4/26 | IOU from last lecture:

$p(x, a)$ = partial type over a with boundedly many solutions.

$B = \{b : p(b, a)\}$ is bounded, we look for a code.

$\Phi = \{\varphi(x, y)$ [x, y in the same sort] : $T \vdash \forall x \neg\varphi(x, x)\}$

It follows: $\forall \varphi \in \Phi \ \exists n_\varphi$ maximal s.t. $\bigwedge_{i<n_\varphi} p(x_i, a) \wedge \bigwedge_{i<j<n_\varphi} \varphi(x_i, x_j)$ consistent.

Continued on next page...

Define $E(z,z') := (z=z') \vee (z,z' \models tp(a) \wedge$

$$\wedge \bigwedge_{\varphi \in \Phi} [\exists x_{<n_\varphi} \bigwedge_{i<n_\varphi} p(x_i, z) \wedge p(x_i, z') \wedge$$

$$\wedge \bigwedge_{i<j<n_\varphi} \varphi(x_i, x_j)\,)\,)$$

Claim: $E(a, a') \iff a' \equiv a \wedge B = \{b \cdot p(b, a')\}$.

Proof: $\Leftarrow$ ✓.

$\Rightarrow$ Clearly $E(a,a') \Rightarrow a \equiv a'$.

By symmetry between a, a' it suffices to show $B \subseteq \cdots$

So let $b \in B$. Assume that $\not\models p(b, a')$.

Let $q(x,z) := tp(b, a')$.

Then $q(x, a') \wedge p(y, a') \wedge x = y$ is inconsistent.

$\Rightarrow \exists z [q(x,z) \wedge p(y,z)] \wedge x = y$ is inconsistent

Therefore $\exists z [\, q(x,z) \wedge p(y,z)] \vdash \varphi(x,y) \in \Phi$. (*)

Since $E(a, a')$, we have $\exists b_0 \cdots b_{n_\varphi - 1}$ s.t.

$\forall i < n_\varphi \quad p(b_i, a) \wedge p(b_i, a')$ (**)

$\forall i < j < n_\varphi \quad \varphi(b_i, b_j)$.

Then (*) + (**) $\Rightarrow \forall i < n_\varphi \quad \varphi(b, b_i)$.

Since $p(b,a)$, contradicts maximality of n_φ. $\square$.

Another semi-IOU:

Let $\mathcal{A}$ be a cofinal class of sets, ie:

(1) it is invariant: if $A \in \mathcal{A}$ & $B \equiv A$ then $B \in \mathcal{A}$.

(2) $\forall A \exists B$ s.t. $A \subseteq B$ and $B \in \mathcal{A}$.

Then the "Freer Theorem" holds even if we only assume
symmetry, trans, ... $\Rightarrow$ simplicity.
the independence theorem for types over distinguished sets,
provided the class of distinguished sets is cofinal.

Only need to reprove $a \mathop{\downarrow'}_{c} b \Rightarrow a \mathop{\downarrow}_{c} b$.

<u>Proof</u> Given $a \mathop{\downarrow'}_{c} b$ and a c-indiscernible sequence $(b_i : i<\omega)$ in $tp(b/c)$.

Let $p(x,y) = tp(a,b/c)$.

Need to show $\bigwedge p(x, b_i)$ is consistent.

We're going to reduce to the case where $c \in \mathcal{A}$ and $(b_i : i<\omega)$ is a Morley sequence$/c$.

Set $\kappa = |T|^+$ (we need $cf(\kappa)$ big enough).

Extend the sequence to $(b_i : i \leq \kappa)$.

Find an increasing sequence $(c_i : i < \kappa)$ st. $c_i \in \mathcal{A}$ and:

- $c \subseteq c_0$
- $b_i \in c_{i+1}$
- $(b_{i+j} : j \leq \kappa)$ is c_i-indiscernible.

Find it by induction:

$$\text{Define } d = \begin{cases} d_i = c & \text{if } i = 0. \\ d_i = c_j b_j & \text{if } i = j+1. \\ d_i = \bigcup_{j<i} c_j & \text{if } i \text{ limit} \end{cases}$$

Find $c_i' \in \mathcal{A}$ st. $d \subseteq c_i'$ (by cofinality).

We know by ind. hyp. $(b_{i+j} : j \leq \kappa)$ is d-indiscernible.
[case by case; easy]

By extn/ext find $(b_{i+j}' : j \leq \kappa)$ which is c'-indiscernible and similar to over d.

Find c_i st. $(c_i, b_{i+j} : j \leq \kappa) \equiv c', (b_{i+j}' : j \leq \kappa)$

By invariance, $c_i \in \mathcal{A}$. So we have our $(c_i : i < \kappa)$.

