2.003SC Engineering Dynamics

Problem Set 1 Solutions

A general approach to problem-solving:

Most problems in dynamics can be reduced to three principal steps.

1. Describe the motion,
2. Apply the appropriate physical laws,
3. Apply the appropriate mathematics.

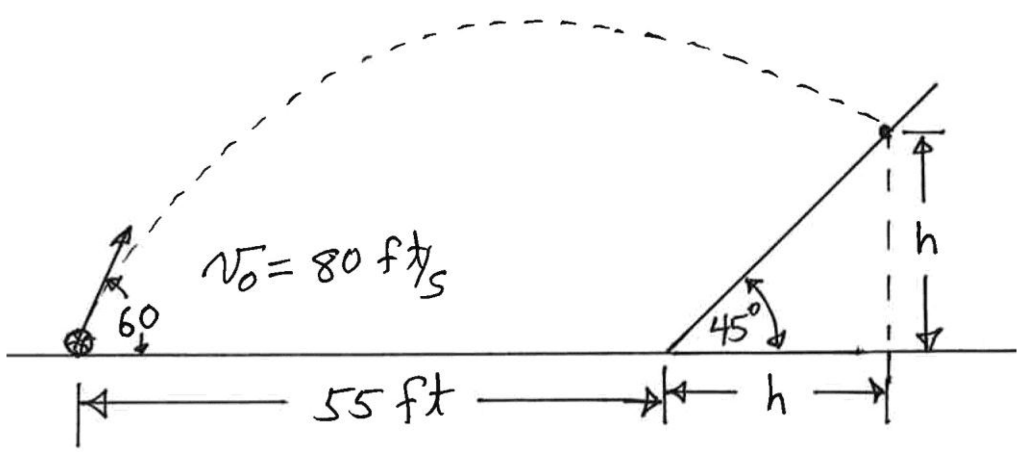
We shall routinely apply these three steps to most of the problems in this course. Beginning with the first problem, this will be done in some detail to provide an example. In later problem sets you will be expected to be able to do this for yourself.

Problem 1

This is a projectile motion problem. There is no air resistance and therefore gravity is the only force acting on the football. We shall see that the solution to this problem requires finding and solving an equation of motion. We begin by applying the first of the three steps.

Describe the motion:

In the figure given with the problem statement, shown below, the unknown location of the landing point of the ball is specified in terms of the unknown height, „h"and horizontal distance „h", measured from a point at the bottom of the stands. „h"is one unknown.



To make this problem easy to solve it is best to use a Cartesian xyz coordinate system located at the launch point of the ball. The final step in describing the motion is to specify the initial conditions of displacement and velocity at time t = 0.

The initial position values are: $x_o=0$ and $y_o=0$.

The initial velocity is given as 80 ft./s at an angle of 60° from the horizontal. This can be broken into x and y vector components as follows:

$$\dot{x}_{ox} = 80\cos(60^\circ)\hat{i} = 40 \text{ ft/s.}\hat{\text{i}}$$
$$\dot{\text{y}}_{ox} = 80\sin(60^\circ)\hat{j} = 69.28 \text{ ft/s.}\hat{\text{j}}$$

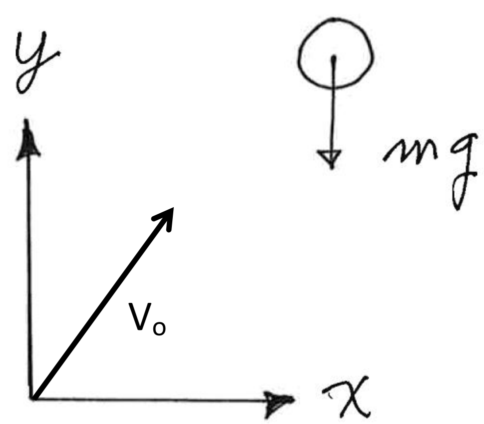
Choose appropriate physical laws:

This projectile motion problem may be solved by direct application of Newton's second law.

