2.003SC
Recitation 3 Notes: V and A of a Point in a Moving Frame

Velocity and Acceleration of a Point in a Moving Frame

The figure below shows a point B in a coordinate system $Axyz$ which is translating and rotating with respect to a fixed coordinate system $OXYZ$.
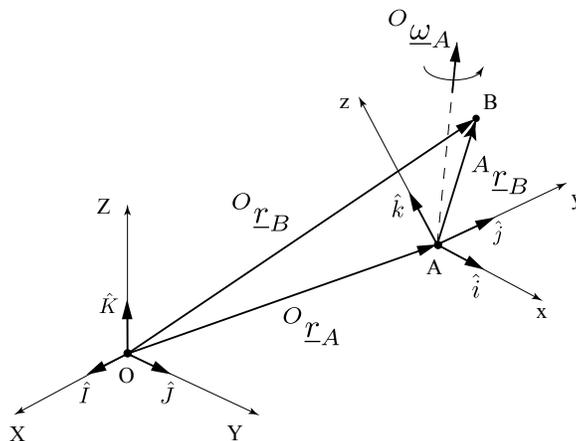
Position

The position of point B with respect to the fixed frame is given by

$$^O\underline{r}_B = {}^O\underline{r}_A + {}^A\underline{r}_B \tag{1}$$

Velocity

The velocity of point B with respect to the fixed frame is given by

$$^O\underline{V}_B = {}^O\underline{V}_A + {}^A\underline{V}_B|_{{}^O\underline{\omega}_A=0} + {}^O\underline{\omega}_A \times {}^A\underline{r}_B \tag{2}$$

where

- $^O\underline{V}_A$ is the velocity of the origin of the moving coordinate system

- $^A\underline{V}_B|_{{}^O\underline{\omega}_A=0}$ is the velocity of point B with respect to point A as seen by an observer attached to the moving coordinate system, i.e. were the moving coordinate system not rotating

- $^O\underline{\omega}_A \times {}^A\underline{r}_B$ is the velocity of point B with respect to point A due to the rotation of the moving coordinate system

Acceleration

The acceleration of B with respect to the fixed frame is given by

$$^O\underline{a}_B = {}^O\underline{a}_A + {}^A\underline{a}_B|_{\omega_A=0} + 2{}^O\underline{\omega}_A \times {}^A\underline{v}_B + {}^O\underline{\dot{\omega}}_A \times {}^A\underline{r}_B + {}^O\underline{\omega}_A \times \left({}^O\underline{\omega}_A \times {}^A\underline{r}_B\right) \tag{3}$$

where

- $^O\underline{a}_A$ is the acceleration of the origin of the moving coordinate system

- $^A\underline{a}_B|_{\omega_A=0}$ is the acceleration of point B as seen by an observer attached to the rotating coordinate system

- $2{}^O\underline{\omega}_A \times {}^A\underline{v}_B$ is the <u>Coriolis</u> acceleration

- $^O\underline{\dot{\omega}}_A \times {}^A\underline{r}_B$ is the <u>Eulerian</u> acceleration

- $^O\underline{\omega}_A \times \left({}^O\underline{\omega}_A \times {}^A\underline{r}_B\right)$ is the <u>centripetal</u> acceleration

Velocity and Acceleration of a Point in a Plane - Polar Coordinates

The figure below shows a point P moving along a path in a fixed coordinate system $OXYZ$. The path is described in polar coordinates, r and θ, with unit vectors $\hat{r}, \hat{\theta}$ in the r-direction and θ-direction, respectively.
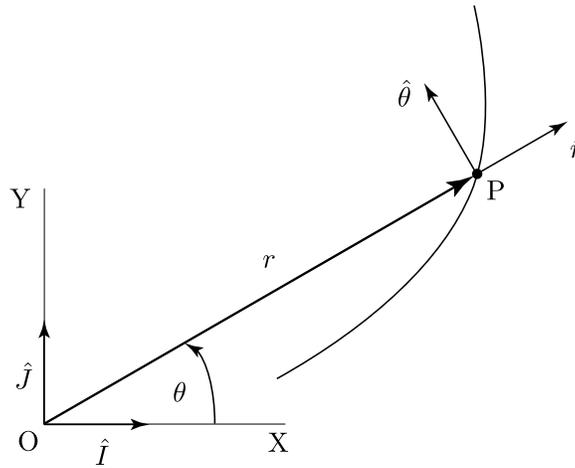
Position

The point's <u>position</u>, $^{O}\underline{r}_{P}$, is given by

$$^{O}\underline{r}_{P} = r\hat{r} \tag{1}$$

Note that $^{O}\underline{r}_{P}$ is a vector with magnitude r in the $\hat{r}$ direction.