So now $\exists A \subseteq \bigcup_{i<\kappa} c_i$ st. $|A| \leq |T|$, $b_\kappa \mathop{\downarrow'}_{A} \bigcup_{i<\kappa} c_i$

But by cofinality, $\exists i$ st $A \subseteq c_i \Rightarrow b_\kappa \mathop{\downarrow'}_{c_i} b_{<\kappa}$.

We conclude $(b_{i+j} : j < \kappa)$ is a Morley sequence $/ c_i$ in $tp(b/c_i)$

We may assume (up to a c-automorphism) that $b = b_\kappa$ and we may assume (up to a b,c-automorphism) that $a \underset{bc}{\mathop{\downarrow}'} c_i$ [extension for $\downarrow'$].

$\Rightarrow a \underset{c}{\mathop{\downarrow}'} b, c_i \Rightarrow a \underset{c_i}{\mathop{\downarrow}'} b$.

This concludes the reduction. Finish as before $\square$.

<u>Theorem</u> Let T be a first order, stable theory in $\mathcal{L}$.

Let $\mathcal{L}_\sigma = \mathcal{L} \cup \{\sigma\}$ where σ = unary function symbol.

Let $T_\sigma = T \cup$ "σ is an automorphism"

Assume that T_σ has a model companion T_A.

Then ① T_A is simple

② If $M \models T_A$ and $a, b, c \in M$. Then $a \underset{c}{\downarrow} b$ in the sense of T_A iff ~~$acl_\sigma(a,c)$~~ $\sigma^{\mathbb{Z}}(a) \underset{\sigma^{\mathbb{Z}}(c)}{\downarrow} \sigma^{\mathbb{Z}}(b)$ in the sense of T.

~~Moreover, T_A is stable~~ iff ~~if~~ A, B are acl^{eq} closed in the

Moreover: T_A is stable iff: [if A, B are acl^{eq}-closed in the sense of T then so is $dcl^{eq}_T(A \cup B)$]
a statement about T.

Proof Define $a \mathop{\downarrow}'_C b \Leftrightarrow \sigma^{\mathbb{Z}}(a) \mathop{\downarrow}^T_{\sigma^{\mathbb{Z}}(C)} \sigma^{\mathbb{Z}}(b)$

verify axioms:
- invariance ✓
- symmetry ✓
- transitivity ✓
- finite character ✓
- local character ✓
- independence thm // ($|T|^+$-saturated) models closed under σ.

this follows from last lecture (Exercise)
- extension:

Fact if A is closed under σ then so is $acl^{eq}_T(A)$. (Trivial).

Write $acl_\sigma(a) := acl_T(\sigma^{\mathbb{Z}}(a))$ = minimal acl-closed, σ-closed set containing a.
(everything in eq)

We know that the isomorphism type of $(acl_\sigma(a), a, \sigma)$ (distinguished el) determines $tp^{T_A}(a)$.

Assume $M \models T_A$ & $a, b, c \in M$, $C = acl_\sigma(c)$, $A = acl_\sigma(ac)$, $B = acl_\sigma(cb)$.

By the proof of the ~~KAPA~~ RAPA, ~~[illegible]~~ $\exists (N, \sigma) \models T_\sigma$ st we can

Recall the proof of the PAPA:

let $N \models T$ sufficiently saturated. Embed A, B, C in N

s.t. $A \underset{C}{\downarrow} B$. let σ_1 be the image of σ on A
σ_2 '' '' on B.

Then $\sigma_1|_C = \sigma_2|_C$ = the image of σ on C ...

blah blah blah ... $\sigma_1 \cup \sigma_2$ extends to an aut of

N: $(N, \sigma) \models T_\sigma$.

By defn of model companion, $\exists (N', \sigma) \models T_A$ s.t.

$(N, \sigma) \subseteq (N', \sigma)$.

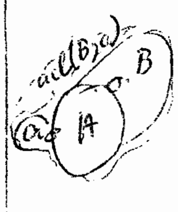
look at defn of dividing.

So in N' we have $\sigma^{\mathbb{Z}} a \underset{C}{\overset{T}{\downarrow}} \sigma^{\mathbb{Z}} b$ $\Leftrightarrow \sigma^{\mathbb{Z}} a \underset{\sigma^{\mathbb{Z}}(c)}{\downarrow} \sigma^{\mathbb{Z}} b$ ie $a \underset{c}{\downarrow} b$.

$C = \mathrm{acl}(\sigma^{\mathbb{Z}}(c))$.

and $tp^{T_A}(a,c)$ $tp^{T_A}(b,c)$ are what we wanted. □

Note:
- If T_0 does not have a model companion, then T_A still exists as a cat and the theorem holds
- If $A = \mathrm{acl}_\sigma(A) \Rightarrow$ types over A are Lascar strong.



