Complex Numbers

• Complex numbers have both real and imaginary components. A complex number $\underline{r}$ may be expressed in Cartesian or Polar forms:

$$\begin{aligned}\underline{r} &= a + jb \text{ (cartesian)} \\ &= |r|e^{\phi} \text{ (polar)}\end{aligned}$$

The following relationships convert from cartesian to polar forms:

$$\begin{aligned}\text{Magnitude } |r| &= \sqrt{a^2 + b^2} \\ \text{Angle } \phi &= \begin{cases} \tan^{-1}\frac{b}{a} & a > 0 \\ \tan^{-1}\frac{b}{a} \pm \pi & a < 0 \end{cases}\end{aligned}$$

• Complex numbers can be plotted on the complex plane in either Cartesian or Polar forms Fig.1.

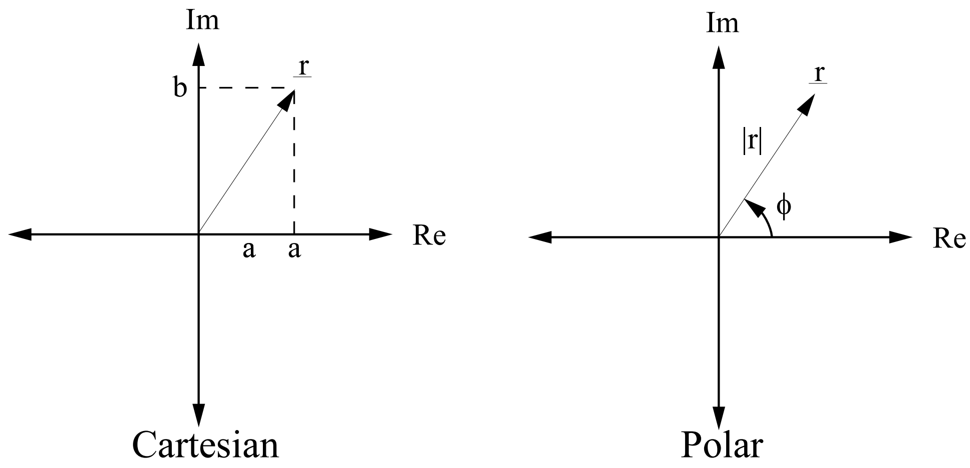

Figure 1: Complex plane plots: Cartesian and Polar forms

Euler's Identity

Euler's Identity states that

$$e^{j\phi} = \cos\phi + j\sin\phi$$

This can be shown by taking the series expansion of *sin*, *cos*, and *e*.

$$\begin{aligned}
\sin\phi &= \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + ... \\
\cos\phi &= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + ... \\
e^{j\phi} &= 1 + j\phi - \frac{\phi^2}{2!} - j\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + j\frac{\phi^5}{5!} + ...
\end{aligned}$$

Combining

$$\begin{aligned}
\cos\phi + j\sin\phi &= 1 + j\phi - \frac{(\phi)^2}{2!} - j\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + j\frac{\phi^5}{5!} + ... \\
&= e^{j\phi}
\end{aligned}$$

Complex Exponentials

• Consider the case where ϕ becomes a function of time increasing at a constant rate ω

$$\phi(t) = \omega t.$$

then $\underline{r}(t)$ becomes

$$\underline{r}(t) = e^{j\omega t}$$

Plotting $\underline{r}(t)$ on the complex plane traces out a circle with a constant radius = 1 (Fig. 2). Plotting the real and imaginary components of $\underline{r}(t)$ vs time (Fig. 3), we see that the real component is $Re\{\underline{r}(t)\} = \cos\omega t$ while the imaginary component is $Im\{\underline{r}(t)\} = \sin\omega t$.
• Consider the variable $\underline{r}(t)$ which is defined as follows:

