Lascar inequalities (Assume T super simple, a,b finite).

I. $SU(a/Ab) + SU(b/A) \leqslant SU(ab/A) \leqslant SU(a/bA) \oplus SU(b/A)$.

Proof By induction on α: $SU(b/A) \geqslant \alpha$ then

$SU(ab/A) \geqslant SU(a/Ab) + \alpha$.

$\alpha = 0$: $\underset{\uparrow \text{exercise}}{SU(ab/A) \geqslant SU(a/A)} \geqslant SU(a/Ab)$.

α limit: By ind hyp. (since α on right-hand-side).

$SU(b/A) \geqslant \alpha+1$ so $\exists c$ s.t. $SU(b/Ac) \geqslant \alpha$, $b \not\downarrow_A c$.

WMA $c \downarrow_{Ab} a$.

$\Rightarrow$ ① $SU(a/Ab) = SU(a/Abc)$

By ind hyp, $SU(ab/Ac) \geqslant SU(a/Abc) + \alpha = SU(a/Ab) + \alpha$.

② $ab \not\downarrow_A c$ ~~= otherwise $a \downarrow bc$~~ since $b \not\downarrow_A c$

$\Rightarrow$ $SU(ab/A) \geqslant SU(ab/Ac) + 1 \geqslant SU(a/Ab) + \alpha + 1$.

Now other inequality, we prove by induction:

$\forall \alpha$ if $SU(ab/A) \geqslant \alpha \Rightarrow SU(a/Ab) \oplus SU(b/A) \geqslant \alpha$.

Again $\alpha = 0$, limit ✓.

Assume $SU(ab/A) \geq \alpha+1$. Then $\exists c$ s.t. $ab \underset{A}{\not\downarrow} c$ & $SU(ab/Ac) \geq \alpha$.

~~Two cases~~. $\Rightarrow SU(a/Abc) \oplus SU(b/Ac) \geq \alpha$ by inc

Either $b \underset{A}{\not\downarrow} c$ or $a \underset{Ab}{\not\downarrow} c$ (otherwise by trans we get a

In either case $SU(a/Ab) \oplus SU(b/A) \geq SU(a/Abc) \oplus SU(b/AC)$

$\geq \alpha+1$

II. If $a \underset{A}{\downarrow} b \Rightarrow SU(ab/A) = \underset{(SU(a/Ab))}{\underset{\parallel}{SU(a/A)}} \oplus SU(b/A)$

Assume $SU(a/A) \oplus SU(b/A) \geq \alpha+1$.

So wlog assume $\exists \beta, \gamma$ s.t. $SU(a/A) \geq \beta+1$, & $SU(b/A) \geq \gamma$ & $\beta \oplus \gamma \geq \alpha$. } requires ordinal

$\exists c$ $a \underset{A}{\not\downarrow} c$ $SU(a/Ac) \geq \beta$.

WMA $c \underset{Aa}{\downarrow} b \Rightarrow ac \underset{A}{\downarrow} b \Rightarrow a \underset{AC}{\downarrow} b$

$\Rightarrow SU(ab/AC) \geq \beta \oplus \gamma$.

Explicitly $SU(ab/Ac) \geq SU(a/Ac) \oplus SU(b/Ac) \geq$

$\Rightarrow SU(ab/A) \geq \beta \oplus \gamma + 1 \geq \alpha+1$. So we have equ writing out ordinal remarks:

Want $\beta \oplus \gamma \geq \alpha+1 \Rightarrow \exists \beta' \ \beta \geq \beta'+1 \quad \beta' \oplus \gamma \geq \alpha$ or $\gamma \ldots$

write $\beta = \sum \omega^{\alpha_i} n_i \qquad \gamma = \sum \omega^{\alpha_i} m_i \qquad \beta \oplus \gamma = \sum \omega^{\alpha_i}(m_i+n_i) \geq \alpha+1$.

~~write~~ $\alpha = \sum \omega^{\alpha_i} k_i$

Fact: $\cancel{\alpha \oplus \beta}$ $(\alpha, \beta) \mapsto \alpha \oplus \beta$ is minimal s.t.

if ~~[illegible]~~ $(\alpha+1) \oplus \beta \geq \alpha \oplus \beta + 1$ and symmetry.