A type-definable group (in T) is given by a partial type $G(x)$ and another partial type $m(x,y,z)$ s.t. the set of realisations of m is the graph of a group operation on the realisations of G, denoted $(G, \circ)$

Let $(G, \circ)$ be definable without parameters in a thick simple cat T.

For a partial type $p(x)$ over A s.t. $p(x) \vdash x \in G$, we define $D_G(p, \Xi) \subseteq \bigcup_{\alpha \in Ord} \Xi^\alpha$ as follows:

- $\emptyset \in D_G(p, \Xi) \Leftrightarrow p$ is consistent.
- if $\xi \in \Xi^\alpha$ & α limit, then $\xi \in D_G(p, \Xi) \Leftrightarrow \forall \beta < \alpha \; \xi|_\beta \in D_G(p, \Xi)$
- let $\xi \in \Xi^{\alpha+1}$, $\xi = \theta \wedge (\varphi, \psi)$ then

  $\xi \in D_G(p, \Xi) \Leftrightarrow \exists g \in G$ & parameter c st.

  $\varphi(x, c)$ divides $/A$ wrt ψ and

  $\theta \in D_G(p(x) \wedge \varphi(g \cdot x, c), \Xi)$

  makes it easier to be in D_G than D.

Since T is simple: $D_G(p, \Xi) \subseteq \Xi^{<|T|^+}$

Write $\varphi'(x,y,z) := \varphi(z \cdot x, y) \wedge G(z) \wedge G(x)$, $\psi'(\bar{y}_{<k}, \bar{z}_{<k}) := \psi(y_{<k}) \wedge z_0 = z_1 = \cdots$

note φ' is type-def not a formula, but it doesn't matter...

Then $\psi'(\bar{y}, \bar{z}) \wedge \bigwedge_{i<k} \varphi'(x, y_i, z_i)$ is contradictory:

otherwise we have $z_0 = z_1 = \ldots =: z$ and $\psi(\bar{y}) \wedge \bigwedge_{i<k} \varphi(z \cdot x, y)$ which is impossible.

$\Rightarrow \exists$ formulas φ'', ψ'' st. $\varphi' \vdash \varphi''$, $\psi' \vdash \psi''$ st.

ψ'' is a k-inconsistency witness for φ''.

So for each pair $(\varphi, \psi) \in \Xi$ choose such $(\varphi''_{\varphi,\psi}, \psi''_{\varphi,\psi}) \in \Xi$.

Now if $((\varphi_i, \psi_i) : i < \alpha) \in D_G(p, \Xi)$ then

$((\varphi''_{(\varphi_i,\psi_i)}, \psi''_{(\varphi_i,\psi_i)}) : i < \alpha) \in D(p, \Xi) \subseteq \Xi^{<|T|^+}$

$\Rightarrow \alpha < |T|^+$.

For $\xi \in \Xi^\alpha$ $(\xi = ((\varphi_i, \psi_i) : i < \alpha))$ and a parameter set A,

we say that $h \in G$ satisfies $\mathrm{div}^G_{A,\xi}$ if

$\exists$ parameters $(c_i : i < \alpha)$ and $(g_i : i < \alpha) \subseteq G$ st.

for all $i < \alpha$ $\varphi_i(x, c_i)$ divides $/ A c_{<i} g_{<i}$ w.r.t. ψ_i and

$h \models \bigwedge_{i<\alpha} \varphi_i(g_i \cdot x, c_i)$.

Using thickness, $\mathrm{div}^{G}_{\xi, A}(x)$ is type-definable.

If $p(x) \vdash x \in G$ is a partial type $/A$ then

$\xi \in D_G(p, \Xi)$ iff $p(x) \wedge \mathrm{div}^{G}_{\xi, A}(x)$ is consistent.

Proof: same.

Proposition: I. The $D_G(-, \Xi)$ is left translation-invariant:

$$D_G(p(x), \Xi) = D_G(p(g \cdot x), \Xi) \quad \forall g \in G.$$

II. TFAE: $(g \in G)$

(i) $g \underset{A}{\perp} B$

(ii) $D_G(g/A, \Xi) = D_G(g/AB, \Xi)$

(iii) $D_G(g/AB, \Xi) \cap m D_G(g/A, \Xi) \neq \emptyset$.

m a maximal elt of $D_G(g/A, \Xi)$.

Proof^{I} $\alpha = 0$, α limit ✓.