$$\underline{r}(t) = e^{\underline{s}t}$$

where $\underline{s}$ is a complex number

$$\underline{s} = \sigma + j\omega$$

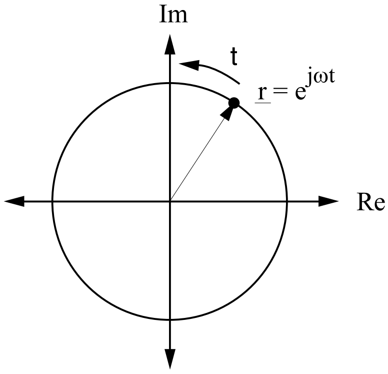

Figure 2: Complex plane plots: $\underline{r}(t) = e^{j\omega t}$

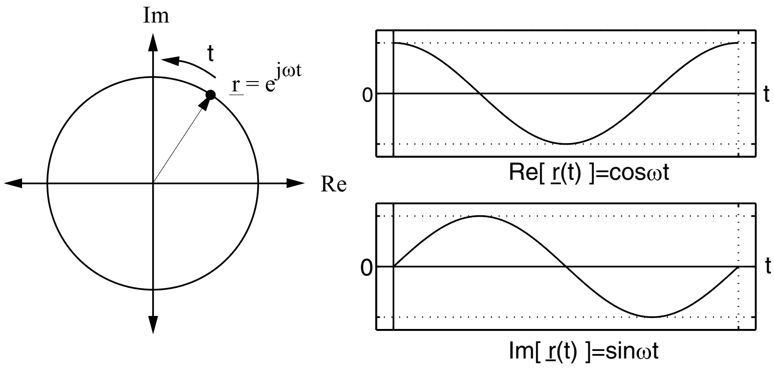

Figure 3: Real and imaginary components of $\underline{r}(t)$ vs time

- What path does $\underline{r}(t)$ trace out in the complex plane ? Consider

$$\underline{r}(t) = e^{\underline{s}t} = e^{(\sigma + j\omega)t} = e^{\sigma t} \cdot e^{j\omega t}$$

One can look at this as a time varying magnitude ($e^{\sigma t}$) multiplying a point rotating on the unit circle at frequency ω via the function $e^{j\omega t}$. Plotting just the magnitude of $e^{j\omega t}$ vs time shows that there are three distinct regions (Fig. 4):

1. $\sigma > 0$ where the magnitude grows without bounds. This condition is unstable.

2. $\sigma = 0$ where the magnitude remains constant. This condition is

called marginally stable since the magnitude does not grow without bound but does not converge to zero.

3. $\sigma < 0$ where the magnitude converges to zero. This condition is termed stable since the system response goes to zero as $t \rightarrow \infty$.

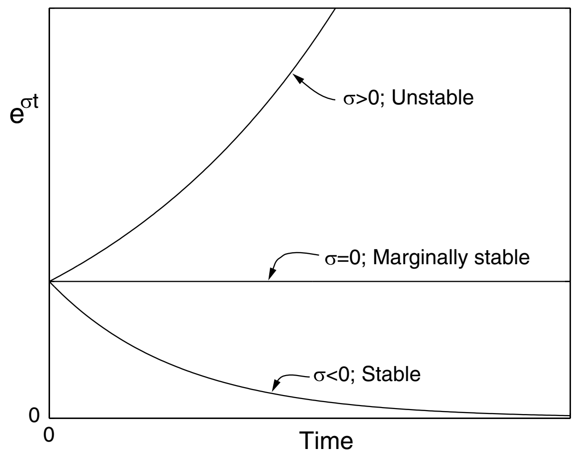

Figure 4: Magnitude $\underline{r}(t)$ for various σ.

Effect of Pole Position

The stability of a system is determined by the location of the system poles. If a pole is located in the $2nd$ or $3rd$ quadrant (which quadrant determines the direction of rotation in the polar plot), the pole is said to be stable. Figure 5 shows the pole position in the complex plane, the trajectory of $\underline{r}(t)$ in the complex plane, and the real component of the time response for a stable pole.

