2.003J/1.053J Dynamics and Control I, Spring 2007
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Recitation 2

Systems of Particles: Linear and Angular Momentum, Solution in MATLAB

Example 2 (continued)

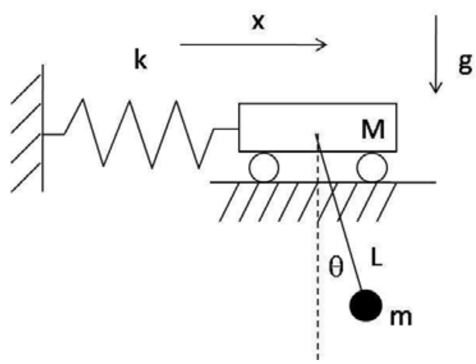

Figure 1: A spring attached to a cart with an attached pendulum. Figure by MIT OCW.

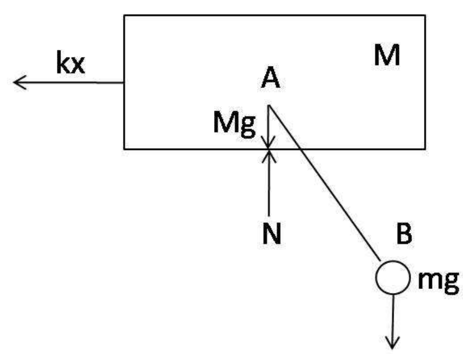

Figure 2: Free body diagram of spring, cart, and pendulum system. Figure by MIT OCW.

Coordinate System: x, θ: Generalized coordinates. Chosen to describe system well.

Kinematics

$r_A = x\hat{\imath} \quad r_B = (x + L\sin\theta)\hat{\imath} - L\cos\theta\hat{\jmath}$
$\dot{r}_A = \dot{x}\hat{\imath} \quad \dot{r}_B = (\dot{x} + L\dot{\theta}\cos\theta)\hat{\imath} + L\dot{\theta}\sin\theta\hat{\jmath}$
$\ddot{r}_A = \ddot{x}\hat{\imath} \quad \ddot{r}_B = (\ddot{x} + L\ddot{\theta}\cos\theta - L\dot{\theta}^2\sin\theta)\hat{\imath} + (L\ddot{\theta}\sin\theta + L\dot{\theta}^2\cos\theta)\hat{\jmath}$
Do not want to introduce unknown forces.

Kinetics

Linear Momentum in x direction

$$\boxed{-kx = M\ddot{x} + m\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta} \tag{1}$$

$$(F_{spring} = F_{M,x} + F_{m,x})$$

Need another equation: Angular momentum for this case. Could also use conservation of energy.
Angular Momentum: Choose A because only mg has moment about A.

$$\underline{\tau}_A = \frac{d}{dt}\underline{H}_A + \underline{v}_A \times \underline{P}$$

$$\underline{\tau}_A = -mgL\sin\theta\hat{k} \tag{2}$$

No moment for M about A because A is the center of mass of M.

$$\begin{aligned}\underline{H}_A &= \underline{AB} \times m\dot{r}_B \\ &= (L\sin\theta\hat{\imath} - L\cos\theta\hat{\jmath}) \times m[(\dot{x} + L\dot{\theta}\cos\theta)\hat{\imath} + (L\dot{\theta}\sin\theta)]\hat{\jmath} \\ &= (mL^2\dot{\theta} + mL\dot{x}\cos\theta)\hat{k}\end{aligned} \tag{3}$$

$$\begin{aligned}\underline{v}_A \times \underline{P} &= \dot{x}\hat{\imath} \times (M\dot{x}\hat{\imath} + m(\dot{x} + L\dot{\theta}\cos\theta)\hat{\imath} + m(L\dot{\theta}\sin\theta)\hat{\jmath}) \\ &= mL\dot{x}\dot{\theta}\sin\theta\hat{k}\end{aligned} \tag{4}$$

Notice: All torques in $\hat{k}$ direction.

$$(2) = \frac{d}{dt}(3) + (4)$$

Substitute and simplify.

$$\boxed{mL^2\ddot{\theta} + mL\ddot{x}\cos\theta + mgL\sin\theta = 0} \tag{5}$$

Discussion

Now we have 2 equations in 2 unknowns. How do we solve? Simulate with MATLAB. This system has certain vibrations.

Equations are nonlinear.

Examples of Linear Terms: $\dot{x}$, $\dot{\theta}$, $\ddot{x}$, $\ddot{\theta}$, x, θ
Combinations of variables: Nonlinear
Operations of variables: $\cos\theta$, $\sin\theta$, θ^2, $\dot{\theta}^2$ (Nonlinear)

In Equation 1: Nonlinear terms are $L\dot{\theta}\cos\theta$ and $-L\dot{\theta}^2\sin\theta$
In Equation 5: Nonlinear terms are $mL\ddot{x}\cos\theta$ and $mgL\sin\theta$

Equation 1 and Equation 5 contain intricate dynamics.

1965: Edward Lorentz at MIT - made a breakthrough in equations predicting weather. Ran simulations on 3 equations.

He could never get the same results twice. Uncertainty with initial conditions, especially due to vacuum tubes used then.
Any small uncertainties can be amplified by equations. "Butterfly effect."
How deterministic is the universe. Not fully deterministic. Cannot know initial condition exactly. H.U.P. (Heisenberg Uncertainty Principle). Then nonlinear equations come in and give different results.

Simulation

To simulate, reorganize equations 1 and 5. First rewrite (5) as

$$\ddot{\theta} = \frac{-1}{L}(\ddot{x}\cos\theta + g\sin\theta)$$

.
Then substitute into Equation 1:

$$\ddot{x}(M + m + m\cos^2\theta) + mg\sin\theta\cos\theta - mL\dot{\theta}^2\sin\theta + kx = 0 \qquad (6)$$

Use Equation 6 to substitute for $\ddot{x}$ in Equation 5 and obtain:

$$\ddot{\theta}(mL^2) + mL\cos\theta\left(\frac{mL\dot{\theta}^2\sin\theta - kx - mg\sin\theta\cos\theta}{M + m + m\cos^2\theta}\right) + mgL\sin\theta = 0 \qquad (7)$$

To solve these numerically:

$$x_1 = x, y_1 = \theta, x_2 = \dot{x} = \dot{x_1}, y_2 = \dot{\theta} = \dot{y_1}$$

2 Second Order Equations → 4 First Order Equations

$$\dot{x_1} = x_2$$

$$\dot{x_2} = \frac{1}{(M + m + m\cos^2 y_1)}[-mg\sin y_1 \cos y_1 + mLy_2^2 \sin y_1 - kx_1]$$

$$\dot{y_1} = y_2$$

$$\dot{y_2} = \frac{-\cos y_1}{L}\left[\frac{mLy_2^2 \sin y_1 - kx_1 - mg\sin y_1 \cos y_1}{M + m + m\cos^2 y_1}\right] - \frac{g}{L}\sin y_1$$

General Form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

where f_1, f_2, f_3, and f_4 are functions of x_1, x_2, y_1, and y_2. Set initial conditions for x_1, x_2, y_1, and y_2. Matlab can solve right-hand side for next time.

Simplest is Euler step-method for solving.

In MATLAB, you will use:

`ode45`

Rest of course: Will have some mathematical analysis of the equations of motion to acquire understanding separate from MATLAB.