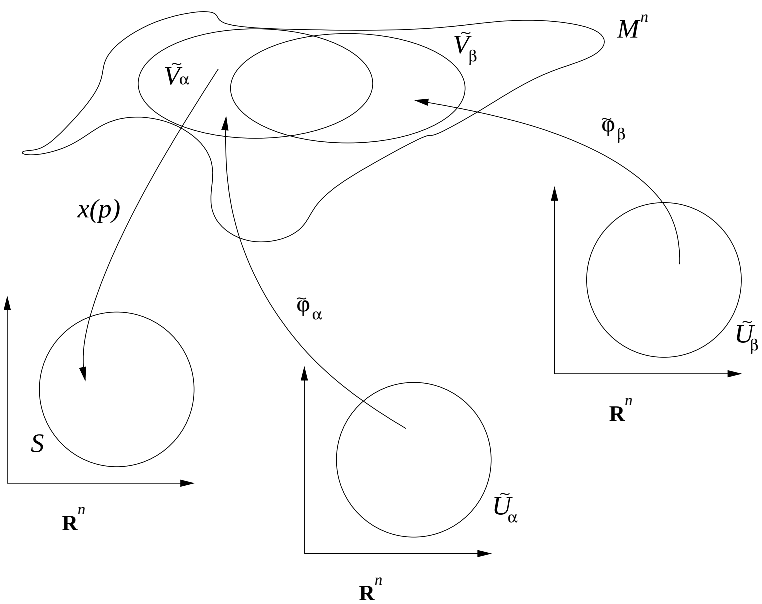


Figure 16.2: Lemma 6.1

The following two lemmas are useful for the proof of Lemma 6.1 in [7].

(Lemma 4.4 in [7]). *Let S be a minimal surface. Every regular point of S has a neighborhood in which there exists a reparametrization of S in terms of isothermal parameters.*

(Lemma 4.5 in [7]). *Let a surface S be defined by $x(u)$, where u_1, u_2 are isothermal parameters, and let $\tilde{S}$ be a reparametrization of S defined by a diffeomorphism $u(\tilde{u})$. Then $\tilde{u_1}, \tilde{u_2}$ are also isothermal parameters if and only if the map $u(\tilde{u}$ is either conformal or anti-conformal.*

(Lemma 6.1 in [7]). *Let S be a regular minimal surface in $\mathbf{R}^n$ defined by a map $x(p) : M \to \mathbf{R}^n$. Then S induces a conformal structure on M.*

Proof. Assume the surface S is orientable, and A be an oriented atlas of M. Let $\tilde{A}$ be the collection of all the maps $(\tilde{U}_\alpha, \tilde{V}_\alpha, \tilde{\varphi}_\alpha) \in A$ such that $\tilde{\varphi}_\beta^{-1} \circ \tilde{\varphi}_\alpha$ preserves orientation wherever defined, and the map $x \circ \tilde{\varphi}_\alpha : \tilde{U}_\alpha \to \mathbf{R}^n$ defines a local surface in isothermal parameters. By Lemma 16.1 the union of $\tilde{V}_\alpha$ equals M, so $\tilde{A}$ is an atlas for M. And by Lemma 16.1 each $\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha$ is conformal wherever defined. So $\tilde{A}$ defines a conformal structure on M. $\square$

With the previous lemma, we can discuss some basic notions connected with conformal structure. If M has a conformal structure, then we can define all concepts which are invariant under conformal mapping, such as analytic maps of one such manifold M into another $\tilde{M}$.

Example 7. *(Stereographic Projection) A* **meromorphic function** *on M is a complex analytic map of M into the Riemann sphere. The latter can be defined as the unit sphere in $\mathbf{R}^3$ with the conformal strucutre defined by a pair of maps*

$$\varphi_1 : x = \left(\frac{2u_1}{|w|^2+1}, \frac{2u_2}{|w|^2+1}, \frac{|w|^2-1}{|w|^2+1} \right), \quad w = u_1 + iu_2 \tag{16.2}$$

$$\varphi_2 : x = \left(\frac{2\tilde{u}_1}{|\tilde{w}|^2+1}, \frac{-2\tilde{u}_2}{|\tilde{w}|^2+1}, \frac{1-|\tilde{w}|^2}{|\tilde{w}|^2+1} \right), \quad \tilde{w} = \tilde{u}_1 + i\tilde{u}_2 \tag{16.3}$$

The map φ_1 is called the **stereographic projection** *from the point $(0,0,1)$, and one can easily show that $\varphi_1^{-1} \circ \varphi_2$ is simply $w = \frac{1}{\tilde{w}}$, a conformal map of $0 < |\tilde{w}| < \infty$ onto $0 < |w| < \infty$.*

Definition 16.1.6. *A* **generalized minimal surface** *S in $\mathbf{R}^n$ is a non-constant map $x(p) : M \to \mathbf{R}^n$, where M is a 2-manifold with a conformal structure defined by an atlas $A = \{U_\alpha, V_\alpha, \varphi_\alpha\}$, such that each coordinate function $x_k(p)$ is harmonic on M, and furthermore*

$$\sum_{k=1}^{n} \phi_k^2(\zeta) = 0 \tag{16.4}$$

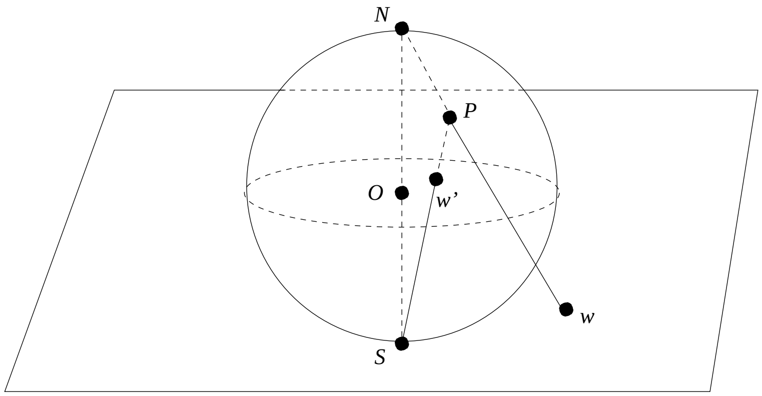


Figure 16.3: Stereographic Projection

where we set for an arbitrary a,

$$h_k(\zeta) = x_k(\varphi_\alpha(\zeta)) \tag{16.5}$$

$$\phi_k(\zeta) = \frac{\partial h_k}{\partial \xi_1} - i\frac{\partial h_k}{\partial \xi_2}, \quad \zeta = \xi_1 + i\xi_2 \tag{16.6}$$

Following is a lemma from Ch.4 in [7]

(Lemma 4.3 in [7]). *Let $x(u)$ define a regular minimal surface, with u_1, u_2 isothermal parameters. Then the function $\phi_k(\zeta)$ defined by 16.6 are analytic, and they satisfy equation*

$$\sum_{k=1}^{n} \phi_k^2(\zeta) = 0 \tag{16.7}$$

and

$$\sum_{k=1}^{n} |\phi_k^2(\zeta)| \neq 0. \tag{16.8}$$

Conversely, let $\phi_1(\zeta), ..., \phi_n(\zeta)$ be analytic functions of ζ which satisfy Eqs. 16.7 and 16.8 in a simply-connected domain D. Then there exists a regular minimal surface $x(u)$ defined over D, such that Eqs. 16.6 are valid.