If the pole is located directly on the imaginary axis, the pole is said to be marginally stable. Figure 6 shows the pole position in the complex plane, the trajectory of $\underline{r}(t)$ in the complex plane, and the real component of the time response for a marginally stable pole.

Lastly, if a pole is located in either the $1st$ or $4th$ quadrant, the pole is said to be unstable. Figure 7 shows the pole position in the complex plane, the trajectory of $\underline{r}(t)$ in the complex plane, and the real component of the time response for an unstable pole.

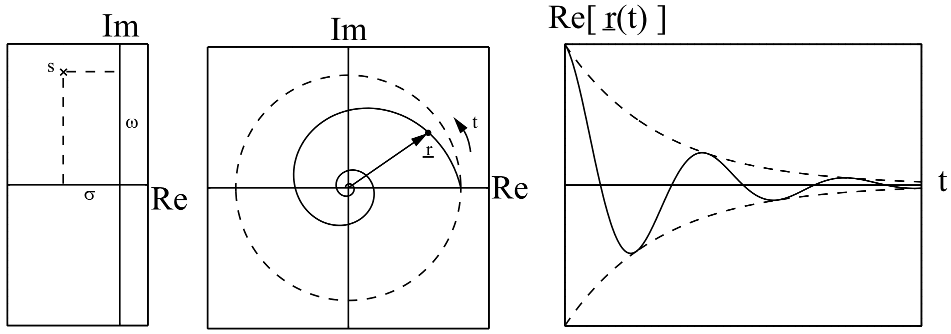

Figure 5: Pole position, $\underline{r}(t)$, and real time response for stable pole.

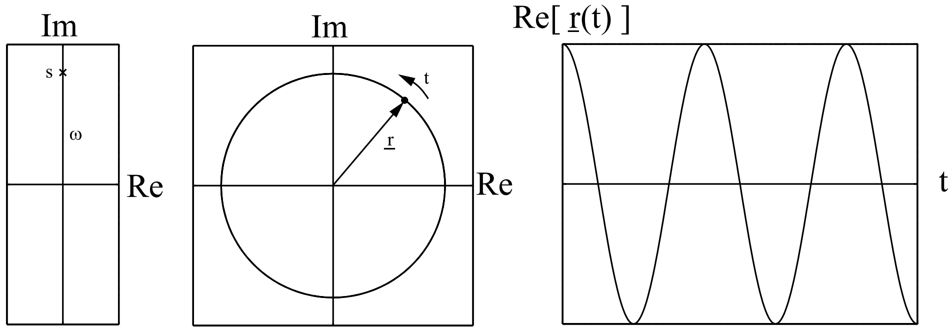

Figure 6: Pole position, $\underline{r}(t)$, and real time response for marginally stable pole.

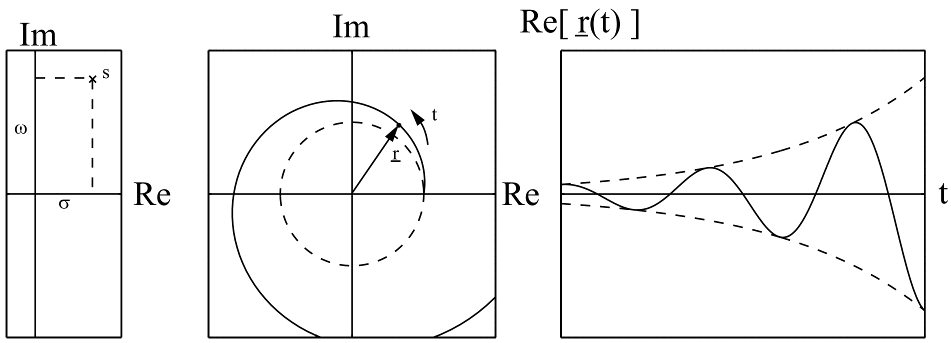

Figure 7: Pole position, $\underline{r}(t)$, and real time response for unstable pole.