$$\sum_{Ext}\vec{F} = \frac{d\vec{P}}{dt} = m\vec{a}$$

Application of Newton"s 2nd law is often facilitated by a free body diagram (fbd), which is shown in the figure below. From the fbd two equations of motion may be deduced, one for each vector component of the acceleration. An equation of motion simply relates external forces to the resulting acceleration according to Newton's second law:

1. $\sum_x F = m\ddot{x}\hat{i} = 0 \Rightarrow \ddot{x} = 0$
2. $\sum_y F = m\ddot{y}\hat{j} = -mg\hat{j} \Rightarrow \ddot{y} = -g$



All that remains is to do the appropriate math:

In both the equations of motion from above the acceleration is a constant and therefore simple to integrate. This leads to two algebraic equations in two unknowns: one is the unknown distance "h" and the other is the time of flight of the ball.

First integrating Equation 1 in the x direction leads to an expression for the velocity in the x direction and then integrating again leads to an equation for the distance traveled in the x direction. Plug in the appropriate values of initial conditions as they come up in the integrals:

3. $\dot{x}(t) = \int_0^t \ddot{x} dt + \dot{x}_o = 0 + \dot{x}_o = 40\,\text{ft/s}$

4. $\text{x(t)} = \int_0^t \dot{x} dt = \dot{x}_o t + x_o = 40t + 0 = 55 + \text{h}$ feet

$\Rightarrow$ 5. h=40t-55 feet

Integrating equation 2, provides the second equation which is needed to find the two unknowns 'h' and 't'.

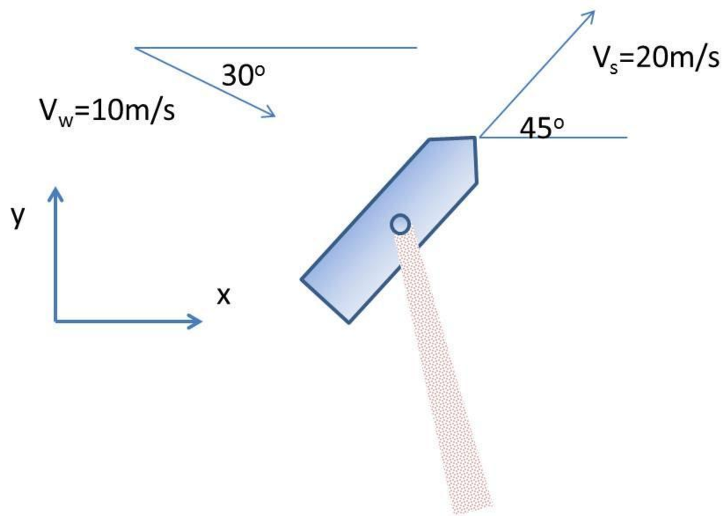
6. $\dot{y}(t) = \int_0^t \ddot{y} dt = -gt + \dot{y}_o$, where g=32.17 ft/s^2

7. $y(t) = \int_0^t \dot{y} dt = -\frac{1}{2} gt^2 + \dot{y}_o t + y_o = -\frac{1}{2} gt^2 + 69.28t + 0 = \text{h}$ (feet)

Equations 5 and 7 provide two equations in two unknowns 'h' and 't' Expressing 'h' in terms of 't' in equation 7 leads to a quadratic equation in 't', which, when solved, provides the time at which the ball strikes the bleachers, t=2.971 seconds. When this value of time is substituted into equation 5 a value for 'h' is found to be a=63.85 feet. The total distance travelled in the x direction is then provided by equation 4. x=55+h=55+63.85=118.85 feet

Problem 2

A ship travels at constant speed of $V_{S/O}$=20 m/s. The wind is blowing at a speed of $V_{W/O}$=10 m/s as shown in the figure. Determine the magnitude and direction of the velocity of the smoke as seen from an observer on the ship. Assume the smoke has a negligible upwards velocity. In other words it travels parallel to the surface of the water.