Corollary 16.1.7. *If S is regular minimal surface, then S is also a generalized minimal surface.*

Proof. We can use the conformal structure defined in Lemma 6.1, and the result follows from Lemma 4.3 in [7] □

Definition 16.1.8. *Let S be a generalized minimal surface, and $\zeta \in S$. The* **branch points** *ζ's with respect to the function ϕ_k correspond to the ζ's at which*

$$\sum_{k=1}^{n} |\phi_k^2(\zeta)| = 0 \tag{16.9}$$

Corollary 16.1.9. *Let S be a generalized minimal surface, and S' be the surface S with branch points with respect to the function ϕ in Eq. 16.6 deleted. Then S' is a regular minimal surface.*

Proof. Let $x(p)$ be the coordinate map of S, where $p \in S$. Since $x(p)$ is non constant, at least one of the function $x_k(p)$ is non constant. That means that the corresponding $\phi_k(\zeta)$ can have at most isolated zeroes, and the equation

$$\sum_{k=1}^{n} |\phi_k^2(\zeta)| = 0 \tag{16.10}$$

can hold at most at the branch points. Since S' consists of the whole surface S without the branch points, S' is a regular minimal surface, from Lemma 4.3 in [7]. □

In the case of $n = 2$ in the definition of a generalized surface, either $x_1 + ix_2$ or $x_1 - x_2$ is a non-constant analytic function $f(\zeta)$. The branch points on the surface satisfy the Eq. 16.9. That is, they satisfy the equation $f'(\zeta) = 0$, which is the inverse mapping.

For large n, the difference between regular and generalized minimal surfaces consists in allowing the possibility of isolated branch points. However, there are theorems where the possible existence of branch points has no effect. The following lemma is one of the example.

(Lemma 6.2 in [7]). *A generalized minimal surface cannot be compact*

Proof. Let S be a generalized minimal surface defined by a map $x(p) : M \to \mathbf{R}^n$. Then each coordinate function $x_k(p)$ is harmonic on M. If M were compact, the function $x_k(p)$ would attain its maximum, hence it would be a constant. This contradicts the assumption that the map $x(p)$ is non-constant. $\square$

16.2 Plateau Problem

One of the prime examples of extending the properties of generalized surface to regular surface is the classical Plateau problem, which is discussed in the appendix of [7].

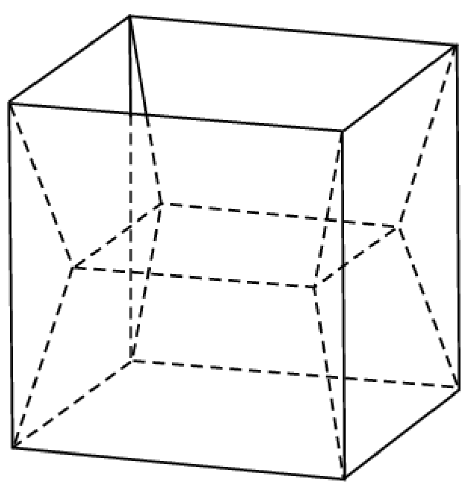


Figure 16.4: A 13-polygon surface obtained for a cubical wire frame

Definition 16.2.1. *An arc $z(t)$ is* **simple** *if $z(t_1) = z(t_2)$ only for $t_1 = t_2$. A* **Jordan curve** *is a simple closed curve.*

Proposition 16.2.2. *(Osserman) Let Γ be an arbitrary Jordan curve in $\mathbf{R}^3$. Then there exists a regular simply connected minimal surface bounded by Γ.*

The existence of a solution to the general case was independently proven by Douglas (1931) [3] and Radò (1933) [8], although their analysis could not exclude the possibility of singularities (i.e. for the case of generalized minimal surface). Osserman (1970) [6] and Gulliver (1973) [4] showed that a minimizing solution cannot have singularities [9].

Table 16.1: Development on the Plateau's problem in 1970-1985

Meeks and Yau	When the curve Γ defined in Prop. 16.2.2 lies on the boundary of a convex body, then the surface obtained is embedded (i.e. without self-intersections).
Gulliver and Spruck	They proved the result from Meeds and Yau under the additional assumption that the total curvature of Γ was at most 4π.

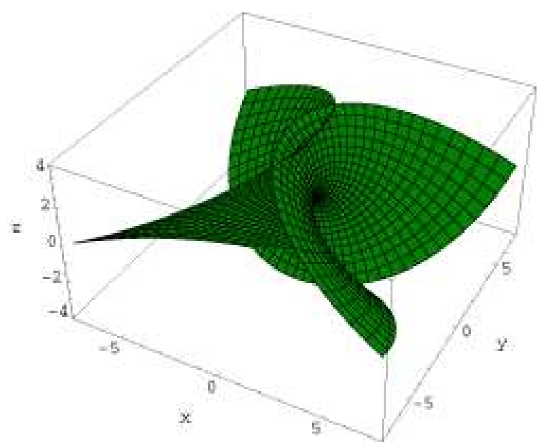


Figure 16.5: An Enneper surface

16.3 Geodesics

A geodesics is analogue to the straight line on a Euclidean plane. In order to define geodesics, we first have to understand the notion of covariant derivative, which is analogue to the usual differentiation of vectors in the plane.

Definition 16.3.1. *A* **vector field** w *in an open set* U *in the regular surface* S *assigns to each* $p \in U$ *a vector* $w(p) \in T_p(S)$*. The vector field is* **differentiable** *at* p *if, for some parametrization* $\boldsymbol{x}(u,v)$ *in* p*, the components* a *and* b *of* $w = a\boldsymbol{x}_u + b\boldsymbol{x}_v$ *in the basis of* $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ *are differentiable functions at* p*. The vector field is differentiable in* U *if it is differentiable for all* $p \in U$*.*

Definition 16.3.2. *Let* w *be a differentiable vector field in an open set* $U \subset S$ *and* $p \in U$*. Let* $y \in T_p(S)$ *and* $\alpha : (-\epsilon, \epsilon) \to U$ *a parametrized curve with* $\alpha(0) = p$ *and* $\alpha'(0) = y$*. Let* $w(t)$ *be the restriction of the vector field* w *to the curve* α*. Then the* **covariant derivative** *at* p *of the vector field* w *relative to the vector* y*, (Dw/dt)(0), is given by the vector obtained by the normal projection of* $(dw/dt)(0)$ *onto the plane* $T_p(S)$*.*

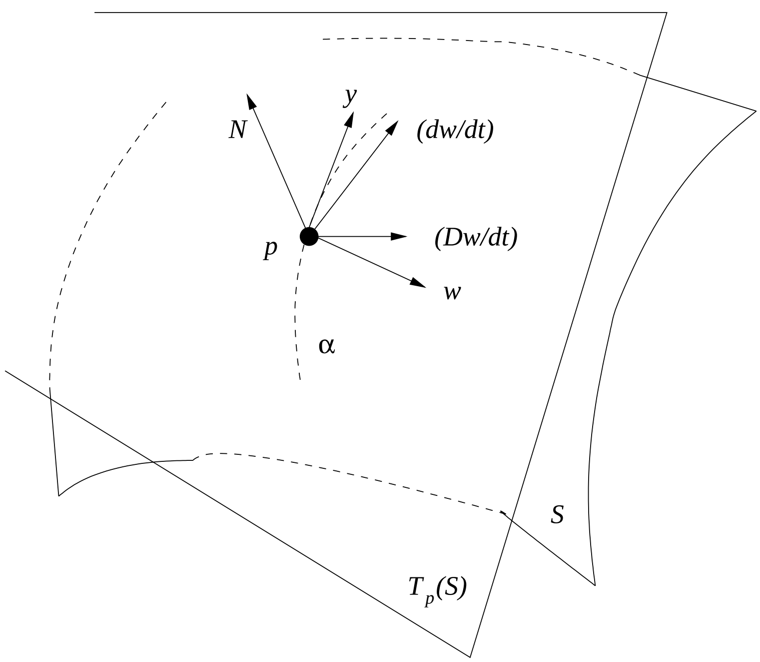


Figure 16.6: The covariant derivative

Definition 16.3.3. *A vector field* w *along a parametrized curve* $\alpha : I \to S$ *is said to be* **parallel** *is* $Dw/dt = 0$ *for every* $t \in I$*.*