Velocity

The velocity of point P with respect to the fixed frame is obtained by differentiation.

$$^{O}\underline{V}_{P} = \underline{v} = {}^{O}\frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\dot{\hat{r}}$$

It can be shown that the time derivatives of the unit vectors, $\hat{r}$ and $\hat{\theta}$, rotating with angular velocity $\dot{\theta}$ are

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} \qquad and \qquad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$

So the <u>velocity</u> of point P is given by

$$\underline{v} = (\dot{r})\hat{r} + (r\dot{\theta})\hat{\theta} \tag{2}$$

or

$$\underline{v} = v_r\hat{r} + v_\theta\hat{\theta} \qquad \text{where} \qquad v_r = \dot{r} \qquad \text{and} \qquad v_\theta = r\dot{\theta}$$
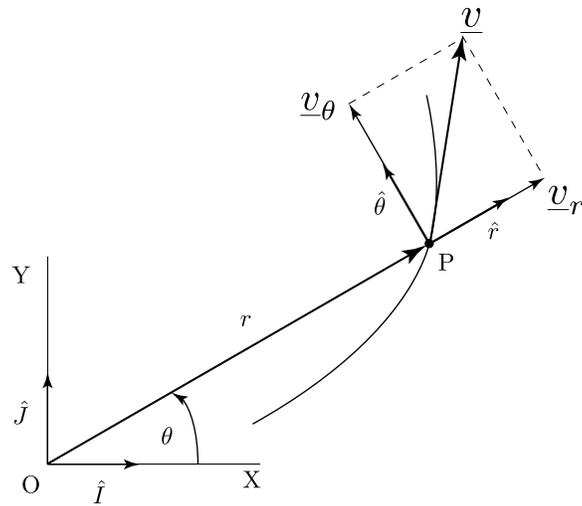
Acceleration

We can differentiate again to obtain the acceleration,

$$^O\underline{a}_P = \underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \tag{3}$$

or

$$\underline{a} = a_r\hat{r} + a_\theta\hat{\theta}$$

where

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad \text{and} \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
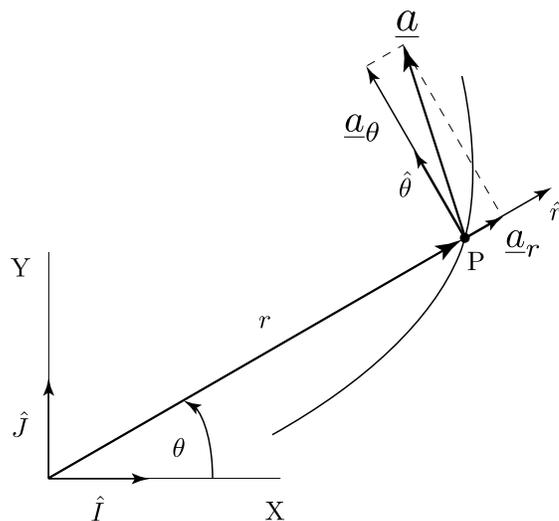
Amusement Park Ride - Problem Statement

A ride at an amusement park consists of four symmetrically located seats, driven to rotate about the vertical axis A at a constant rate of ω_2 rev/min, with respect to the supporting arm, r_1 . The vertical axis, O , is driven by another motor at a constant rate of ω_1 rev/min, with respect to ground. All four seats are a distance, r_2 , from the A axis and the O and A axes are parallel as shown in the figure. Assume the existence of a fixed coordinate system, $OXYZ$, attached to the ground and a rotating coordinate system, $Axyz$, attached to the component holding the passengers as shown the figure. Find the acceleration with respect to ground of each of the four passengers at the instant shown.
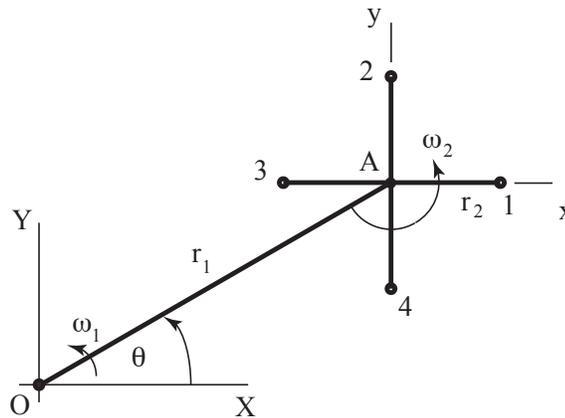
Amusement Park Ride - Solution - Accelerations