$\xi = \theta \wedge (\varphi, \psi) \in D_G(p(x), \Xi)$, p is over A

So there are h, c s.t. $\varphi(x, c)$ divides $/A$ wrt ψ

and $\theta \in D_G(p(x) \wedge \varphi(h \cdot x, c), \Xi)$.

$\Rightarrow$ $\theta \in D_G(p(g \cdot x) \wedge \varphi(gh \cdot x, c), \Xi)$

Since g was fixed from beginning, we may assume $g \in A$, so

$p(g\cdot x)$ is also over A and ... $\Rightarrow \xi \in D_G(p(g\cdot x, \Xi)$

use version of lemma that says doesn't matter what A is.

This proves $\subseteq$, so $D_G(p(g\cdot x), \Xi) \subseteq D_G(p(g^{-1}\cdot g\cdot x, \Xi)$ ∎

II. (i) $\Rightarrow$ (ii) Same as for $D(\_, \Xi)$.

(ii) $\Rightarrow$ (iii) ✓

(iii) $\Rightarrow$ (i) If $g \not\downarrow_A B$ then there is $c \in AB$, $\varphi(x,y)$, $\psi(y_{<k})$

s.t. $\varphi(x,c)$ divides$/A$ wrt ψ and $g \models \varphi(x,c)$.

$\Rightarrow$ $g \models \varphi(1\cdot x, c)$.

$\Rightarrow$ $\forall \xi \in D_G(g/AB, \Xi)$: $\xi \wedge (\varphi, \psi) \in D_G(g/A, \Xi)$

contradicting (iii) ∎

<u>Defn</u> $g \in G$ is left-generic over A if whenever $h \in G$ and $g \downarrow_A h$ then $g\cdot h \downarrow_A h$.

$g \in G$ is right-generic ... $h\cdot g \downarrow_A h$.

$g \in G$ is left + right generic.

Prop TFAE for $g \in G$:

(i) g is ~~left~~ right-generic $/A$

(ii) g is generic $/A$ and $D_G(g/A, \exists) = D_G(\underset{\text{w/o params}}{G}, \exists)$.

(iii) $m D_G(G, \exists) \cap D_G(g/A, \exists) \neq \emptyset$.

or g is generic $/A$ $\iff$ generic $/\emptyset$ and $g \downarrow A$.

Proof (of Prop) (i) $\Rightarrow$ (ii)

Assume g is ~~left~~ right-generic. Let $\xi \in D_G(G)$.

Then $G(x) \wedge \mathrm{div}^{G}_{A,\xi}(x)$ is consistent, so there is a realisation $h \in G$ st. $\xi \in D_G(h/A, \exists)$.

WMA $g \underset{G}{\downarrow} h \Rightarrow \overset{hg}{\cancel{gh}} \underset{A}{\downarrow} h$

$\xi \in D_G(h/A, \exists) = D_G(h/A \cup \{\overset{hg}{\cancel{gh}}\}, \exists) = D_G(\overset{(hg)^{-1}\cdot h}{\cancel{h \cdot (gh)^{-1}}}/A \cup \{gh\}, \exists)$.

[If $p = tp(h/A \cup \{\overset{hg}{\cancel{gh}}\})$ then $tp(\overset{(hg)^{-1}\cdot h}{\cancel{h\cdot(gh)^{-1}}}/-) = p(\underset{hg\cdot x}{\cancel{x\cdot(gh)}})$ ✓]

.... $\Rightarrow$ $\xi \in D_G(g^{-1}/A \cup \{\overset{hg}{\cancel{gh}}\}, \exists) \subseteq D_G(g/A, \exists)$.

$\Rightarrow$ $D_G(g^{-1}/A) = D_G(G)$.

Assuming we already proved (iii) $\Rightarrow$ (i)

g^{-1} is ~~left~~ right generic$/A \Rightarrow g$ is ~~right~~ left generic$/A$.

$\Rightarrow D_G((g^{-1})^{-1}, \Xi) = D_G(G)$.

(ii) $\Rightarrow$ (iii) ✓.

(iii) $\Rightarrow$ (i) Assume $\xi \in mD_G(G, \Xi) \cap D_G(g/A, \Xi)$.

Let $h \in G$, $h \underset{A}{\downarrow} g$.

Then $\xi \in D_G(g/A, \Xi) = D_G(g/Ah, \Xi) = D_G(hg/Ah, \Xi)$

$\leq D_G(hg/A, \Xi) \subseteq D_G(G, \Xi)$.

By maximality of ξ in $D_G(G, \Xi)$, it is also maximal in $\Rightarrow hg \underset{A}{\downarrow} h$. □

5/3. Remember: $g \in G$ generic over $A \Leftrightarrow D_G(g/A, \Xi) = D_G(G, \Xi)$

$\Leftrightarrow D_G(g/A\xi, \Xi) \cap mD_G(G, \Xi) \neq \phi$

$\Leftrightarrow g$ is generic$/\phi$ and $g \downarrow A$.