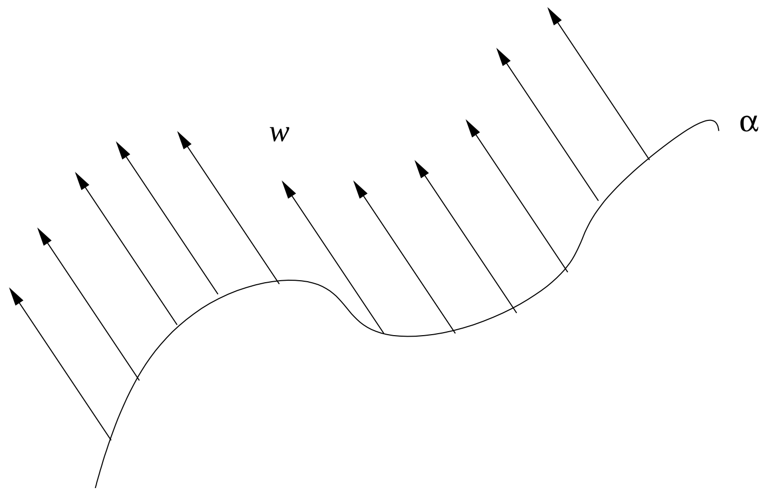


Figure 16.7: A parallel vector field w along the curve α.

Definition 16.3.4. *A non-constant, parametrized curve $\gamma : I \to S$ is said to be* **geodesic** *at $t \in I$ if the field of its tangent vectors $\gamma'(t)$ is parallel along γ at t, that is*

$$\frac{D\gamma'(t)}{dt} = 0 \tag{16.11}$$

From Eq. 16.11, we know that $|\gamma'(t)|$ =constant, thus the arc length s is proportional to the parameter t, and thus we can reparametrize γ with parameter s. Note also that Eq. 16.11 implies that $\alpha''(s) = kn$ is normal to the tangent plane, or parallel to the normal to the surface. Therefore another way to define a geodesic is a regular curve which its principal normal at each point p along the curve is parallel to the normal to S at p.

Below are some examples of geodesics:

Example 8 (Geodesics on the sphere S^2). *The great circles C of a sphere S^2 are obtained by intersecting the sphere with a plane that passes through the center O of the sphere. The principal normal at a point $p \in C$ lies in the direction of the line that connects p to O, the center of C. Since this is also the direction of the normal at p, the great circles are geodesics.*

Example 9 (Geodesics on a right circular cylinder over the circle $x^2 + y^2 = 1$). *It is clear that the circles obtained by the intersection of the cylinder with planes that are normal to the axis of the cylinder are geodesics. The straight lines of the cylinder are also geodesics. To find other geodesics on the cylinder C, consider the parametrization*

$$\boldsymbol{x}(u, v) = (\cos u, \sin u, v) \tag{16.12}$$

of the cylinder in a point $p \in C$, with $\boldsymbol{x}(0, 0) = p$. Then $\boldsymbol{x}$ is an isometry that maps a neighborhood U of $(0, 0)$ of the uv-plane into the cylinder. Since the condition of being a geodesic is local and invariant by isometries, the image of straight lines in U under the map $\boldsymbol{x}$ should be a geodesic on C. Since a straight line on the uv-plane can be expressed as

$$u(s) = as, \quad v(s) = bs, \quad a^2 + b^2 = 1, \tag{16.13}$$

it follows that a geodesic of the cylinder is locally of the form

$$(\cos as, \sin as, bs) \tag{16.14}$$

which is a helix.

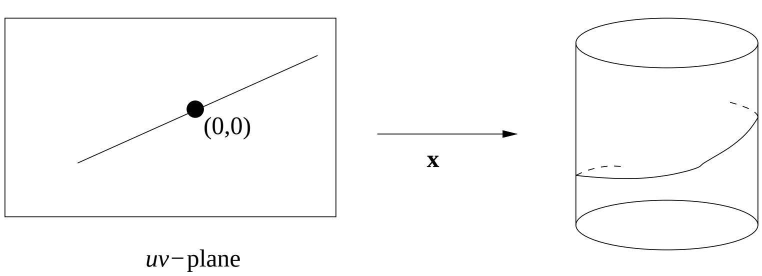


Figure 16.8: Geodesics on a cylinder

16.4 Complete Surfaces

In order to study regular surfaces globally, we need some global hypothesis to ensure that the surface cannot be extended further as a regular surface. Compactness serves this purpose, but it would be useful to have a weaker hypothesis than compctness which could still have the same effect.

Definition 16.4.1. *A regular (connected) surface S is said to be* **extendable** *if there exists a regular (connected) surface $\bar{S}$ such that $S \subset \bar{S}$ as a proper subset. If there exists no such $\bar{S}$, then S is said to be* **nonextendable**.

Definition 16.4.2. *A regular surface S is said to be* **complete** *when for every point $p \in S$, any parametrized geodesic $\gamma : [0, \epsilon) \to S$ of S, starting from $p = \gamma(0)$, may be extended into a parametrized geodesic $\bar{\gamma} : \mathbf{R} \to S$, defined on the entire line $\mathbf{R}$.*

Example 10 (Examples of complete/non-complete surface).

1. *The plane is a complete surface.*

2. *The cone minus the vertex is a noncomplete surface, since by extending a generator (which is a geodesic) sufficiently we reach the vertex, which does not belong to the surface.*

3. *A sphere is a complete surface, since its parametrized geodesics (the great circles) may be defined for every real value.*

4. *The cylinder is a complete surface since its geodesics (circles, lines and helices) can be defined for all real values*

5. *A surface $S - \{p\}$ obtained by removing a point p from a complete surface S is not complete, since there exists a geodesic of $S - \{p\}$ that starts from a point in the neighborhood of p and cannot be extended through p.*

Proposition 16.4.3. *A complete surface S is nonextendable.*

Proof. Let us assume that S is extendable and obtain a contradiction. If S is extendable, then there exists a regular (connected) surface $\bar{S}$ such that $S \subset \bar{S}$. Since S is a regular surface, S is open in $\bar{S}$. The boundary $\mathrm{Bd}(S)$ of S is nonempty, so there exists a point $p \in \mathrm{Bd}(S)$ such that $p \notin S$.

Let $\bar{V} \subset \bar{S}$ be a neighborhood of p in $\bar{S}$ such that every $q \in \bar{V}$ may be joined to p by a unique geodesic of $\bar{S}$. Since $p \in \mathrm{Bd}(S)$, some $q_0 \in \bar{V}$ belongs to S. Let $\bar{\gamma} : [0,1] \to \bar{S}$ be a geodesic of $\bar{S}$, with $\bar{\gamma}(0) = p$ and $\bar{\gamma}(1) = q_0$. It is clear that $\alpha : [0, \epsilon) \to S$, given by $\alpha(t) = \bar{\gamma}(1-t)$, is a geodesic of S, with $\alpha(0) = q_0$, the extension of which to the line $\mathbf{R}$ would pass through p for $t = 1$. Since $p \notin S$, this geodesic cannot be extended, which contradicts the hypothesis of completness and concludes the proof. □

Proposition 16.4.4. *A closed surface $S \subset \mathbf{R}^3$ is complete*

Corollary 16.4.5. *A compact surface is complete.*

Theorem 16.4.6 (Hopf-Rinow). *Let S be a complete surface. Given two points $p, q \in S$, there exists a nimimal geodesic joining p to q.*

16.5 Riemannian Manifolds

Definition 16.5.1. *A* **Riemannian structure** *on M, or a C^q-***Riemannian metric** *is a collection of matrices G_a, where the elements of the matrix G_a are C^q-functions on $V_\alpha, 0 \leq q \leq r-1$, and at each point the matrix G_α is positive definite, while for any α, β such that the map $u(\tilde{u}) = \varphi_\alpha^{-1} \circ \varphi_\beta$ is defined, the relation*

$$G_\beta = U^T G_\alpha U \tag{16.15}$$

must hold, where U is the Jacobian matrix of the transformation $\varphi_\alpha^{-1} \circ \varphi_\beta$.