$\beta \oplus \gamma = \sum \omega^{\alpha_i}(m_i + n_i) \geq \left(\sum \omega^{\alpha_i} k_i\right) + 1$.

Let j be least s.t. $k_j < m_j + n_j$

~~[illegible]~~

If $n_j = 0 \quad k_j < m_j$.

Let $\gamma' = \sum_{i<j} \omega^{\alpha_i} m_i + \cancel{\omega^{\alpha_j} k_j} \sum_{i \geq j} \omega^{\alpha_i} k_i$

so $\gamma \geq \gamma' + 1 \quad$ & $\gamma' \oplus \beta \geq \alpha$.

Otherwise define $\beta' = \sum_{i<j} \omega^{\alpha_i} n_i + \omega^{\alpha_j}(n_j - 1) + \sum_{i>j} \omega^{\alpha_i} k_i$

so $\beta \geq \beta' + 1 \quad$ & $\beta' \oplus \gamma \geq \alpha$.

Friday 1pm

One more inequality (whose proof is the messiest).

III ("Higher ~~...~~ exponent symmetry").

Assume $SU(a/A) \geq SU(a/Ab) + \omega^{\alpha}\cdot n$.
(ie very dependent on b).

Then $SU(b/A) \geq SU(b/Aa) + \omega^{\alpha}\cdot n$.

This property is useful.

Small Claim: Assume $SU(a/B) = \omega^{\alpha}$ and $B \subseteq C$ & $a \underset{B}{\downarrow} C$ &

$b \underset{B}{\equiv} a$ s.t. $b \underset{B}{\not\downarrow} C$ $\Rightarrow a \underset{C}{\downarrow} b$.

Proof Assume $a \underset{C}{\not\downarrow} b$. Then $\omega^{\alpha} = SU(a/C) \leq SU(ab/C)$

$\leq \underbrace{SU(a/bC)}_{<\omega^{\alpha}} \oplus \underbrace{SU(b/C)}_{<\omega^{\alpha}} < \omega^{\alpha}$. □.

4/7. Defn (1) For every two contradicting formulas $\varphi(x,y)$, $\psi(x,y)$ (could be tuples)
$\in \omega \cup \{\infty\}$
define $R(p(x), \varphi, \psi, 2)$ inductively as follows:

- $R(p, \varphi, \psi, 2) \geq 0$ if $p(x)$ is consistent.
- $R(p, \varphi, \psi, 2) \geq n+1$ if $\exists b$ st. $R(p(x) \wedge \varphi(x,b), \varphi, \psi, 2) \geq n$

  and $R(p(x) \wedge \psi(x,b), \varphi, \ldots) \geq n$.

(2) The pair (φ, ψ) is stable if $R(\overset{T}{\underset{i.e.}{x=x}}, \varphi, \psi, 2) < \infty$.

(3) φ is stable if (φ, ψ) is stable $\forall \psi$ contradicting φ.

T is stable if all formulas are.

<u>Defn</u> let $p \in S(A)$, $\varphi(x; y)$ a formula. (finite tuples since talking about a single formula.)

A φ-definition for p (over A) is a partial type $d_\varphi p(y)$ $/A$

satisfying:

- $|d_\varphi p| \leq |T|$.
- $\forall b \in A$ (of the length of y), $\varphi(x, b) \in p$ iff $\models d_\varphi p(b)$.

(2) A definition of $p(x)$ is a set $\{d_\varphi p : \varphi(x, y)\}$

such that each $d_\varphi p$ is a φ-def for p.

(3) A good defn for p is a definition $\{d_\varphi p\}$ st.

$\forall B$ the type $q = \{\varphi(x, b) : \varphi(x, y), b \in B$ st. $\models d_\varphi(b)\}$

is a complete consistent type.

(4) p is (well) definable if it has a (good) definition.

(a)

Now: If $\varphi(x,b) \in p$ then $\chi(x,c) \wedge \varphi(x,b) \in p$

$\Rightarrow \rho_\psi(b)$ is true.

On the other hand if $\psi(x,b) \in p$ then $\chi(x,c) \wedge \psi(x,b) \in p$

$\Rightarrow \rho_\psi(b)$ is false (o.w. $R(\chi, \varphi, \psi, 2) \geq n+1$)

Let $d_\varphi p(y) = \{\rho_\psi(y) : \psi$ contradicting $\varphi\}$.