<u>Cor</u> Let $gen(G) = \{$generics (over ϕ)$\}$.

Then $gen(G)$ is type definable and nonempty:

If $\xi \in mD_G(G, \Xi)$ then $gen(G) = \{g \models div^G_{\xi, \phi}\ \xi\}$.

Assume now that G is type definable in a f.o. supersimple theory.

$(SU(g/A) \geq \alpha+1 \Leftrightarrow \exists b$ s.t. $SU(g/Ab) \geq \alpha$ and $g \not\downarrow_A b)$.

<u>Prop</u> For $g \in G$, $g \in gen(G) \Leftrightarrow SU(g)$ is maximal in $\{SU(h) : h \in G\}$.

<u>Proof</u> Let $g \in G$ & generic, $h \in G$ (any elt),

we want to show $SU(g) \geq SU(h)$.

Fact: $SU(g/A) = SU(g/Ab) \Leftrightarrow g \downarrow_A b$ (we have proved)

Fact: If a, b are interdefinable (even interalgebraic) $/A$ then $SU(a/A) = SU(a,b/A) = SU(b/A)$

Exercise

(This follows from Lascar inequalities $+ SU(a/bA) = SU(b/aA) = 0$)

We may assume $g \downarrow h$ (only care about type of h).

Since g is generic: $gh \downarrow h$.

So $SU(g) = SU(g/h) = SU(g \cdot h/h) = SU(gh) \geq$

$\geq SU(gh/g) = SU(h/g) = SU(h)$.

$\Rightarrow SU(g)$ is maximal.

Conversely assume $SU(g)$ is maximal. Assume $g \downarrow h$.

Then: $SU(g) = SU(g/h) = SU(gh/h) \leq SU(gh) \leq SU(g)$

~~But~~ by maximality

$\Rightarrow$ equality so $gh \downarrow h$.

Cor if T is supersimple and its models admit two distinct definable group structures, then both groups have the same generics.

Not true if T is not supersimple

Examples (1) ~~Abelian~~ Additive group of a vector space: g is generic$/A$

iff $\underbrace{g \downarrow A \text{ and } g \neq 0}_{g \notin \langle A \rangle}$.

(2) Additive group of an ACF.

g is generic $\Leftrightarrow$ g is transcendental $/A$.

Also for multiplicative group, also have

More generally, G is an algebraic group then G is definable in ACF and $g \in G$ generic$/_{\text{params}}$ $\Leftrightarrow$ generic in the sense of algebraic geom. $/$same params.

③. Theory $T = Th$(probability algebras).

$\mathcal{U}$ = Algebra of borel sets of $[0,1]^k$ modulo null measure sets.

$\mathcal{L}$ is $\{\wedge, \vee, {}^c, 1, 0, \mu(\cdot) \geq r, \mu(\cdot) \leq r\}$ measure fn

Δ = {all positive qf formulas}.

Define $a \oplus b := a \triangle b$ (addition of the Boolean ring)

$G = (\mathcal{U}, \oplus)$.

Then this is stable and $a \in G$ is generic $\Leftrightarrow \mu(a) = \frac{1}{2}$.
generic/A $\Leftrightarrow \mathbb{P}(a|_{\langle A \rangle}) \equiv \frac{1}{2}$.

G is still type-definable $/\emptyset$ but let $H < G$ type-def $/A$.

Say $[G:H] < \infty$ if the index of H in G is bounded.

TFAE: ① $[G:H] < \infty$

② $\exists g \in H$ which is generic $/A$ for G.

~~③ $\forall B \supseteq A$, $\forall g \in H$: g is generic $/B$ for G~~

③ $D_G(G, \equiv) = D_G(H, \equiv)$

④ $m D_G(G, \equiv) \cap D_G(H, \equiv) \neq \emptyset$.

<u>Proof</u> ① $\Rightarrow$ ②. Let $\overset{\in H}{g}$ be generic $/A$, let $p = \operatorname{stp}(g/A)$.

H induces an equiv relation on G & we can view $G/H = \{gH : g \in G\}$ as a set of equivalence classes ie hyperimaginaries.

By assumption, $gH \in \operatorname{bdd}(A)$.

So $p(x) \vdash$ "$x \in gH$".

Let $g' \models p$ st. $g' \downarrow_A g$. Then $g' \in gH$ so $g^{-1}g' \in H$.