Chapter 17

Complete Minimal Surfaces

Reading:

- Osserman [7] Pg. 49-52,
- Do Carmo [2] Pg. 325-335.

17.1 Complete Surfaces

In order to study regular surfaces globally, we need some global hypothesis to ensure that the surface cannot be extended further as a regular surface. Compactness serves this purpose, but it would be useful to have a weaker hypothesis than compctness which could still have the same effect.

Definition 17.1.1. *A regular (connected) surface S is said to be* **extendable** *if there exists a regular (connected) surface $\bar{S}$ such that $S \subset \bar{S}$ as a proper subset. If there exists no such $\bar{S}$, then S is said to be* **nonextendable**.

Definition 17.1.2. *A regular surface S is said to be* **complete** *when for every point $p \in S$, any parametrized geodesic $\gamma : [0, \epsilon) \to S$ of S, starting from $p = \gamma(0)$, may be extended into a parametrized geodesic $\bar{\gamma} : \mathbf{R} \to S$, defined on the entire line $\mathbf{R}$.*

Example 11 (Examples of complete/non-complete surfaces). *The following are some examples of complete/non-complete surfaces.*

1. *The plane is a complete surface.*

2. *The cone minus the vertex is a noncomplete surface, since by extending a generator (which is a geodesic) sufficiently we reach the vertex, which does not belong to the surface.*

3. *A sphere is a complete surface, since its parametrized geodesics (the great circles) may be defined for every real value.*

4. *The cylinder is a complete surface since its geodesics (circles, lines and helices) can be defined for all real values*

5. *A surface $S - \{p\}$ obtained by removing a point p from a complete surface S is not complete, since there exists a geodesic of $S - \{p\}$ that starts from a point in the neighborhood of p and cannot be extended through p.*

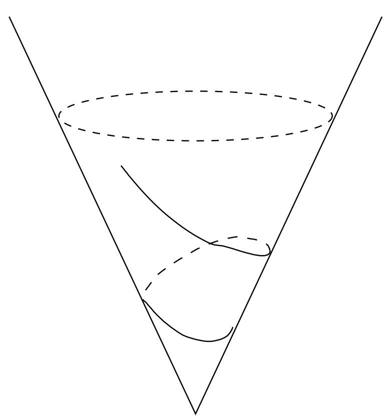


Figure 17.1: A geodesic on a cone will eventually approach the vertex

Proposition 17.1.3. *A complete surface S is nonextendable.*

Proof. Let us assume that S is extendable and obtain a contradiction. If S is extendable, then there exists a regular (connected) surface $\bar{S}$ such that $S \subset \bar{S}$. Since S is a regular surface, S is open in $\bar{S}$. The boundary $\mathrm{Bd}(S)$ of S is nonempty, so there exists a point $p \in \mathrm{Bd}(S)$ such that $p \notin S$.

Let $\bar{V} \subset \bar{S}$ be a neighborhood of p in $\bar{S}$ such that every $q \in \bar{V}$ may be joined to p by a unique geodesic of $\bar{S}$. Since $p \in \mathrm{Bd}(S)$, some $q_0 \in \bar{V}$ belongs to S. Let $\bar{\gamma} : [0,1] \to \bar{S}$ be a geodesic of $\bar{S}$, with $\bar{\gamma}(0) = p$ and $\bar{\gamma}(1) = q_0$. It is clear that $\alpha : [0, \epsilon) \to S$, given by $\alpha(t) = \bar{\gamma}(1-t)$, is a geodesic of S, with $\alpha(0) = q_0$, the extension of which to the line $\mathbf{R}$ would pass through p for $t = 1$. Since $p \notin S$, this geodesic cannot be extended, which contradicts the hypothesis of completness and concludes the proof. $\square$

Proposition 17.1.4. *A closed surface $S \subset \mathbf{R}^3$ is complete*

Corollary 17.1.5. *A compact surface is complete.*

Theorem 17.1.6 (Hopf-Rinow). *Let S be a complete surface. Given two points $p, q \in S$, there exists a nimimal geodesic joining p to q.*

17.2 Relationship Between Conformal and Complex-Analytic Maps

In surfaces, conformal maps are basically the same as complex-analytic maps. For this section, let $U \subset \mathbf{C}$ be a open subset, and $z \in U$.

Definition 17.2.1. *A function $f : U \to \mathbf{C}$ is* **conformal** *if the map df_z preserves angle and sign of angles.*

Proposition 17.2.2. *A function $f : U \to \mathbf{C}$ is conformal at $z \in U$ iff f is a complex-analytic function at z and $f'(z) \neq 0$.*

Proof. Let B be the matrix representation of df_z in the usual basis. Then f is conformal $\Leftrightarrow B = cA$ where $A \in SO(2)$ and $c > 0$. Thus

$$BB^T = c^2 I \quad \Leftrightarrow \quad B^T = (\det B)B^{-1} \tag{17.1}$$

Let $z = x + iy$ and $f(z) = f(x, y) = u(x, y) + iv(x, y)$, then

$$B = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \tag{17.2}$$

where $u_y = \frac{\partial u}{\partial y}$. However, from Eq. 17.1, we have

$$\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} v_y & -u_y \\ -v_x & u_x \end{pmatrix} \tag{17.3}$$

which implies the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x. \tag{17.4}$$

Thus f is complex-analytic. □

17.3 Riemann Surface

Definition 17.3.1. *A* **Riemann Surface** *M is a 1-dim complex analytic manifold, i.e. each $p \in M$ has a neighborhood which is homeomorphic to a neighborhood in* $\mathbf{C}$*, and the transition functions are complex analytic.*

In order to study Riemann surface, one needs to know the basic of harmonic and subharmonic functions.

Table 17.1: The analogues of harmonic and subharmonic functions on $\mathbf{R}$

$\mathbb{R}$	$\mathbb{C}$
Linear	Harmonic
Convex	subharmonic

Definition 17.3.2. *A function $h : \mathbf{R} \to \mathbf{R}$ is* **harmonic** *iff it is in the form $h(x) = ax + b$, where $a, b \in \mathbf{R}$. In other words, $\Delta h = 0$ where $\Delta = \frac{d^2}{dx^2}$.*

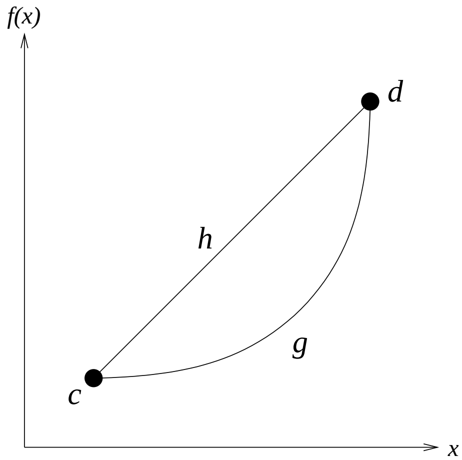


Figure 17.2: A graphical representation of a harmonic function h and a subhamonic function g in $\mathbb{R}$.