The key to doing this problem is to use the standard vector velocity equation:

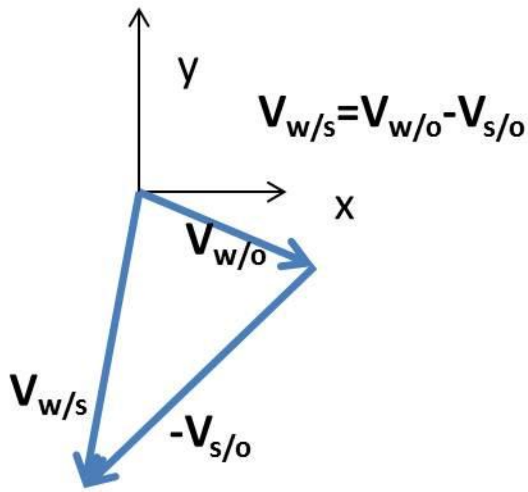
$\vec{V}_{W/O} = \vec{V}_{W/S} + \vec{V}_{S/O}$ where
$\vec{V}_{W/O} =$ the velocity of the wind w.r.t. the inertial frame,
$\vec{V}_{W/S} =$ the velocity of the wind w.r.t the ship,
$\vec{V}_{S/O} =$ the velocity of the ship w.r.t. the inertial frame.

An important fact is that the smoke travels with the wind at the velocity of the wind.

Therefore
$\vec{V}_{W/O}$ also = the velocity of the smoke w.r.t. the inertial frame,
$\vec{V}_{W/S}$ also = the velocity of the smoke w.r.t the ship.

Solving the velocity equation for the unknown $V_{W/S}$, the velocity of the wind with respect to the ship one arrives at:

$\vec{V}_{W/S} = \vec{V}_{W/O} - \vec{V}_{S/O}$ This is shown in the diagram below.



$\vec{V}_{W/S} = \vec{V}_{W/O} - \vec{V}_{S/O}$ where

$\vec{V}_{W/O} = 10\sqrt{\frac{3}{2}}\hat{i} + 10\frac{1}{2}\hat{j}$

$\vec{V}_{S/O} = 20\sqrt{2}\hat{i} + 20\sqrt{2}\hat{j}$

Therefore

$\vec{V}_{W/S} = [10\sqrt{\frac{3}{2}} - 20\sqrt{2}]\hat{i} + [5 - 20\sqrt{2}]\hat{j}$

Problem 3.

The problem is to find the velocity of the bar riding without slip on the top of the roller.

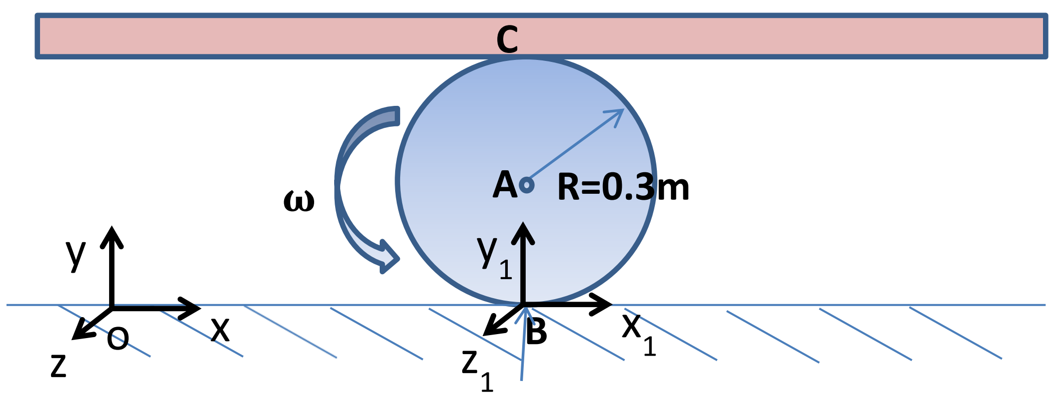
Describe the motion

Assign an inertial coordinate system, O_{xyz}, as shown in the figure. It is fixed to the ground.

