Problem Set 2

IAP 2019 18.S097: Applied Category Theory

Good academic practice is expected. In particular, cooperation is encouraged, but assignments must be written up alone, and collaborators and resources consulted must be acknowledged. Please let us know if you consult the Solutions section in the book.

We suggest that you attempt all problems, but we do not expect all problems to be solved.

Question 1. *Meets and joins and adjoints too.*

Given a preorder P, recall that $P \times P$ is the preorder with pairs (p, q) of elements of P as elements, such that $(p_1, q_1) \leq (p_2, q_2)$ if $p_1 \leq p_2$ and $q_1 \leq q_2$.

(a) Show that the function $\delta \colon P \to P \times P$ sending p to (p, p) is a monotone map.
(b) Suppose that P has meets. Show that δ has a right adjoint.
(c) Suppose that δ has a left adjoint. What does that say about P?
(d) Assume P has all meets and joins. Show that $P \times P$ does too.
(e) In the case that P has all meets and joins, the adjoint functor theorem (Theorem 1.115) tells us that the diagonal preserves meets and joins. Check that this is the case.

This means that we can express meets and joins, examples of limits and colimits, in terms of Galois connections, or adjunctions. It's in general the case that one way of thinking about universal properties can be phrased in terms of others.

Question 2. *Upper sets and the Yoneda lemma.*

Let $(P, \leq)$ be a preorder, and write P^{op} for the *opposite preorder*, where we say that $p \leq q$ in P^{op} iff $q \leq p$ in P.

Given a preorder $(P, \leq)$, an *upper set* in P is a subset U of P satisfying the condition that if $p \in U$ and $p \leq q$, then $q \in U$. "If p is an element then so is anything bigger." Write $\mathsf{U}(P)$ for the set of upper sets in P. We can give the set U an order by letting $U \leq V$ if U is contained in V.

(a) Show that the set $\uparrow p := \{p' \in P \mid p \leq p'\}$ is an upper set, for any $p \in P$.
(b) Show that this construction defines a monotone map $\uparrow \colon P^{\mathrm{op}} \to \mathsf{U}(P)$.
(c) Show that if $p \leq p'$ in P if and only if $\uparrow(p') \subseteq \uparrow(p)$.
(d) Conclude that p and p' are isomorphic if and only if $\uparrow(p') = \uparrow(p)$.
(e) Draw a picture of the map $\uparrow$ in the case where P is the preorder $(b \geq a \leq c)$

This is known as the *Yoneda lemma* for preorders. The if and only if condition proved in (c) implies that, up to equivalence, to know an element is the same as knowing its upper set—that is, knowing its web of relationships with the other elements of the preorder. The general Yoneda lemma is a powerful tool in category theory. The equality of upper sets for isomorphic elements is part of the reason we consider preorders and posets to be 'the same'.

Question 3. *Skeletons and daggers.*

Consider the following definitions.

- A *skeletal* $\mathcal{V}$-category is one in which if $I \leq \mathcal{X}(x,y)$ and $I \leq \mathcal{X}(y,x)$, then $x = y$.
- The *opposite* of a $\mathcal{V}$-category $\mathcal{X}$, denoted $\mathcal{X}^{\mathrm{op}}$, is the category with the same objects as $\mathcal{X}$ but with $\mathcal{X}^{\mathrm{op}}(x,y) := \mathcal{X}(y,x)$.
- A $\mathcal{V}$-category $\mathcal{X}$ is a *dagger* $\mathcal{V}$-category if the identity function is a $\mathcal{V}$-functor $\dagger\colon \mathcal{X} \to \mathcal{X}^{\mathrm{op}}$.

(a) Show that a skeletal preorder is a poset.
(b) Show that a skeletal dagger preorder is a set.
(c) Consider the monoidal poset $R = (\mathbb{R}_{\geq 0}, \leq, +, 0)$ of nonnegative real numbers. Show that a skeletal dagger R-category is a metric space in the usual sense (Definition 2.51).

Thus the relationship between preorders and sets is similar to the relationship between Lawvere metric spaces and metric spaces.

Question 4. *Functors between preorders.*

Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{V}$-categories with sets of objects X and Y respectively. A $\mathcal{V}$-functor is a function $f\colon X \to Y$ such that $\mathcal{X}(x_1, x_2) \leq \mathcal{Y}(f(x_1), f(x_2))$ for all objects $x_1, x_2 \in X$.

(a) We said that preorders are **Bool**-categories. Prove that monotone maps are **Bool**-functors.
(b) We also said that preorders are just categories with at most one morphism between any two objects. Prove that functors (Definition 3.35) between preorders are monotone maps.

Question 5. *A presentation.*

Write down all the morphisms in the category presented by the following diagram:

$$\begin{tikzcd} \overset{A}{\bullet} \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "j"] & \overset{B}{\bullet} \arrow[d, "h"] \\ \underset{C}{\bullet} \arrow[r, "i"'] & \underset{D}{\bullet} \end{tikzcd}$$

$$f \mathbin{;} h = j = g \mathbin{;} i$$

Question 6. *Isomorphisms.*

An *isomorphism* is a morphism $f\colon A \to B$ such that there exists a morphism $g\colon B \to A$ satisfying $f \mathbin{;} g = \mathrm{id}_A$ and $g \mathbin{;} f = \mathrm{id}_B$ (Definition 3.28).

Let G be a graph, and let $\mathbf{Free}(G)$ be the corresponding free category. Somebody tells you that the only isomorphisms in $\mathbf{Free}(G)$ are the identity morphisms. Is that person correct? Why or why not?

Question 7. *Monoids as one object categories.*

Remember from Lecture 3 that a *monoid* consists of a set M, a function $*\colon M \times M \to M$ called the *monoid multiplication*, and an element $e \in M$ called the *monoid unit*, such that, when you write $*(m, n)$ as $m * n$, i.e. using infix notation, the equations

$$m * e = m, \qquad e * m = m, \qquad (m * n) * p = m * (n * p) \tag{1}$$

hold for all $m, n, p \in M$.

(a) Someone tells you that a monoid is the same as a category with one object. What does this mean?
(b) Consider the monoid $(\mathbb{N}, +)$ as a one object category. This means that $(1+2)+3$ is a morphism. Draw its wiring diagram.

Thus categories are generalizations of *both* monoids and preorders!

Question 8. *Migrate some data.*

Pick a topic of your own interest.

(a) Design two simple database schemas $\mathcal{A}$ and $\mathcal{B}$ to organize information about two closely related aspects of your topic.
(b) Describe the relationship between them as a functor $F\colon \mathcal{A} \to \mathcal{B}$.
(c) Write down an instance I of the schema $\mathcal{B}$.
(d) Write down the pullback of I along F as an instance of the schema $\mathcal{A}$.

You may want to consult Section 3.4.1 as an example.

Question 9. *Grade the p-set.*

Give a grade to this problem set, taking into account how much you learned, how interesting or fun it was, and how much time you spent on it. Explain your grade.