8.5 The Paley Graph

Let p be a prime such that $p \cong 1 \mod 4$. The Paley graph of order p is a graph on p nodes (each node associated with an element of $\mathbb{Z}_p$) where (i,j) is an edge if $i-j$ is a quadratic residue modulo p. In other words, (i,j) is an edge is there exists a such that $a^2 \cong i-j \mod p$. Let $\omega(p)$ denote the clique number of the Paley graph of order p, meaning the size of its largest clique. It is conjectured that $\omega(p) \lesssim \mathrm{pollywog}(n)$ but the best known bound is $\omega(p) \leq \sqrt{p}$ (which can be easily obtained). The only improvement to date is that, infinitely often, $\omega(p) \leq \sqrt{p}-1$, see [BRM13].

The theta function of a graph is a Semidefinite programming based relaxation of the independence number [Lov79] (which is the clique number of the complement graph). As such, it provides an upper bound on the clique number. In fact, this upper bound for Paley graph matches $\omega(p) \leq \sqrt{p}$.

Similarly to the situation above, one can define a degree 4 sum-of-squares analogue to $\theta(G)$ that, in principle, has the potential to giving better upper bounds. Indeed, numerical experiments in [GLV07] seem to suggest that this approach has the potential to improve on the upper bound $\omega(p) \leq \sqrt{p}$

Open Problem 8.4 *What are the asymptotics of the Paley Graph clique number $\omega(p)$? Can the the SOS degree 4 analogue of the theta number help upper bound it?* [34]

Interestingly, a polynomial improvement on Open Problem 6.4. is known to imply an improvement on this problem [BMM14].

References

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