Then $|d_\varphi p| \leq |T|$ & $d_\varphi p(y)$ is over A.

If $\varphi(x,b) \in p$ then $\models d_\varphi p(b)$ from (a).

If $\varphi(x,b) \notin p$ then since p is complete

$\exists \psi$ contradicting φ s.t. $\psi(x,b) \in p$.

So $\not\models \rho_\psi(b) \Rightarrow \not\models d_\varphi p(b)$.

(2) $\Rightarrow$ (3): count possible definitions.

(3) $\Rightarrow$ (4): eg take $\lambda = 2^{|T|}$ so $(\lambda + |T|)^{|T|} = \lambda$.

(4) $\Rightarrow$ (1): Assume $\neg$(1) and let λ be any cardinal.

let κ be least s.t. $2^\kappa > \lambda$. so $\kappa \leq \lambda$

So $2^{<\kappa} = \sum_{\mu<\kappa} 2^\mu \leq \lambda \cdot \lambda = \lambda$.

So by assumption we have φ, ψ contradictory s.t. $R(x=x, \varphi, \psi, 2) = \infty$.

So by compactness we find $\{a_\tau : \tau \in 2^\kappa\}$ and $\{b_\sigma : \sigma \in 2^{<\kappa}\}$ s.t. $\forall \tau \in 2^\kappa, \alpha < \kappa$ we have if $\tau(\alpha) = 0$ then $\varphi(a_\tau, b_{\tau|_\alpha})$

if $\tau(\alpha) = 1$ then $\psi(a_\tau, b_{\tau|_\alpha})$.

Let $B = \{b_\sigma\}$ then $|B| = 2^{<\kappa} \leq \lambda$

But we found $2^\kappa > \lambda$ contradictory φ-types over B. $\square$

<u>Corollary</u> TFAE:

① T stable

② ~~[illegible]~~ Every complete type is definable.

③ $\forall A$, $|S(A)| \leq (|A| + |T|)^{|T|}$

④ $\exists \lambda$ s.t. $|A| \leq \lambda \Rightarrow |S(A)| \leq \lambda$.

<u>Sketchy proof</u> (1) $\Rightarrow$ (2) ✓. $\Rightarrow$ ③ notice $[(|A|+|T|)^{|T|}]^{|T|} = (|A|+|T|)^{|T|}$.

③ $\Rightarrow$ ④. Take $\lambda = 2^{|T|}$ & use proof of ④ $\Rightarrow$ ① in theorem. $\square$.

<u>Defn</u> Let $A \subseteq B$, $p \in S(B)$. Then p is <u>non-splitting</u> over A if ~~[illegible]~~ $\forall \varphi(x,y)$ & $b, c \in B$ of length of y, then if $b \equiv_A c$ then $\varphi(x,b) \in p$ iff $\varphi(x,c) \in p$.

In other words, if $a \models p$ and $b \equiv_A c$ $(b, c \in B)$

then $b \equiv_{Aa} c$.

<u>Defn</u> Let $\kappa > |T|$. A set $\mathcal{M} \subseteq \mathcal{U}$ is κ-saturated if

$\forall A \subseteq M$ $|A| < \kappa$, $\forall p \in S(A)$, p is realised in $\mathcal{M}$

<u>Fact</u> $\forall A \ \exists \mathcal{M} \supseteq A$ s.t. $\mathcal{M}$ is $|T|^+$-saturated.

<u>Lemma</u> Let $\mathcal{M}$ be $|T|^+$saturated, $p \in S(\mathcal{M})$ definable.

Then (i) p has a unique definition (up to equivalence)

(ii) the unique definition is good.

- $\forall B$, let $p|_B$ be the type resulting from the application of the definition to B.

(iii). $\forall B \supseteq \mathcal{M}$, $p|_B$ is a nonsplitting extension of p.

<u>Proof</u> (i) Assume $\{d_\varphi p\}$ and $\{d'_\varphi p\}$ are both definitions & not equivalent.