Definition 17.3.3. *A function $g : \mathbf{R} \to \mathbf{R}$ is* **convex** *if for every interval $[c, d] \subset \mathbf{R}$, $g(x) < h(x)$ for $x \in (c, d)$ where h is the linear function such that $h(c) = g(c)$ and $h(d) = g(d)$.*

Definition 17.3.4 (Second definition of convex functions). *If $g : \mathbf{R} \to \mathbf{R}$ is* **convex** *and $g \le \tilde{h}$ on (c, d) for $\tilde{h}$ a harmonic function, then either $g < \tilde{h}$ or $g \equiv \tilde{h}$ there.*

Subharmonic functions on $\mathbb{C}$ are just the equivalents of convex functions on $mathbbR$.

Definition 17.3.5. *A function $g : M \to \mathbb{R}$ is* **subharmonic** *on a Riemann surface M if*

1. *g is constant.*
2. *For any domain D and any harmonic functions $h : D \to \mathbb{R}$, if $g \le h$ on D, then $g < h$ on D or $g = h$ on D.*
3. *The difference $g - h$ satisfies the maximum principle on D, i.e. $g - h$ cannot have a maximum on D unless it is constant.*

Definition 17.3.6. *A Riemann surface M is* **hyperbolic** *if it supports a non-constant negative subharmonic function.*

Note: *If M is compact, then all constant functions on M that satisfy the maximum principle are constant. Therefore M is not hyperbolic.*

Definition 17.3.7. *A Riemann surface M is* **parabolic** *if it is not compact nor hyperbolic.*

Theorem 17.3.8 (Koebe-Uniformization Theorem). *If M is a simply connected Riemann surface, then*

1. *if M is compact, M is conformally equivalent to the sphere.*
2. *if M is parabolic, M is conformally equivalent to the complex plane.*
3. *if M is hyperbolic, M is conformally equivalent to the unit disc on the complex plane. But note that the disc has a hyperbolic metric*

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}. \tag{17.5}$$

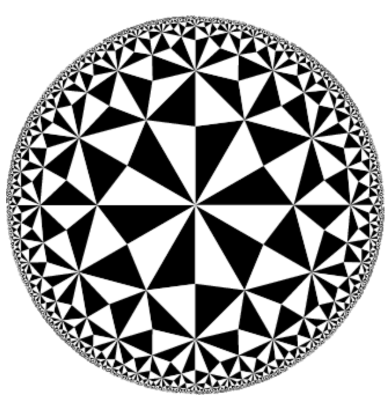


Figure 17.3: The Poincaré Hyperbolic Disk [9]

Table 17.2: Categorization of Riemann surfaces

Type	Conformally equivalent to	Remark
Hyperbolic	sphere	supports a non-constant negative subharmonic function
Compact	$\mathbb{C}$	
Parabolic	$D = \{z \in \mathbb{C} \| \|z\| < 1\}$	Not hyperbolic and not compact

17.4 Covering Surface

Definition 17.4.1. *A* **covering surface** *of a topological 2-manifold M is a topological 2-manifold $\tilde{M}$ and a map*

$$\rho : \tilde{M} \to M \tag{17.6}$$

such that ρ is a local homeomorphic map.

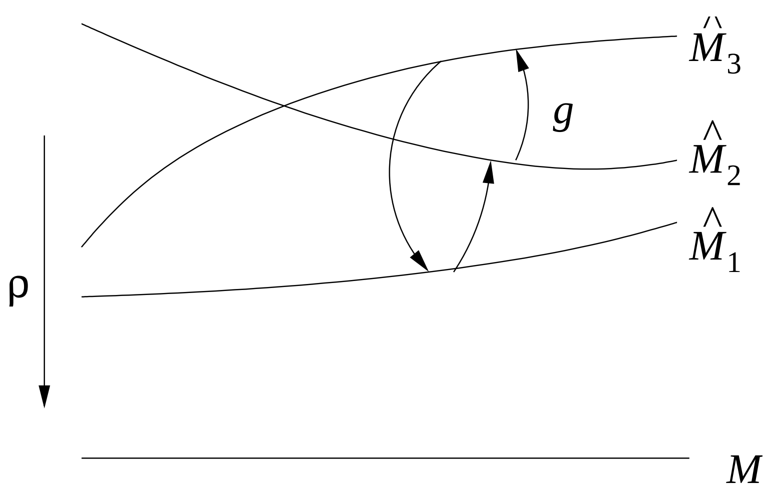


Figure 17.4: Covering surfaces

Definition 17.4.2. *A covering transformation of $\tilde{M}$ is a homeomorphism $g : \tilde{M} \to \tilde{M}$ such that $\rho \circ g = \rho$*

This forms a group G.

Proposition 17.4.3. *Every surface (2-manifold) M has a covering space $(\hat{M}, \rho)$ such that $\tilde{M}$ is simply connected, and*

$$\hat{M}/G \cong M \tag{17.7}$$

Chapter 18

Weierstrass-Enneper Representations

18.1 Weierstrass-Enneper Representations of Minimal Surfaces

Let M be a minimal surface defined by an isothermal parameterization $x(u,v)$. Let $z = u + iv$ be the corresponding complex coordinate, and recall that

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}), \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$$

Since $u = 1/2(z + \overline{z})$ and $v = -i/2(z - \overline{z})$ we may write

$$x(z,\overline{z}) = (x^1(z,\overline{z}), x^2(z,\overline{z}), x^3(z,\overline{z}))$$

Let $\phi = \frac{\partial x}{\partial z}, \phi^i = \frac{\partial x^i}{\partial z}$. Since M is minimal we know that ϕ^is are complex analytic functions. Since x is isothermal we have

$$(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0 \tag{18.1}$$

$$(\phi^1 + i\phi^2)(\phi^1 - i\phi^2) = -(\phi^3)^2 \tag{18.2}$$

Now if we let $f = \phi^1 - i\phi^2$ and $g = \phi^3/(\phi^1 - i\phi^2)$ we have

$$\phi^1 = 1/2f(1-g^2), \phi^2 = i/2f(1+g^2), \phi^3 = fg$$

Note that f is analytic and g is meromorphic. Furthermore fg^2 is analytic since $fg^2 = -(\phi^1 + i\phi^2)$. It is easy to verify that any ϕ satisfying the above equations and the conditions of the preceding sentence determines a minimal surface. (Note that the only condition that needs to be checked is isothermality.) Therefore we obtain:

Weierstrass-Enneper Representation I If f is analytic on a domain D, g is meromorphic on D and fg^2 is analytic on D, then a minimal surface is defined by the parameterization $x(z, \overline{z}) = (x^1(z, \overline{z}), x^2(z, \overline{z}), x^3(z, \overline{z})$, where

$$x^1(z, \overline{z}) = Re \int f(1-g^2)dz \tag{18.3}$$

$$x^2(z, \overline{z}) = Re \int if(1+g^2)dz \tag{18.4}$$

$$x^3(z, \overline{z}) = Re \int fgdz \tag{18.5}$$

Suppose in WERI g is analytic and has an inverse function g^{-1}. Then we consider g as a new complex variable $\tau = g$ with $d\tau = g'dz$ Define $F(\tau) = f/g'$ and obtain $F(\tau)d\tau = fdz$. Therefore, if we replace g with τ and fdz with $F(\tau)d\tau$ we get

Weierstrass-Enneper Representation II For any analytic function $F(\tau)$, a minimal surface is defined by the parameterization $x(z, \overline{z}) = (x^1(z, overlinez), x^2(z, \overline{z}), x^3(z, \overline{z}))$,

where

$$x^1(z,\overline{z}) = Re\int F(\tau)(1-\tau^2)dz \tag{18.6}$$

$$x^2(z,\overline{z}) = Re\int iF(\tau)(1+\tau^2)dz \tag{18.7}$$

$$x^3(z,\overline{z}) = Re\int F(\tau)\tau dz \tag{18.8}$$

This representation tells us that any analytic function $F(\tau)$ defines a minimal surface.

class exercise Find the WERI of the helicoid given in isothermal coordinates (u, v)

$$x(u,v) = (sinhusinv, -sinhucosv, -v)$$

Find the associated WERII. (answer: $i/2\tau^2$) Show that $F(\tau) = 1/2\tau^2$ gives rise to catenoid. Show moreover that $\tilde{\phi} = -i\phi$ for conjugate minimal surfaces x and $\tilde{x}$.