~~If g is generic$/B$, $g \downarrow h$~~

~~Fact product of independent generics is generic.~~

<u>Proof of Fact</u>. Assume g, h are generic $/B$, $g \downarrow_B h$. $(g \downarrow B, h \downarrow B)$

Then g, h are generic $/\emptyset$, $g \downarrow h$, $g, h \downarrow B$.

So suffices to prove it $/\emptyset$.

$g \downarrow h \Rightarrow g \cdot h \downarrow g$ (h generic)

$$D_G(gh, \Xi) = D_G(gh/g, \Xi) = D_G(h/g, \Xi)$$
$$= D_G(h, \Xi)$$

Fact: If g is generic$/B$, $g \underset{B}{\downarrow} h \Rightarrow gh, hg$ are generic$/B$.

Proof of Fact: Suffices to prove that hg is generic.
(then $(gh)^{-1} = h^{-1}g^{-1}$).

So $hg \underset{B}{\downarrow} h$.

$D_G(hg/B, \Xi) = D_G(hg/Bh, \Xi) = D_G(g/Bh, \Xi) =$
$D_G(g/B, \Xi) = D_G(G, \Xi)$.

(2) $\Rightarrow$ (3): Let g witness (2). Then $D_G(H, \Xi) \geq D_G(g/A, \Xi)$
$= D_G(G, \Xi)$.

(3) $\Rightarrow$ (4) ✓.

(4) $\Rightarrow$ (1). By contrapositive.

Assume that $[G:H] = \infty$.

We can find an A-indiscernible sequence $(g_i : i < \omega)$ st.
$g_iH \neq g_jH$ for $i \neq j$.

Let $\xi \in D_G(H, \Xi)$.

We can find $h \in H$ st. $\xi \in D_G(h/A, \Xi)$ and
we may assume $h \underset{A}{\downarrow} g_0$.

Then $\xi \in D_G(h/Ag_0, \Xi)$

~~Let p(x,y) = tp(h, g/A). Then p(x,y) ⊢~~

So $\xi \in D_G(h/Ag_0, \Xi) = D_G(g_0^{\circ}h/Ag_0, \Xi)$.

Let $p(x y) = tp(g_0^{\circ}h, g_0/A)$. Then $p(xy) \vdash x \in yH$.

Then $\xi \in D_G(p(x, g_0), \Xi)$, and $p(x, g_0)$ divides $/A$ since $\bigwedge x \in g_i H$ is contradictory.

Then ξ is not maximal in $D_G(g_0 h/A g_0, \Xi)$ and a fortiori not maximal in $D_G(G, \Xi)$. $\square$

<u>Defn</u>: $G_A^0 := \bigcap \{H : H \text{ is type-definable over } A, [G:H] < \infty\}$

is the A-connected component of G.

Want to prove ① $[G : G_A^0] < \infty$ and ② $G \trianglerighteq G_A^0$.

Proof ① Since we only need to consider boundedly many H's.

② Since G_A^0 has bounded index, it has boundedly many conjugacy classes.

If $g G_A^0 = h G_A^0$ then $g G_A^0 g^{-1} = h G_A^0 h^{-1}$.

Let $H' = \bigcap\{\text{all conjugates of } G_A^0\}$. Then $[G : H'] < \infty$

and H' is A-invariant. $\Rightarrow$ H' is type definable $/A$ $\Rightarrow$
$G_A^0 \leq H'$. $\Rightarrow G \trianglerighteq G_A^0$.

Exercise (Similar): $G^0_A = G^0_{\mathrm{bdd}(A)}$.

Assume $H < G$, $[G:H] < \infty$, H typedefinable $/A$.

Then $g \in H$ is generic for G $/A$ iff for H.

<u>Proof</u> $\Rightarrow$ ✓

$\Leftarrow$ We know that $\exists h \in H$ which is generic $/A$ for G.

Let $g \in H$ be generic $/A$ for H. WTS: g is generic for G.

We may assume $h \underset{A}{\downarrow} g$.

Since g is generic for H and $h \in H$:

$hg \underset{A}{\downarrow} h$ and hg is generic for G. (since h is generic for G)

$hg \underset{A}{\downarrow} h^{-1} \Rightarrow \underset{h^{-1}(hg)}{\overset{\shortparallel}{g}}$ is generic for G

<u>Fact</u> if g is generic, $g \downarrow h$ then gh & hg are generic.

<u>Cor</u> Every $a \in G$ is a product of ~~of~~ two generics.

Just choose $g \downarrow a$ generic, $a = g(g^{-1}a)$

In other words: $G = \mathrm{gen}(G)^2$

Prop Let $X \subseteq G$ type-definable $/\emptyset$. (Otherwise add the parameters to the language)
Assume that for all $a, b \in X$ if $a \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}} b$ then $a^{-1}b \in X$.
Then $Y := X^2 < G$ and every generic of Y is in X.