So $\exists \varphi$ st. $d_\varphi p \not\equiv d'_\varphi p$.

ie $\exists b$ (not in $\mathcal{M}$) s.t. say $\models d_\varphi p(b)$ &
$\not\models d'_\varphi p(b)$

~~So there exists~~

So $tp(b/M)$ contradicts over $d_\varphi' p(y)$.

$\Rightarrow \exists c \in M$ & $\chi(y,z)$ s.t. $\models \chi(b,c)$ and

$\chi(y,c)$ contradicts $d_\varphi' p(y)$.

Let A = set of parameters used in $d_\varphi p$, then

$A \subseteq M$, $|A| \leq |T|$.

By $|T|^+$-saturation $\exists b' \in M$ st. $b' \underset{Ac}{\equiv} b$.

Then $\models d_\varphi p(b')$ & $\not\models d_\varphi' p(b')$ (because $\chi(b',c)$).

So $d_\varphi p$, $d_\varphi' p$ do not ~~def~~ define the same φ-type in M.

(ii) Let B be any set. ~~[illegible]~~

We want to prove $p|_B$ is a complete consistent type.

~~consistent~~ Let A be, as above, the set of parameters.

consistent: if not, there are $\varphi_i(x_i, b_i) \in p|_B$ $i<n$ s.t.

$\bigwedge \varphi_i(x, b_i)$ is inconsistent.

By saturation, find $\bar{b}' \underset{A}{\equiv} \bar{b}$, $\bar{b}' \in M$.

Then $\varphi_i(x, b_i') \in p$ $\forall i$ and $\bigwedge \varphi_i(x, b_i')$ is inconsistent

$\#$

complete: Assume not. Then $\exists b \in B$ and $\varphi(x,y)$ s.t.

$\varphi(x,b) \notin p|_B$ and $\forall \psi$ contradicting φ, $\psi(x,b) \notin p|_B$.

find $b' \underset{A}{\equiv} b$ in $\mathcal{M}$ ....etc..

(iii) not enough time, so exercise!

<u>Recall</u>: $A \subseteq B$ $p \in S(B)$: p is nonsplitting if $\forall b, b' \in B$ if $b \equiv_A b'$ then $p|_b, p|_{b'}$ are conjugates $/A$.

M $|T|^+$-saturated $p \in S(M)$ definable then:
(i) unique definition (ii) good defn
(iii) $\forall B \supseteq M$ $p|_B$ is nonsplitting$/M$.

<u>Assume</u> $p(x, B)$ is a partial type$/B$, invariant under automorphisms fixing A.

Then $\exists$ $q(x, A) \equiv p(x, B)$ st. $|q| \leq |p| + |T|$.

$[q(x, A) := \text{"}\exists C \text{ st. } C \underset{A}{\equiv} B \wedge p(x, C)\text{"}]$.

<u>Remarks</u>

I. Assume $p \in S(A)$ has a good definition. Then $\forall B \supseteq A$, $p|_B$ (following that defn) dnd $/A$.

<u>Proof</u> Assume $(B_i : i < \omega)$ is indisc. in $tp(B/A)$.

let $a \models p|_{\cup B_i}$ $\square$.

II Assume that $p \in S(A)$ has nonsplitting extensions to every set $B \supseteq A$. Then p is Lascar strong.

<u>Proof</u> Let $N \supseteq A$ be $(|A| + |T|)^+$-saturated.

Let $q \in S(N)$ be a nonsplitting extension of p.

Let $a, b \models p$. We need: $a \overset{Ls}{\underset{A}{\equiv}} b$.

We may assume that $a, b \in N$ (realise $tp(ab/A)$ in N).

By induction on $i < \omega$ find $c_i \in N$ s.t. $c_i \models q|_{Aabc_{<i}}$

Then each $a, c_0, c_1, c_2, \ldots$ and $b c_0 c_1 \ldots$ is A-indiscernible since nonsplitting (so we have pairs) & induction.

(ie each $a, c_0, c_1, \ldots$ has same type / A (p).
Since q is nonsplitting each pair, each triplet, ...)

$\Rightarrow a \overset{Ls}{\underset{A}{\equiv}} b$. $\square$

<u>Corollary</u> (T stable). Every type over a ~~[illegible]~~ $|T|^+$-saturated model is Lascar strong.