The velocity of the bar is the same as the velocity of the point of contact between the bar and the roller. Thus the problem reduces to finding the velocity of a point on a translating and rotating rigid body. Such problems will come up many times during this course on dynamics and therefore a general means of dealing with such problems is useful.

In general the approach is to attach a coordinate system to the body with its origin at a point whose velocity is known with respect to an inertial frame or is easy to obtain. In this case the velocity of the point of contact with the ground at B is known; it is zero, because the ground is not moving. This type of point is known as an instantaneous center of rotation. In this problem this is the only point on the roller that has a known velocity.

A second coordinate system B_{x1y1z1} is attached to the roller at B. This is a moving coordinate system which rotates with the roller.



This problem is solved by using the standard form of the velocity equation for a point on a translating and rotating rigid body. The standard velocity equation is given below and defined carefully one time in this first problem set.

$\vec{V}_{C/O} = \vec{V}_{B/O} + \vec{V}_{C/B} + \vec{\omega} \times \vec{r}_{C/B}$, where

$\vec{V}_{C/O}$ = velocity of point C in the O_{xyz} frame

$\vec{V}_{B/O}$ = velocity of point B in the O_{xyz} frame

$\vec{V}_{C/B}$ = velocity of point C relative to point B in the B_{x1y1z1} frame which is attached to the roller

A comment about notation: Take for example $\mathbf{V_{C/O}}$. It is a vector, indicated in an ordinary sentence by making it in **bold** or by putting an arrow over the symbol. In equations vectors will usually be indicated with an arrow over the character. The diagonal '/' symbol means with respect to. Hence C/O is intended to mean the velocity of point C with respect to the O_{xyz} inertial frame of reference. Note: the velocity of a point is always defined with respect to a frame of reference. However, it is correct to compute the relative velocity between two points with respect to a reference frame as shall be demonstrated in a moment. With these definitions the meaning of $\mathbf{V_{C/O}}$ and $\mathbf{V_{B/O}}$ are clear. This leaves two terms in the velocity equation requiring further explanation: $\mathbf{V_{C/B}}$ and $\vec{\omega}\times\vec{r}_{C/B}$.

$\mathbf{V_{C/B}}$ is defined as the velocity of point C with respect to the B_{x1y1z1} frame, which is fixed to the roller. In his book "Advanced Dynamics" Prof. James Williams uses the notation $\mathbf{V_{rel}}$ for this particular term, because it means the velocity of point C „relative" to the rigid body as you could observe it if you were sitting at a fixed point on the rigid body and rotating and translating with the rigid body. If point C is also a fixed point on the body, then $\mathbf{V_{C/B}} = 0$. It is possible to have a situation in which point C moves relative to the rigid body. Imagine an insect clinging to the roller at point C. The insect is scrambling to get out of the way to keep from getting crushed under the bar. Let"s say that it is moving in the radial direction toward the center at 0.05 m/s. Then $\vec{V}_{C/B} = -0.05\,\hat{j}_1\, m/s$ where $\hat{j}_1$ is the unit vector in the y_1 direction in the B_{x1y1z1} frame which is attached to the body. Another way of defining the meaning of $\mathbf{V_{C/B}}$ is to think of it as a partial derivative in which the effects of the rate of rotation of the position vector $\mathbf{r_{C/B}}$ are excluded from the process of taking the derivative. In other words, while taking the derivative, the rate of the rotation of the vector $\mathbf{r_{C/B}}$ is for the purposes of the partial derivative assumed to be zero. Hence:

$\vec{V}_{C/B} \equiv \left(\dfrac{\partial\vec{r}_{C/B}}{\partial t}\right)_{|\omega=0}$ It is also true that this term is the relative velocity between points B and the insect at C as seen from the O_{xyz} inertial frame, excluding the contribution of rotation. The contribution of rotation to the relative velocity must be accounted for separately by the term $\vec{\omega}\times\vec{r}_{C/B}$, which expresses the relative velocity between points C and B, caused only by the rotation of the rigid body and as seen from the point of view of an observer in the non-rotating O_{xyz} inertial frame. From the point of view of an observer attached to the body the relative velocity caused by rotation appears to be zero.