Eg: $X = \mathrm{gen}(G)$

Proof Let $X' = X \cap X^{-1}$. Then X' satisfies the assumptions.

Also: $X \subset (X')^2$. Let $a \in X$. Find $b, c \in X$ st. $\{a, b, c\}$ are independent. Then $b^{-1}a \in X$, also $a^{-1}b \in X$, $b^{-1}a \in X'$.

Since $b^{-1}a \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}} c$: $(b^{-1}a)^{-1} \cdot c \in X$ and by same argument $(b^{-1}a)^{-1}c \in X'$. So $c \in (X')^2$.

So X, X' generate the same subgroup.

So WMA $X = X^{-1}$!

Choose $\xi \in D_G(X, \Xi)$ maximal, $d \in X$ st. $\xi \in D_G(d, \Xi)$.

Let $a, b, c \in X$. We may assume $a, b, c \mathop{\smile\hspace{-0.9em}|\hspace{0.6em}} d$.

Then $d^{-1}c \in X$ and $b \cdot d \in X$.

Also, $\xi \in D_G(d, \Xi) = D_G(d/a, b, \Xi) = D_G(bd/ab, \Xi)$
and $D_G(bd/ab) \subseteq D_G(bd, \Xi) \subseteq D_G(X, \Xi)$.

So ξ is maximal in $D_G(bd, \Xi) \Rightarrow bd \downarrow a,b$.

Therefore $abd \in X$ so $abc = (abd)(d^{-1}c) \in X^2$

$\Rightarrow X^3 \subseteq X^2$ So $X^2 = Y < G$.

Let $g \in Y = X^2$ be generic. $y = ab$ for $a, b \in X$.

WMA $d \downarrow a, b, g$.

By same argument: $abd = gd \in X$.

Since g is generic and $g \downarrow d$ we have $gd \downarrow d$

$\Rightarrow g = (gd)d^{-1} \in X$. $\square$

Let $p(x) \vdash x \in G$ be a Lascar strong type. Say over $\emptyset$.

The left pre stabiliser of p is $S(p) = \{ a \in G \mid$

$\exists g \models p \; g \downarrow a$ st. $ag \models p \}$

<u>Lemma</u> $S(p)$ is type definable. $S(p) = S(p)^{-1}$.

if $a, b \in S(p)$ and $a \downarrow b$ then $ab \in S(p)$.

<u>Proof</u> Choose $\xi \in mD_G(p, \Xi)$.

Then $S(p) = \{ a : \exists y \, (p(y) \wedge p(a \cdot y) \wedge \mathrm{div}^G_{\xi, a}(y))$.

Assume $a \in S(p)$ and let $g \downarrow a$ witness this.

Then: $D_G(g, \Xi) = D_G(g/a, \Xi) = D({}^{a}g/a, \Xi)$

$$\subseteq D_G(ag, \Xi) = D_G(g, \Xi) = D_G(p, \Xi).$$

so $ag \downarrow a$. In other words $a^{-1} \downarrow ag$ and

$a^{-1} \cdot ag \models p$ so $a^{-1} \in S(p)$.

Assume $a, b \in S(p)$, $a \downarrow b$. let $g \downarrow a$ witness $a \in S(p)$

and $g' \downarrow b$ witness $b \in S(p)$.

We have $a \downarrow b$, $g \downarrow a$, $g' \downarrow b$, $g \overset{LS}{\equiv} g'$

So $\exists g'' \downarrow a, b$ st. $ag'', bg'' \models p$.

By previous argument: $ag'' \downarrow a$.

$g'' \downarrow a, b \Rightarrow g'' \downarrow_a b \Rightarrow ag'' \downarrow_a b \Rightarrow ag'' \downarrow a, b$

$\Rightarrow ag'' \downarrow ba^{-1}$

so $ba^{-1} \in S(p)$ $\square$.

Therefore $Stab(p) := S(p)^2$ is a group and

$S(p)$ contains all of its generics.

Prop
With the previous assumptions (ie p is a Lstp$/\emptyset$),

(1) p is generic $\Leftrightarrow$ (3) $[G:\mathrm{Stab}(p)] < \infty \Leftrightarrow$ (2) $\mathrm{Stab}(p) = G^0_\emptyset$.

(1) $\Rightarrow$ (2): Assume that p is generic.

Let $g, g' \models p$ st. $g \downarrow g'$.

Then $g'g^{-1} \downarrow g \Rightarrow g'g^{-1} \in S(p) \subseteq \mathrm{Stab}(p)$.

Also $g'g^{-1}$ is generic for G.