Notational convention We have two Fs here: The F of the first fundamental form and the F in WERII. In order to avoid confusion well denote the latter by T and hope that Oprea will not introduce a parameter using the same symbol. Now given a surface $x(u, v)$ in R^3 with $F = 0$ we make the following observations:

i. x_u, x_v and $N(u, v)$ constitute an orthogonal basis of R^3.

ii. N_u and N_v can be written in this basis coefficients being the coefficients of matrix dNp

iii. x_uu, x_vv and x_uv can be written in this basis. One should just compute the dot products $\langle x_{uu}, x_u\rangle, \langle x_{uu}, x_v\rangle, \langle x_{uu}, N\rangle$ in order to represent x_{uu} in this basis. The same holds for x_{uv} and x_{vv}. Using the above ideas one gets the following equations:

$$x_{uu} = \frac{E_u}{2E}x_u - \frac{E_v}{2G} + eN \tag{18.9}$$
$$x_{uv} = \frac{E_v}{2E}x_u + \frac{G_v}{2G} + fN \tag{18.10}$$
$$x_{vv} = \frac{-G_u}{2E}x_u + \frac{G_v}{2G} + gN \tag{18.11}$$
$$N_u = -\frac{e}{E}x_u - \frac{f}{G}x_v \tag{18.12}$$
$$N_v = -\frac{f}{E}x_u - \frac{g}{G}x_v \tag{18.13}$$

Now we state the Gausss theorem egregium:

Gausss Theorem Egregium The Gauss curvature K depends only on the metric $E, F = 0$and G:

$$K = -\frac{1}{2\sqrt{EG}}(\frac{\partial}{\partial v}(\frac{E_v}{\sqrt{EG}}) + \frac{\partial}{\partial u}(\frac{G_u}{\sqrt{EG}}))$$

This is an important theorem showing that the isometries do not change the Gaussian curvature.

proof If one works out the coefficient of x_v in the representation of $x_{uuv} - x_{uvu}$ one gets:

$$x_{uuv} = []x_u + [\frac{E_uG_u}{4EG} - (\frac{E_v}{2G})_v - \frac{E_vG_v}{4G^2} - \frac{eg}{G}]x_v + []N \tag{18.14}$$
$$x_{uvu} = []x_u + \frac{E_v}{2E}x_{uu} + (\frac{G_u}{2G})_u x_u v + f_u N + fN_u \tag{18.15}$$
$$x_{uvu} = []x_u + [-\frac{E_vE_v}{4EG} + (\frac{G_u}{2G})_u + \frac{G_uG_u}{4G^2} - \frac{f^2}{G}]x_v + []U \tag{18.16}$$

Because the x_v coefficient of $x_{uuv} - x_{uvu}$ is zero we get:

$$0 = \frac{E_uG_u}{4EG} - (\frac{E_v}{2G})_v - \frac{E_vG_v}{4G^2} + \frac{E_vE_v}{4EG} - (\frac{G_u}{2G})_u - \frac{GuGu}{4G^2} - \frac{eg - f^2}{G}$$

dividing by E, we have

$$\frac{eg-f^2}{EG} = \frac{E_uG_u}{4E^2G} - \frac{1}{E}(\frac{E_v}{2G})_v - \frac{E_vG_v}{4EG^2} + \frac{E_vE_v}{4E^2G} - \frac{1}{E}(\frac{G_u}{2G})_u - \frac{G_uG_u}{4EG^2}$$

Thus we have a formula for K which does not make explicit use of N:

$$K = -\frac{1}{2\sqrt{EG}}(\frac{\partial}{\partial v}(\frac{\partial E_v}{\partial\sqrt{EG}}) + \frac{\partial}{\partial u}(\frac{G_u}{\sqrt{EG}}))$$

Now we use Gausss theorem egregium to find an expression for K in terms of T of WERII

$$K = -\frac{1}{2\sqrt{EG}}(\frac{\partial}{\partial v}(\frac{E_v}{\sqrt{EG}}) + \frac{\partial}{\partial u}(\frac{G_u}{\sqrt{EG}})) \tag{18.17}$$

$$= -\frac{1}{2E}(\frac{\partial}{\partial v}(\frac{E_v}{E}) + \frac{\partial}{\partial u}(\frac{E_u}{E})) \tag{18.18}$$

$$= -\frac{1}{2E}\Delta(lnE) \tag{18.19}$$

Theorem The Gauss curvature of the minimal surface determined by the WER II is

$$K = \frac{-4}{|T|^2(1+u2+v2)^4}$$

where $\tau = u + iv$. That of a minimal surface determined by WER I is:

$$K = \frac{4|g'|^2}{|f|^2(1+|g|^2)^4}$$

In order to prove this thm one just sees that $E = 2|\phi|^2$ and makes use of the equation (20). Now we prove a proposition that will show WERs importance later.

Proposition Let M be a minimal surface with isothermal parameterization $x(u,v)$. Then the Gauss map of M is a conformal map.

proof In order to show N to be conformal we only need to show $|dNp(x_u)| =$

$\rho(u,v)|x_u|, |dNp(x_v)| = \rho(u,v)|x_v|$ and $dNp(x_u).dNp(x_v) = \rho^2 x_u.x_v$ Latter is trivial because of the isothermal coordinates. We have the following eqns for $dNp(x_u)$ and $dNp(x_v)$

$$dNp(x_u) = N_u = -\frac{e}{E}x_u - \frac{f}{G}x_v \tag{18.20}$$

$$dNp(x_v) = N_v = -\frac{f}{E}x_u - \frac{g}{G}x_v \tag{18.21}$$

By minimality we have $e + g = 0$. Using above eqns the Gauss map is conformal with scaling factor $\frac{\sqrt{e^2+f^2}}{E} = \sqrt{|K|}$ It turns out that having a conformal Gauss map almost characterizes minimal surfaces:

Proposition Let M be a surface parameterized by $x(u,v)$ whose Gauss map $N : M \longrightarrow S^2$ is conformal. Then either M is (part of) sphere or M is a minimal surface.

proof We assume that the surface is given by an orthogonal parameterization (F = 0) Since $x_u.x_v = 0$ by conformality of N $N_u.N_v = 0$ using the formulas (13) (14) one gets $f(Ge + Eg) = 0$ therefore either $e = 0$ (at every point) or $Ge + eG = 0$(everywhere). The latter is minimal surface equality. If the surface is not minimal then $f = 0$. Now use $f = 0$, confomality and (13), (14) to get

$$\frac{e^2}{E} = N_u.N_u = \rho^2 E, \frac{g^2}{G} = N_v.N_v = \rho^2 G$$

Multiplying across each equation produces

$$\frac{e^2}{E^2} = \frac{g^2}{G^2} \Rightarrow \frac{e}{G} = \pm\frac{g}{G}$$

The last equation with minus sign on LHS is minimal surface equation so we may just consider the case $e/E = g/G = k$. Together with $f = 0$ we have $N_u = -kx_u$ and $N_v = -kx_v$ this shows that x_u and x_v are eigenvectors of the differential of the Gauss map with the same eigenvalue. Therefore any point on M is an umbilical point. The only surface satisfying this property is sphere so were done.