(1st order: don't need T stable ~~[illegible]~~ or $|T|^+$-saturated).

Defn A class of sets (in the universal domain) $\mathcal{A}$ is cofinal $\forall B \ \exists A \in \mathcal{A} \ A \supseteq B$.

Eg let $\mathcal{M}_{|T|^+} = \{ |T|^+\text{-saturated models} \}$

Then $\mathcal{M}_{|T|^+}$ is cofinal.

Prop Assume that for every final tuple a, for every increasing sequence $(A_i : i < |T|^+)$ in $\mathcal{A}$, $\exists j < |T|^+$ st $a \underset{A_j}{\downarrow} \bigcup_{i<|T|^+} A_i$. Then T is simple.

Proof Assume ~~|T| is a~~ T is not simple $\Rightarrow$

$\exists a, b_i : i < |T|^+$ st, $a \underset{b_{<i}}{\not\downarrow} b_i \ \forall i$. let $p_i(x, b_{\le i}) := tp(a/b_{\le i})$

Find $(A_i : i < |T|^+)$ increasing in $\mathcal{A}$ and $c_i : i < |T|^+$

and st. $\bar{c} \equiv \bar{b}$, $c_i \in A_{i+1}$ st. if $a'\bar{c} \equiv a\bar{b}$ then

$a' \underset{A_i}{\not\downarrow} c_i \quad \forall i$. ($\Rightarrow$ contradiction)

Construction: $i = 0$: A_0 = anything in $\mathcal{A}$.

i limit: anything containing $\bigcup_{j<i} A_j$.

$i+1$: $A_{i+1} \supseteq A_i c_i$.

Now choose c_i.

We have A_i, $c_{<i}$, want to find c_i.

Since $a \underset{b_{<i}}{\not\downarrow} b_i$ $\exists$ $b_{<i}$ indiscernible sequence $(d_{i,j} : j<\omega)$ witnessing it. Since by ind hyp, $c_{<i} \equiv b_{<i}$,

find $(e_{i,j} : j<\omega)$ s.t.: $\overline{e_i}\, c_{<i} \equiv \overline{d_i}\, b_{<i}$.

By extension/extraction, we may assume $(e_{i,j} : j<\omega)$ is A_i-indiscernible.

$c_i := e_{i,0}$.

(1) $c_{\leq i} \equiv b_{\leq i}$ (since $d_{i,0} \underset{b_{<i}}{\equiv} b_i$).

Let ~~$p_i(x, y_{<i}$~~ $p_i(x, y, \bar{z}) := tp(a\, b_i\, b_{<i})$.

Then $\bigwedge_j p(x, b_{i,j}, c_{<i})$ is inconsistent and the sequence $\{e_{i,j} : j<\omega\}$ is A_i-indiscernible

$\Rightarrow$ if $a' \models p(x, c_i, c_{<i})$ then $a_i \underset{A_i}{\not\downarrow} c_i$ $\square$

Now we can prove the main theorem...

Theorem: TFAE:

(1) T stable
(2) T simple and Lascar strong types are stationary (ie have unique nd extn to every set)
(3) T simple and every type has a bounded multiplicity (ie $\exists \lambda$ s.t. p has at most λ-many nondividing extn to any set).

Proof (1) $\Rightarrow$ (2): Assume T stable.

We know $\mathcal{M}_{|T|^+}$ is cofinal.

Assume a is a finite tuple, $(\mathcal{M}_i : i < |T|^+$ is an increasing sequence of $|T|^+$-saturated models.

Then $M = \bigcup \mathcal{M}_i$ is $|T|^+$-saturated $\Rightarrow$ $tp(a/\mathcal{M})$ has a good defn.

This good defn uses only $|T|$ parameters and is therefore over $\mathcal{M}_j$ for some $j < |T|^+$.

Let $p = tp(a/M_j)$. ~~It also has a unique good definition.~~

The same def is a def for p.

So $tp(a/\mathcal{M}) = p|_{\mathcal{M}}$. $\Rightarrow a \underset{\mathcal{M}_j}{\downarrow} \mathcal{M}$

Let $p(x, A)$ be Lascar strong & nonstationary.