It is the sum of the two terms $\vec{V}_{C/B} \equiv \left(\dfrac{\partial\vec{r}_{C/B}}{\partial t}\right)_{|\omega=0}$ and $\vec{\omega}\times\vec{r}_{C/B}$ that yields the total relative velocity between the insect at C and the point B, as seen from the inertial frame of reference. Putting this all together in this problem we arrive at the answer that was originally sought, the velocity of point C with respect to the O_{xyz} inertial frame.

$\vec{V}_{C/O} = \vec{V}_{B/O} + \vec{V}_{C/B} + \vec{\omega} \times \vec{r}_{C/B}$, where

$\vec{V}_{B/O} =$ 0 because it is an instantaneous center or rotation,

$\vec{V}_{C/B} =$ velocity of point C relative to point B in the B_{x1y1z1} frame

which is attached to the roller = 0.

$$\therefore \quad \vec{V}_{C/O} = 0 + 0 + \vec{\omega} \times \vec{r}_{C/B} = 4\frac{\text{radians}}{\text{sec}}\hat{k} \times 2(0.3m)\hat{j} = -2.4\frac{m}{s}\hat{i}$$

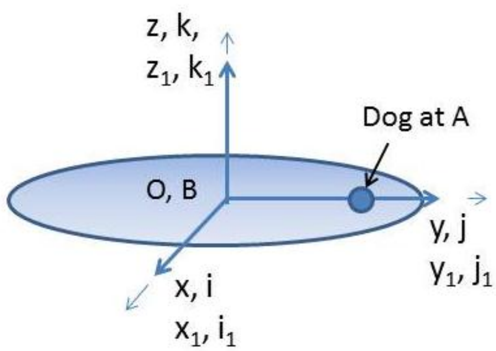
If we wished to compute the velocity of the insect then $\mathbf{V}_{\mathbf{C/B}}$=- - 0.05 m/s $\mathbf{j}_{\mathbf{1}}$, and $\mathbf{V}_{\mathbf{C/O}}$ = (-2.4$\mathbf{i}_{\mathbf{1}}$ – 0.05$\mathbf{j}_{\mathbf{1}}$)m/s.

This problem was entirely kinematics and vector mathematics. No physical laws needed to be invoked.

The use of the concept of an instantaneous center of rotation made this problem particularly simple, because it meant only a single term in the velocity equation had to be evaluated because the others were conveniently zero.

Problem 4

A dog runs outwards in a straight radial line on a rotating platform. At the instant shown the dog is at position 'A'at a radius of r=5.0 feet form B. At this time the dog has an outward radial speed of 2.0ft/s. The platform rotates at a rate of $\vec{\omega} = 0.5\frac{\text{rad}}{s}\hat{k}$. Find the speed of the dog relative to the O_{xyz} fixed inertial frame of reference. The B_{x1y1z1} frame is attached to the platform.



Describe the motion:

The fixed inertial reference frame, O_{xyz} is located at the center of the platform. It has unit vectors $\hat{i}, \hat{j}, \hat{k}$. A rotating reference frame, B_{x1y1z1} is also located at the center but is fixed to the platform and rotates with it. This frame has unit vectors $\hat{i}_1, \hat{j}_1, \hat{k}_1$. At the particular moment in time being considered, the dog is at point A and the axes of the fixed and rotating systems are coincident. The x and x_1 axes line up with one another, as do the y and y_1 axes. The z and z_1 are coincident at all times. This means their unit vectors also line up and are the same at this moment in time.

The velocity equation for use when working with rotating and translating coordinate systems is appropriate to use in this case.