Steographic Projection: $St : S^2 - N \longrightarrow R^2$ is given by $St(x,y,z) = (x/(1-z), y/(1-z), 0)$ We can consider the Gauss map as a mapping from the surface to $C \cup \infty$ by taking its composite with steographic projection.Note that the resulting map is still conformal since both of Gauss map and Steographic are conformal. Now we state a thm which shows that WER can actually be attained naturally:

Theorem Let M be a minimal surface with isothermal parameterization $x(u,v)$ and WER (f,g). Then the Gauss map of M, $G : M \longrightarrow C \cup \infty$ can be identified with the meromorphic function g.

proof Recall that

$$\phi^1 = \frac{1}{2}f(1-g^2), \phi^2 = i2f(1+g^2), \phi^3 = fg$$

We will describe the Gauss map in terms of ϕ^1, ϕ^2 and ϕ^3.

$$\begin{aligned} x_u \times x_v &= ((x_u \times x_v)^1, (x_u \times x_v)^2, (x_u \times x_v)^3) && (18.22) \\ &= (x_u^2 x_v^3 - x_u^3 x_v^2, x_u^3 x_v^1 - x_u^1 x_v^3, x_u^1 x_v^2 - x_u^2 x_v^1) && (18.23) \end{aligned}$$

and consider the first component $x_u^2 x_v^3 - x_u^3 x_v^2$ we have

$$x_u^2 x_v^3 - x_u^3 x_v^2 = 4Im(\phi^2 \overline{\phi}^3)$$

Similarly $(x_u \times x_v)^2 = 4Im(\phi^2 \overline{\phi}^1)$ and $(x_u \times x_v)^3 = 4Im(\phi^1 \overline{\phi^2})$ Hence we obtain

$$x_u \times x_v = 4Im(\phi^2\overline{\phi^3}, \phi^3\overline{\phi^1}, \phi^1\overline{\phi^2}) = 2Im(\phi \times \overline{\phi})$$

Now since $x(u,v)$ is isothermal $|x_u \times x_v| = |x_u||x_v| = E = 2|\phi|^2$. Therefore we have

$$N = \frac{x_u \times x_v}{|x_u \times x_v|} = \frac{\phi \times \overline{\phi}}{|\phi|^2}$$

Now

$$\begin{aligned}
G(u,v) &= St(N(u,v)) && (18.24)\\
&= St(\frac{x_u \times x_v}{|x_u \times x_v|}) && (18.25)\\
&= St(\frac{\phi \times \overline{\phi}}{|\phi|^2}) && (18.26)\\
&= St(2Im(\phi^2\overline{\phi^3}, \phi^3\overline{\phi^1}, \phi^1\overline{\phi^2})|\phi|^2) && (18.27)\\
&= (\frac{2Im(\phi^2\overline{\phi^3})}{|\phi|^2 - 2Im(\phi^1\overline{\phi^2})}, \frac{2Im(\phi^3\overline{\phi^1})}{|\phi|^2 - 2Im(\phi^1\overline{\phi^2})}, 0) && (18.28)
\end{aligned}$$

Identifying (x, y) in R^2 with $x + iy \in C$ allows us to write

$$G(u,v) = \frac{2Im(\phi^2\overline{\phi^3}) + 2iIm(\phi^3\overline{\phi^1})}{|\phi|^2 - 2Im(\phi^1\overline{\phi^2})}$$

Now its simple algebra to show that

$$G(u,v) = \frac{\phi^3}{\phi^1 - i\phi^2}$$

But that equals to g so were done.

Chapter 19

Gauss Maps and Minimal Surfaces

19.1 Two Definitions of Completeness

We've already seen do Carmo's definition of a complete surface — one where every partial geodesic is extendable to a geodesic defined on all of $\mathbb{R}$. Osserman uses a different definition of complete, which we will show to be equivalent (this is also exercise 7 on page 336 of do Carmo).

A *divergent curve* on S is a differentiable map $\alpha\colon [0,\infty) \to S$ such that for every compact subset $K \subset S$ there exists a $t_0 \in (0,\infty)$ with $\alpha(t) \notin K$ for all $t > t_0$ (that is, α leaves every compact subset of S). We define the *length* of a divergent curve as $\lim_{t\to\infty} \int_0^t |\alpha'(t)|\, dt$, which can be unbounded. Osserman's definition of complete is that every divergent curve has unbounded length. We will sketch a proof that this is an equivalent definition.

First, we will assume that every divergent curve in S has unbounded length and show that every geodesic in S can be extended to all of $\mathbb{R}$. Let $\gamma\colon [0,\epsilon) \to S$ be a geodesic that cannot be extended to all of $\mathbb{R}$; without loss of generality assume specifically that it cannot of be extended to $[0,\infty)$. Then the set of numbers x such that γ can be extended to $[0,x)$ is nonempty (because it contains ϵ) and bounded above (because it cannot be extended to $[0,\infty)$), so it has an inf R. We note that since we can extend γ to $[0, R-\delta)$ for all (small) δ, we can in fact extend it to $\gamma' \colon [0, R) \to S$. Because γ' has constant speed, it must tend to a limit point $q \in \mathbb{R}^n$ (by completeness of $\mathbb{R}^n$ using a standard topological definition of completeness involving Cauchy sequences). Let $\alpha \colon [0,\infty) \to S$ be defined by $\alpha(t) = \gamma'(R(1-e^{-t})$. Then α is just a reparametrization of γ', so it has the same length as γ', which is (because γ' is a geodesic) a constant multiple of R and thus bounded. So if we can show that α is a divergent curve, we will have a contradiction. Clearly q is also a limit point of α, since it is a reparametrization of γ'. If $q \in S$, then a regular neighborhood of q is contained in S and we could have extended the geodesic further, so $q \in Bd(S) - S$. So if α is in a compact (and thus closed) subset of S for arbitrarily large values of t, q must be in that set too, which is a contradiction. So in fact every geodesic can be extended to $\mathbb{R}$.

Next we assume that every geodesic can be extended to all of $\mathbb{R}$ and show that every divergent curve has unbounded length. Let α be a divergent curve with bounded length. Then we have for any $k > 0$ that $\lim_{n\to\infty} \int_n^{n+k} |\alpha'(t)|\, dt = 0$ — that is, points on α get arbitrarily close to each other, so because $\mathbb{R}^3$ is complete (in the Cauchy sense) α has a limit point q in $\mathbb{R}^3$. q cannot lie on S, because otherwise (the image under a chart of) a closed ball around q would be a compact set that α doesn't leave, and we know that α is divergent. So $q \notin S$. I don't quite see how to finish the proof here, but if it's true that if S is *any* surface (not just a complete one) then $S - \{p\}$ is not complete (in the geodesic sense), then this implies that our surface is not complete. I'm not sure if that's true though.

19.2 Image of S under the Gauss map

One very important consequence of the WERI representation is that the Gauss map $N \colon S \to S^2$ is just the function g, with S^2 the Riemann sphere;

that is, if $p\colon S^2 \to \mathbb{C} \cup \{\infty\}$ is stereographic projection, then $g = p \circ N$. Nizam proved this in his notes; the proof is mostly a matter of working through the algebra.