The general expression for the acceleration of B with respect to the fixed frame is given by

$$^{O}\underline{a}_B = {}^{O}\underline{a}_A + {}^{A}\underline{a}_B|_{\omega_A=0} + 2({}^{O}\underline{\omega}_A \times {}^{A}\underline{v}_B) + {}^{O}\underline{\dot{\omega}}_A \times {}^{A}\underline{r}_B + {}^{O}\underline{\omega}_A \times \left({}^{O}\underline{\omega}_A \times {}^{A}\underline{r}_B\right) \tag{1}$$

Correspondence

- point B in equation (1) will correspond to each of the passengers at points 1 through 4, in turn

- the angular velocity of the moving frame, $^{O}\underline{\omega}_A = \omega_1 + \omega_2$

The first term, $^O\underline{a}_A$, describes the acceleration of point A with respect to frame O as shown below.

$$|^O\underline{a}_A| = r_1\omega_1{}^2$$

or
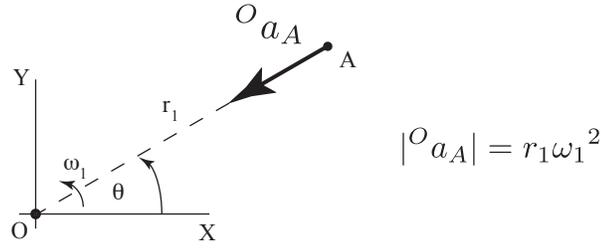
$$^O\underline{a}_A = [-r_1\omega_1^2 cos\theta]\hat{I} + [-r_1\omega_1^2 sin\theta]\hat{J}$$

The second term, $^A\underline{a}_B|_{\omega_A=0}$, is zero because the point of interest is not accelerating within the moving frame, i.e. $^A\underline{a}_B = 0$.

The third term, $2(^O\underline{\omega}_A \times {}^A\underline{v}_B)$, is zero because the point of interest is not translating within the moving frame, i.e. $^A\underline{v}_B = 0$.

The fourth term, $^O\underline{\dot{\omega}}_A \times {}^A\underline{r}_B$, is zero because the angular accelerations are zero.

Only the last term, $^O\underline{\omega}_A \times (^O\underline{\omega}_A \times {}^A\underline{r}_B)$, i.e. the centripetal acceleration, is non-zero as shown below.

$$|^A\underline{a}_i| = r_2(\omega_1 + \omega_2)^2$$
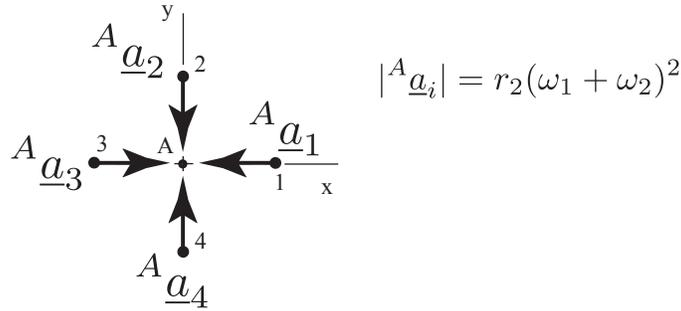
or

$$^A\underline{a}_1 = -r_2(\omega_1 + \omega_2)^2\hat{I}$$
$$^A\underline{a}_2 = -r_2(\omega_1 + \omega_2)^2\hat{J}$$
$$^A\underline{a}_3 = r_2(\omega_1 + \omega_2)^2\hat{I}$$
$$^A\underline{a}_4 = r_2(\omega_1 + \omega_2)^2\hat{J}$$

so

$$^O\underline{a}_1 = [-r_1\omega_1^2 cos\theta - r_2(\omega_1 + \omega_2)^2]\hat{I} + [-r_1\omega_1^2 sin\theta]\hat{J}$$
$$^O\underline{a}_2 = [-r_1\omega_1^2 cos\theta]\hat{I} + [-r_1\omega_1^2 sin\theta - r_2(\omega_1 + \omega_2)^2]\hat{J}$$
$$^O\underline{a}_3 = [-r_1\omega_1^2 cos\theta + r_2(\omega_1 + \omega_2)^2]\hat{I} + [-r_1\omega_1^2 sin\theta]\hat{J}$$
$$^O\underline{a}_4 = [-r_1\omega_1^2 cos\theta]\hat{I} + [-r_1\omega_1^2 sin\theta + r_2(\omega_1 + \omega_2)^2]\hat{J}$$

Block Sliding on Rotating Bar - Problem Statement

The bar OA rotates about a horizontal axis through O with a constant counterclockwise velocity $\omega = 3$ rad/sec, in the presence of gravity. As it passes the position $\theta = 0$, a small block of mass, m , is placed on it at a radial distance $r = 18$ in. If the block is observed to slip at $\theta = 50^o$,

1. determine the coefficient of static friction, μ_s , between the block and the bar.