So $\mathrm{Stab}(p)$ contains a generic so

$\mathrm{Stab}(p) > G^0_{\mathrm{bdd}\,\emptyset} = G^0_\emptyset$. (from exercise).

Since p is a Lstp and $G^0_\emptyset$ has boundedly many right cosets.

$p(x)$ says in which right coset of $G^0_\emptyset$ x lies.

Therefore, if $a \in S(p)$ then $a = (ag) \cdot g^{-1} \in G^0_\emptyset$.

So $\mathrm{Stab}(p) < G^0_\emptyset \Rightarrow$ ~~Stab~~ $\mathrm{Stab}(p) = G^0_\emptyset$.

(2) $\Rightarrow$ (3) ✓

(3) $\Rightarrow$ (1) $\mathrm{Stab}(p)$ has bounded index it contains a generic of G, call it h.

Then h is also a generic of $Stab(p) \Rightarrow h \in S(p)$.

So $\exists g \downarrow h$ st. $g, hg \models p$.

Since h is generic, hg is generic $\Rightarrow$ p is generic. $\square$.

Assume T is stable.

<u>Lemma</u> G^0_ϕ has a unique generic type $/ \phi$.

<u>Proof</u> Assume not. Then the same is true over $\mathrm{bdd}\,\phi$.

So we may assume we work over $\mathrm{bdd}\,\phi$.

Let p, q be two generic types of G^0_ϕ.

Then $Stab(p) = Stab(q) = G^0_\phi$.

So $p(x) \vdash x \in S(q)$ (since p is generic type of $Stab(q)$)

and $q(x) \vdash x \in S(p)$.

Therefore $\exists\, g \models p$ and $h \models q$ st. $g \downarrow h$ and $gh \models q$.

But p & q are stationary.

So for all $g \models p$ $h \models q$ if $g \downarrow h \Rightarrow gh \models q$.

Similarly: let $p^{-1} := tp(g^{-1})$ where $g \models p$ &
$q^{-1} := tp(h^{-1})$ $(h \models q)$.

By same argument for all $h \models q$ & $g \models p$

namely for all $h^{-1} \models q^{-1}$ & $g^{-1} \models p^{-1}$ then

$$g^{-1} \perp h^{-1} \quad \underbrace{h^{-1}g^{-1}}_{\substack{\| \\ (gh)^{-1}}} \models p^{-1}.$$

so $p^{-1} = q^{-1}$ so $p = q$ $\square$

able)

'' **Theorem** The connected component G_A^0 does not depend on A. We call it G^0.

G is connected (ie $G = G^0$) iff G has a unique generic type.

Proof Let A be any set of parameters.

Assume that $G_\phi^0 \neq G_A^0$.

Let p be a generic type of G_A^0.

Then p is also generic for G and $p \vdash x \in G_\phi^0$. we proved this

Therefore p does not divide $/\phi$.

Let $q = p|_{bdd(\phi)}$.

So p is the unique non div extn of q.

Let $a \in G^\circ_\emptyset$. Let g be generic for $G^\circ_\emptyset$ ($g \models q$),

and $g \downarrow a, A$.

Then $g \models p$. Now $g \downarrow a, A \Rightarrow ag \underset{a}{\downarrow} A$

$\Rightarrow ag \downarrow aA$

and ag is generic for $G^\circ_\emptyset \Rightarrow ag \models q \Rightarrow ag \models p$.

So $g, ag \models p \Rightarrow a = (ag)\cdot g^{-1} \in G^\circ_A$.

~~[illegible]~~ ($\Leftarrow$:) Let $g \models p$ (unique generic type). If G is not connected: ① $p \vdash x \in G^\circ$ since:
~~let $g' \downarrow g$, $g' \models p$. Then~~ let p' = generic type of G°. Then $p' = p$. ② Let $a \in G \setminus G^\circ$. WMA $a \downarrow g$.
Then ag is generic $\Rightarrow ag \models p$. But $ag \in aG^\circ$. ※.

5/10. Making one point clearer from last time:

Assume T is stable, G type-def group.

$G^\circ_\emptyset = G^\circ_{bdd(\emptyset)}$ and each has a unique generic type over $\emptyset$, $bdd(\emptyset)$ resp.

Let p be the generic type of $G^\circ_\emptyset$. Then every extn of p to $bdd(\emptyset)$ is non-dividing and therefore a generic type of $G^\circ_{bdd(\emptyset)}$ [$p \vdash x \in G^\circ_\emptyset = G^\circ_{bdd(\emptyset)}$].

So p has a unique extension to $bdd(\emptyset)$.

So even though p is $/\emptyset$ it is Lascar strong.