$\vec{V}_{A/O} = \vec{V}_{B/O} + \vec{V}_{A/B} + \vec{\omega}\times\vec{r}_{A/B}$, where

$\vec{V}_{A/O}$ = unknown velocity of point A(dog) in the O_{xyz}inertial frame.

$\vec{V}_{B/O} = 0$ = velocity of translation of the moving coordinate system B_{x1y1z1}, which is attached to the platform at O, but rotates.

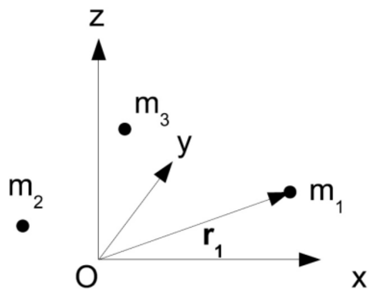
$\vec{V}_{A/B} = 2\text{ ft/s}\,\hat{\text{j}}_1$ = velocity of point A(dog) relative to the rotating B_{x1y1z1} frame.

$\vec{\text{r}}_{A/B} = 5\text{ ft}\,\hat{\text{i}}_1$ = position vector of the dog at A in the B_{x1y1z1} frame.

$\vec{\omega}\times\vec{r}_{A/B} = 0.5\text{rad/s}\,\hat{\text{k}}\times 5\text{ ft}\,\hat{\text{j}}_1 = -2.5\text{ft/s}\,\hat{\text{i}}_1$

$\vec{V}_{A/O} = 0 + 2\text{ ft/s}\,\hat{\text{j}}_1 - 2.5\text{ft/s}\,\hat{\text{i}}_1 = 2\text{ ft/s}\,\hat{\text{j}} - 2.5\text{ft/s}\,\hat{\text{i}}$

Problem 5



The data for the problem are: m_1=3m, m_2=m, m_3=5m, m=1 kg.

$R_{1/0}$=(2i, 2j, 5k), $R_{2/0}$=(-2i, 4j, 3k), $R_{3/0}$=(4i, 0j, 2k)

$V_{1/0}$=(1i, -4j, 3k)V_o, $V_{2/0}$=(i, 0j, 0k)V_o, $V_{3/0}$=(-2i,-j, -k)

V_o=10 m/s.

a. Find the center of mass of the three particles:

$$r_{G/O}=\frac{\sum_i m_i r_{i/O}}{\sum_i m_i}=\frac{[3m(2i,02j,5k)+m(-2i,4j,3k)+5m(4i,0j,0k)]}{9m}=2.67i+1.11j+3.11k$$

b. Compute the total linear momentum of the system of particles.

$$\vec{P}=\sum_i m_i V_{i/O}=3m(i,-4j,3k)10\frac{kg-m}{\sec}+m(i,0j,0k)10\frac{kg-m}{\sec}+5m(-2i,-1j,-1k)10\frac{kg-m}{\sec}$$

$$\vec{P}=[-60i-170j+40k]\frac{kg-m}{\sec}$$

c. Compute the velocity of the center of mass:

$$V_{G/O}=\frac{\sum_i m_i V_{i/O}}{\sum_i m_i}=(-6.67i-18.89j+4.44k)m/s$$

$$|V_{G/O}|=20.52\,m/s$$

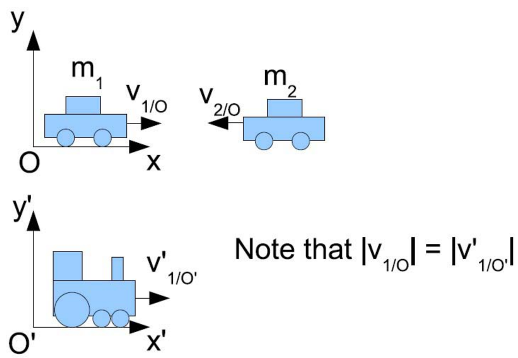
d. Compute the total kinetic energy of the system:

$$T_{total}=\frac{1}{2}\sum_i m_i V_{i/O}\bullet V_{i/O}=5450\frac{kg-m^2}{s^2}$$

If one computes $\frac{1}{2}M_{total}V_G\bullet V_G=\frac{1}{2}9kg(20.52)^2=1894.81\frac{kg-m}{s^2}$

Which is considerably less than the sum of the kinetic energies taken one particle at a time.