2. determine the torque applied to the bar at O.
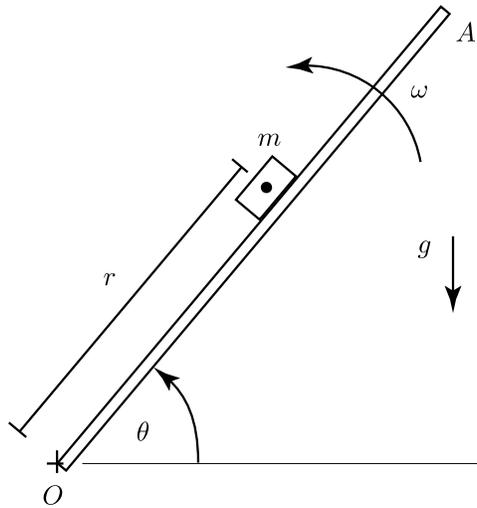
Block Sliding on Rotating Bar - Solution

The free body diagram, depicting the external forces acting on the mass is shown in the figure below.
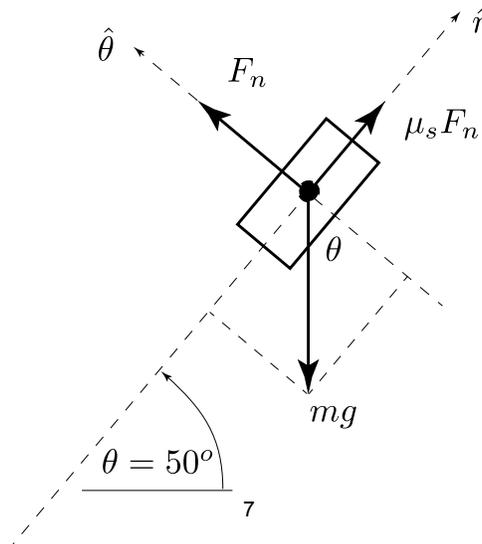
Note:

- the unit vectors $\hat{r}$ and $\hat{\theta}$

- the mass's weight (i.e. the force gravity exerts on the mass), $F_w = mg$

- the normal force (of the bar on the mass), F_n

- the friction force (of the bar on the mass), $F_f = \mu_s F_n$

Accelerations

In polar coordinates, the acceleration of the mass is given by

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \tag{1}$$

or

$$\underline{a} = (-r\omega^2)\hat{r} + (0)\hat{\theta} \tag{2}$$

Newton's Laws

Applying Newton's Laws in the $\hat{r}$ and $\hat{\theta}$ directions,

$$\Sigma \underline{F}_r = m\underline{a}_r$$
$$\Sigma \underline{F}_\theta = m\underline{a}_\theta$$

we know that

$$\mu_s F_n - mgsin\theta = -mr\omega^2$$
$$F_n - mgcos\theta = 0$$

Coefficient of Static Friction

Eliminating F_n, we find that

$$\mu_s = tan\theta - \frac{r\omega^2}{gcos\theta}$$
$$\mu_s = 0.54$$

Torque

The free-body diagram for the bar is
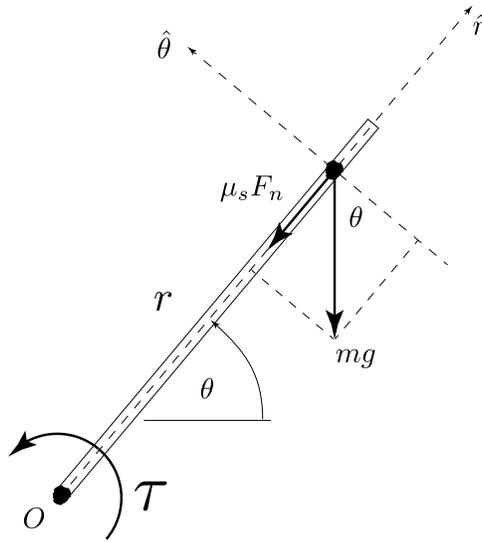
Summing torques in the $\hat{k}$ direction,

$$\tau - mg\cos\theta = 0$$

The torque applied to the bar at O is

$$\underline{\tau} = rF_n\hat{k} = r(mg\cos\theta)\hat{k}$$

2.003SC / 1.053J Engineering Dynamics
Fall 2011