So $\exists b$, q_0, q_1 in $S(Ab)$ $q_0(x b), q_1(x b)$ both non-dividing extensions of p & $q_0 \neq q_1$ & $q_0|_{cb} \neq q_1|_{cb}$ for $c \in A$.

Pick your favourite cardinal λ.

Find $(b_i : i < \lambda)$ indep $/A$ in $tp(b/A)$.

$\forall \xi \in 2^\lambda$, we can find using successive applications of the independence theorem $a_\xi \underset{A}{\downarrow} \bar{b}$ st.

$a_\xi \models \bigwedge_{i<\lambda} q_{\xi(i)}(x, b_i)$. $\Rightarrow 2^\lambda$ distinct types over $c, b_{<\lambda}$ & $|c, b_{<\lambda}| = \lambda$.

$\Rightarrow$ not stable.

(2) $\Rightarrow$ (3): Let $p \in S(A)$. Then $\lambda = |\{\text{ext. of } p \text{ to } bdd(A)\}|$ is the multiplicity of p.

(3) $\Rightarrow$ (1): count types.

For every set A st. $|A| \leq |T|$ and for every $p(x, A) \in S(A)$, by assumption p has at most d_p nondividing extns to any set, and this only depends on $p(x, Y)$.

let $\lambda = \sup\{\lambda_p : \forall p(x,Y)$ st. x is finite, $|Y| \leq |T|\}$.

let $\mu = \lambda^{|T|}$.

let $|B| \leq \mu$.

Every type p over B dnd over some $A \subseteq B$ st. $|A| \leq |T|$.

This gives us $\mu^{|T|}$ possibilities.

So p is a nd extn of $p|_A$ to B: at most λ possibilities.

& finally since $|A| \leq |T|$, $2^{|T|}$ possibilites for $p|_A$.

So we have $\underbrace{\mu^{|T|}}_{\text{choose } A} \cdot \underbrace{2^{|T|}}_{\text{choose } p|_A} \cdot \underbrace{\lambda}_{\text{choose } p} = \mu$.

so stable $\square$.

Minor remark:

Stationarity $\Rightarrow$ ind thm.

let $p \in S(A)$. Then p stationary $\Leftrightarrow$

$\forall b, c$ & $\forall q_0 \in S(Ab)$, $q_1 \in S(Ac)$ n.d. $/A$,

$q_0 \cup q_1$ dnd $/A$.

$\Rightarrow$ same thing with $b \underset{A}{\perp} c$, which is ind thm.

4/12. <u>Defn</u>: A formula $\varphi(x,y)$ [x,y in the same sort], possibly with hidden parameters is <u>thin</u> if $\bigwedge_{i<j<\omega} \varphi(x_i, x_j)$ is inconsistent.

<u>Fact</u>: $d_A(a,b) \leq 1$ iff a,b satisfy no thin form over A.

<u>Proof</u>: Let $p(x,y) = tp(a,b/A)$.

Then $d_A(a,b) \leq 1$ iff $\bigwedge_{i<j<\omega} p(x_i, x_j)$ cons

iff

Fix x (finite length), A. Let $\{\varphi_i(x,y) : i<\lambda\}$ enumerate all thin formulas $/A$. (Rmk: $\lambda = |A| + |T|$).

We will consider a tree indexed by $I \subseteq \bigcup_{\alpha \in \mathrm{Ord}} \lambda^\alpha$

(I indexes a tree means that if $\sigma \in I \cap \lambda^\alpha$ then $\sigma|_\beta \in I$ $\forall \beta < \alpha$.)

On the nodes $\sigma \in I$, we put a_σ in the sort of x s.t.: if $\sigma \in I \cap \lambda^\alpha$, $\beta < \alpha$, then $\varphi_{\sigma(\beta)}(a_{\sigma|_\beta}, a_\sigma$.

If $\sigma \in I \cap \lambda^{\alpha}$ then $\forall i < \lambda$: $\{\beta < \alpha : \sigma(\beta) = i\}$ is finite, since φ_i is thin.

$\Rightarrow |\alpha| \leq \lambda \Rightarrow \alpha < \lambda^{+}$

$\Rightarrow I \subseteq \lambda^{<(\lambda^{+})} \quad (:= \bigcup_{\alpha < \lambda^{+}} \lambda^{\alpha})$.