Lemma 19.2.1 (Osserman Lemma 8.4). *A minimal surface $S \subset \mathbb{R}^3$ that is defined on the whole plane is either a plane or has an image under the Gauss map that omits at most two points.*

Proof. We can find a WERI representation unless $\phi_1 = i\phi_2$ and $\phi_3 = 0$, but this means that x_3 is constant so that S is a plane. Otherwise, g is meromorphic in the entire plane, so by Picard's Theorem it takes on all values with at most two exceptions or is constant; so the Gauss map either takes on all values except for maybe two or is constant, and the latter case is a plane. □

Theorem 19.2.2 (Osserman Theorem 8.1). *Let S be a complete regular minimal surface in $\mathbb{R}^3$. Then S is a plane or the image of S under N is dense in the sphere.*

Proof. If the image is not everywhere dense then it omits a neighborhood of some point, which without loss of generality we can assume to be $N = (0, 0, 1)$. If we can prove that x is defined on the entire plane, then by the previous lemma we have our result. I do not entirely understand the proof, but it involves finding a divergent path of bounded length. □

We note that this implies Bernstein's Theorem, since a nonparametric minimal surface misses the entire bottom half of the sphere. So how many points can we miss?

Theorem 19.2.3 (Osserman Theorem 8.3). *Let E be an arbitrary set of $k \leq 4$ points on the unit sphere. Then there exists a complete regular minimal surface in $\mathbb{R}^3$ whose image under the Gauss map omits precisely the set E.*

Proof. We can assume (by rotation) that E contains the north pole $N = (0,0,1)$. If in fact $E = \{N\}$, then we can take $f(\zeta) = 1$ and $g(\zeta) = \zeta$, which clearly obey the properties that f and g must (as they are both analytic); since g takes on all values in $\mathbb{C}$, N must take on all values of $S^2 - \{N\}$ by inverse stereographic projection. (This is called Enneper's surface.) Otherwise let the points of $E - \{N\}$ correspond to points $w_m \in \mathbb{C}$ under stereographic projection. Let

$$f(\zeta) = \frac{1}{\prod(\zeta - w_m)}, g(\zeta) = \zeta$$

and use WERI with the domain $\mathbb{C} - \{w_1, \ldots, w_{k-1}\}$. Clearly g takes on all values except for the points w_m, so the image of the Gauss map omits only the values in E. f and g are both analytic (since the points where it looks like f would have poles are not in the domain). It remains to show that the surface is complete. We can show that in general the path length of a curve C equals $\int_C \frac{1}{2}|f|(1+|g|^2)|d\zeta|$. The only way a path can be divergent here is if it tends towards ∞ or one of the points w_m; in the former case the degree of $|f|(1+|g|^2)$ is at least -1 (because there are at most three terms on the bottom of f), so it becomes unbounded; in the latter case g goes to a constant and $|f|$ becomes unbounded, so every divergent curve has unbounded length and the surface is complete. $\square$

It has been proven by Xavier (see p 149 of Osserman) that no more than six directions can be omitted, and as of the publication of Osserman it is not known whether five or six directions can be omitted.

19.3 Gauss curvature of minimal surfaces

Nizam showed that the Gauss curvature of a minimal surface depends only on its first fundamental form as $K = -\frac{1}{2g_{11}}\Delta(\ln g_{11})$; doing the appropriate calculations (starting with $g_{11} = 2|\phi|^2$ shows that we can write it in terms of

f and g as

$$K = -\left(\frac{4|g'|}{|f|(1+|g|^2)^2}\right)^2$$

This implies that the Gauss curvature of a minimal surface is non-positive everywhere (which is not surprising, since $K = k_1 k_2 = -k_1^2$). It also implies that it can have only isolated zeros unless S is a plane. This is because K is zero precisely when the analytic (according to Osserman, though I don't see why) function g' has zeros, which is either isolated or everywhere. But if g' is identically zero, then g is constant, so N is constant, so S is a plane.

Consider, for an arbitrary minimal surface in $\mathbb{R}^3$, the following sequence of mappings:

$$D \xrightarrow{x(\zeta)} S \xrightarrow{N} S^2 \xrightarrow{p} \mathbb{C}$$

where p is stereographic projection onto the w-plane. The composition of all of these maps is g, as we have seen. Given a differentiable curve $\zeta(t)$ in D, if $s(t)$ is the arc length of its image on S, then (as mentioned above)

$$\frac{ds}{dt} = \frac{1}{2}|f|(1+|g|^2)|\frac{d\zeta}{dt}|$$

The arc length of the image in the w-plane is simply

$$abs\frac{dw}{dt} = |g'(\zeta)||\frac{d\zeta}{dt}|$$

because the composed map is g. If $\sigma(t)$ is arc length on the sphere, then by computation on the definition of stereographic projection we can show that

$$\frac{d\sigma}{dt} = \frac{2}{1+|w|^2}|\frac{dw}{dt}|$$

(note that $|w|$ here is the same as $|g|$. So dividing through we find that

$$\frac{\frac{d\sigma}{dt}}{\frac{ds}{dt}} = \frac{4|g'|}{|f|(1+|g|^2)^2} = \sqrt{|K|}$$

So there is a natural definition of Gauss curvature in terms of the Gauss map.

We define the total curvature of a surface to be the integral $\iint K$. We can show that this is in fact equal to the negative of spherical area of the image under the Gauss map, counting multiple coverings multiply.

19.4 Complete manifolds that are isometric to compact manifolds minus points

Theorem 19.4.1 (Osserman 9.1). *Let M be a complete Riemannian 2-manifold with $K \leq 0$ everywhere and $\iint |K| < \infty$. Then there exists a compact 2-manifold $\hat{M}$ and a finite set $P \subset \hat{M}$ such that M is isometric to $\hat{M} - P$.*

(Proof not given.)

Lemma 19.4.2 (Osserman 9.5). *Let x define a complete regular minimal surface S in $\mathbb{R}^3$. If the total curvature of $S$is finite, then the conclusion of the previous theorem holds and the function $g = p \circ N$ extends to a meromorphic function on $\hat{M}$.*

Proof. We already know that $K \leq 0$. This implies that $\iint |K| = |\iint K|$, the absolute value of the total curvature, which is finite. So the previous theorem holds. The only way that g could fail to extend is if it has an essential singularity at a point of P, but that would cause it to assume (almost) every value infinitely often, which would imply that the spherical area of the image of the Gauss map is infinite, which contradicts our assumption of finite total curvature. $\square$

Theorem 19.4.3 (Osserman 9.2). *Let S be a complete minimal surface in $\mathbb{R}^3$. Then the total curvature of S is $-4\pi m$ for a nonnegative integer m, or $-\infty$.*

Proof. Since $K \leq 0$, either $\iint K$ diverges to $-\infty$, or it (the total curvature) is finite. Because K is preserved by isometries, we apply the previous lemma and see that the total curvature is the negative of the spherical area of the image under g of $\hat{M} - P$. Because g is meromorphic, it is either constant or takes on each value a fixed number of times m. So either the image is a single point (so the total curvature is $-4\pi 0$) or an m-fold cover of the sphere (so the total curvature is $-4\pi m$). $\square$

Bibliography

[1] L. Ahlfors. *Complex Analysis*. McGraw Hill, 1979.

[2] M. Do Carmo. *Differential Geometry of Curves and Surfaces*. Prentice Hall, 1976.

[3] J. Douglas. Solution of the problem of plateau. *Trans. Amer. Math. Soc.*, 33:263–321, 1931.

[4] R. D. Gulliver. Regularity of minimizing surfaces of prescibed mean curvature. *Ann. of Math.*, 97:275–305, 1973.

[5] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*. American Mathematical Society, 2001.

[6] R. Osserman. A proof of the regularity everywhere of teh classical solution to plateau's problem. *Ann. of Math.*, 91:550–569, 1970.

[7] R. Osserman. *A Survey of Minimal Surfaces*. Dover Phoenix Editions, 1986.

[8] T. Radó. *On the problem of Plateau*. Eugenbnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1933.

[9] E. W. Weisstein. Hyperbolic metric.