Moreover by Zorn's lemma, we may assume tree is maximal.

Now let $A \subseteq \mathcal{M}$ where $\mathcal{M}$ is λ^{+}-saturated (ie $(|A| + |T|)^{+}$-sat)

By induction on $\alpha < |\lambda|^{+}$, for every $\sigma \in I \cap \lambda^{\alpha}$,

find $b_{\sigma} \in \mathcal{M}$ st. $tp(b_{\sigma}, (b_{\sigma|_{\beta}} : \beta < \alpha) / A)$

$$= tp(a_{\sigma}, (\underbrace{a_{\sigma|_{\beta}} : \beta < \alpha}_{\leq \lambda \text{ parameters}}) / A)$$

so ~~preserving~~ realising types of branches in $\mathcal{M}$.

$\Rightarrow$ the tree $(b_{\sigma} : \sigma \in I)$ has the property (+) as well,

and is maximal as such.

Done with construction.

Now if a is in the sort of x, then $\exists\, b_{\sigma} \in \mathcal{M}$ st.

$d_A(a, b_{\sigma}) \leq 1$ ie st. a, b_{σ} satisfy no thin formula $/A$. (If not, we can add a to the tree.)

Consequences

1. If $\mathcal{M}$ is $|T|^+$-saturated, then types over M are Lascar strong.

In fact: if $a \equiv_M b \Rightarrow d_M(a,b) \le 2$.

Proof $\forall A \subseteq M$ finite, $\exists c \in M$ s.t. $d_A(a,c) \le 1$.

Since $a \equiv_M b$, $d_A(b,c) \le 1$.

$\Rightarrow d_A(a,b) \le 2$.

Since this is true $\forall A \subseteq M$ finite, by compactness: $d_M(a,b) \le 2$.

[We only use thickness, ie that $d_Z(x,y) \le 1$ is type-defn]

2. If $A \subseteq M$ and M is $(|A|+|T|)^+$-sat., then all Lascar strong types (of finite tuples) over A are realised in M.

Proof $\forall a\ \exists c \in M$ s.t. $d_A(a,c) \le 1$.

3. "co-heir" property

Assume a finite, M is $|T|^+$-saturated, $B \supseteq M$, T stable.

Then $a \underset{M}{\downarrow} B \iff \forall A \subseteq B$ s.t. $|A| \le |T|$, $tp(a/A)$ is realised in M.

(*)

[Namely $a \underset{M}{\perp} B$ iff all sufficiently small bits of $tp(a/B)$ are realised in M]

(*) = an analogue of "$tp(a/B)$ is a coheir of $tp(a/M)$"

<u>Proof</u> $\Rightarrow$: let $A \subseteq B$ s.t. $|A| \leq |T|$.

Then by local character, $\exists C \subseteq M$ s.t. $|C| \leq |T|$

s.t. $A \underset{C}{\perp} M$.

So: $a \underset{M}{\perp} B \Rightarrow a \underset{M}{\perp} A \Rightarrow aM \underset{C}{\perp} A$

So by α, $\exists\, b \in M$ satisfying ~~tp~~ $lstp(a/C)$.

$\Rightarrow a, b \underset{C}{\perp} A$

So by stationarity of lstp: $a \underset{AC}{\equiv} b \Rightarrow a \underset{A}{\equiv} b$

$\Leftarrow$: ~~It suffices to prove $a \underset{M}{\perp} A$ for all $A \subseteq B$ finite.~~
~~(ie finite character).~~

~~let A be such.~~

~~let $b \in M$ st. $a \underset{A}{\equiv} b$. so $tp(a/A) = tp(b/A)$~~

It suffices to prove $\forall A \subseteq B$ finite that $tp(a/A)$ dnd $/M$.

let A be such.

let $b \in M$ st $a \underset{A}{\equiv} b$ & $tp(a/A) = tp(b/A) =: p(x, A)$.

Let $(A_i : i<\omega)$ be an M-indiscernible sequence in $tp(A/M)$. $\Rightarrow \bigwedge_{i<\omega} p(b, A_i)$. $\square$.

<u>Canonical Bases and Stationarity</u> (T stable)

p is stationary $\Leftrightarrow$ lstp.

Let $p \in S(A)$ be stationary. Then its unique n.d. extn to ~~any $|T|^+$-s~~ a type over any $|T|^+$-sat. model has a definition which is good.

For every $M \subseteq N$ $|T|^+$-saturated, containing A (the parameters of p), $p|_M$ and $p|_N$ have the same definition.

$\Rightarrow \forall M, N \supseteq A$ and are $|T|^+$-saturated, $p|_M$ and $p|_N$ have the same definition (embed into 3rd).

So p has a "unique good definition" — this is <u>the</u> good definition of p, say $\{d_\varphi p\}$.

Any automorphism fixing A pointwise necessarily fixes the family of n.d. extensions of p setwise and therefore fixes $\{d_\varphi p\}$ $\Rightarrow$ $\{d_\varphi p\}$ can be taken with parameters in A.

Moreover p does not divide over $Cb(p)$ and $p|_{Cb(p)}$ is a lstp & so stationary.

So it follows $\{d_\varphi p\} = \{d_\varphi\, p|_{Cb(p)}\}$

$\Rightarrow$ $\{d_\varphi p\}$ are over $Cb(p)$.

Alternatively, let q be another stationary type in the same variables. Then p & q have a common n.d. extension

$(p \parallel_1 q)$ $\Leftrightarrow$ they have same definition.
 ↑ an equivalence relation.

$\Leftrightarrow$ $p \parallel q$

So an automorphism fixes $Cb(p)$ $\Leftrightarrow$ it fixes $p/\parallel$

$\Leftrightarrow$ it fixes $\{d_\varphi p\}$

So canonical base of p is a <u>canonical parameter</u> for the defn.

Now assume T is (stable) and first order (in particular we have negations.)

If M is a model of T & $p \in S(M)$ then p has a defn over M.

• in M

Recall $R(-,\varphi,\psi,\lambda)$. Here we only need to consider $R(-,\varphi,\neg\varphi,\lambda) \to$ replace with $R(-,\varphi,\lambda)$.

Since for each φ we consider a single rank $R(-,\varphi,\lambda)$ (and not $R(-,\varphi,\psi,\lambda)$ $\forall\psi$ contradicting φ): same argument as before yields: T stable $\Leftrightarrow$ Every type $p \in S(A)$ has a definition where $d_\varphi p$ is a single formula $\forall\varphi$.

Now: T stable & first order.

So same arguments as before work when M is a model (and not necessarily $|T|^T$-saturated)

e.g. $p(x) \in S(M)$ has unique defn, ~~####~~ it is good, etc.

If ~~####~~ $B \supseteq M$, $q \in S(B)$, $q \supseteq p$, then q is a co-heir of p if $\forall \varphi(x,b) \in q$ is realised in M.

Then q dnd over $M \Leftrightarrow$ is a co-heir.

Let $\psi(x,y)$ be any formula.

Let $E(y,y') := \forall x \; \psi(x,y) \leftrightarrow \psi(x,y')$.

Then b/E is an imaginary and is a canonical parameter for $\psi(x,b)$.

$f(b/E) = b/E \iff f(\psi(x,b)) \equiv \psi(x,b)$.

Let p be stationary. Let c_φ be a canonical parameter for $d_\varphi p$.

Then an automorphism fixes $\{c_\varphi\} \iff$ fixes the definition $\iff$ fixes $Cb(p)$.

Conclusion: $\{c_\varphi\}$ is a canonical base for p.

<u>Cor</u> Let $A \subseteq \mathfrak{U}^{eq}$, a any tuple. Then $tp(a/acl^{eq}(A))$ is Lascar strong

<u>Pf</u>

We know that $p = tp(a/bdd(A))$ is Lascar strong.

Let $c = Cb(p) = \{c_\varphi\}$ = canonical params of def.

Then $\forall \varphi$: $c_\varphi \in dcl(bdd(A)) \Rightarrow c_\varphi \in bdd(A)$

$\Rightarrow c_\varphi \in acl^{eq}(A)$ (using negations)

$\Rightarrow$ an automorphism fixing $acl^{eq}(A)$ fixes also $Cb(p)$

$\Rightarrow$ fixes $p|_{\mathfrak{U}} \Rightarrow$ fixes p. $\square$