Problem set 1, Problem 6:



The problem data is given as:

m_1=1000kg , m_2=2000kg

$V_{1/O}$=25i m/s, $V_{2/O}$=-25i m/s, $V'_{1/O}$=25i m/s

a. Compute the change in the total linear momentum of the two body system and the change in the total kinetic energy of the system before and after the collision.

i. By considering the two cars as one system, it is possible to say that the total system linear momentum is constant, because there are no net external forces. This also means that the velocity of the center of mass before and after the collision must remain constant.

$$\vec{P}_{before} = \vec{P}_{after} = 1000kg(25\frac{m}{s})\hat{i} - 2000kg(25\frac{m}{s})\hat{i} = -25000(\frac{kg-m}{s})\hat{i} = (m_1 + m_2)V_{G/O}$$

Therefore the final velocity of the two cars stuck together is the velocity of the center of mass. $V_{G/O,after} = 25000\hat{i}\frac{kg-m}{s}/(m_1+m_2) = 8.3333\frac{m}{s}\hat{i} = V_{G/O,before}$ There is no change in the system linear momentum before and after the collision.

ii. The total kinetic energy is given by the sum of the energies of the two cars. After the collision the two cars stick together and the kinetic energy is computed using the velocity of the center of mass.

$$T_{before} = \frac{1}{2}m_1V_{1/O}^2 + \frac{1}{2}m_2V_{2/O}^2 = 937,500\frac{kg-m^2}{s^2}$$

$$\mathrm{T}_{after} = \frac{1}{2}(m_1+m_2)V_{G/O}^2 = 104,167\frac{kg-m^2}{s^2}$$

$$\Delta T = \mathrm{T}_{after} - T_{before} = -833,333\frac{kg-m^2}{s^2}$$

The system loses energy as heat during the collision.

b. Compute the change in the linear momentum and the change in the kinetic energy during the collision from the point of view of an observer traveling in a reference frame attached to a train travelling at constant velocity equal to that of m_1. First, describe the motion. Even though this is simple enough that one can reason out the correct relative velocities, use the vector velocity equation involving a moving but not rotating coordinate system. This develops good practice that will be useful in more difficult relative velocity problems.

$$\vec{V}_{train/O} = \vec{V}_{1/O} = 25\frac{m}{s}\hat{i}$$

$$\vec{V}_{1/O} = \vec{V}_{train/O} + \vec{V}_{1/train}$$
$$\Rightarrow \vec{V}_{1/train} = 0$$

$$\vec{V}_{2/O} = \vec{V}_{train/O} + \vec{V}_{2/train}$$
$$\Rightarrow \vec{V}_{2/train} = \vec{V}_{2/O} - \vec{V}_{train/O} = \vec{V}_{2/O} - \vec{V}_{1/O} = -50\frac{m}{s}\hat{i}$$

With the above we have the velocities of the two cars as observed from the train.

i. As before the change in linear momentum before and after the collision as seen from the train must stay constant. The train is travelling at constant speed and therefore is an inertial reference frame. Since there are no net external forces acting on the system of two cars, the momentum must stay constant.

$$\vec{P}_{before/train} = \vec{P}_{after/train} = m_1\vec{V}_{1/train} + m_2\vec{V}_{2/train} = 1000kg \bullet 0\frac{m}{s}\hat{i} - 2000kg \bullet 50\frac{m}{s}\hat{i} = -100,000\frac{kg-m}{s}\hat{i}$$
$$\vec{P}_{after/train} = (m_1 + m_2)\vec{V}_{G/train}$$
$$\Rightarrow \vec{V}_{G/train} = -33.33\frac{m}{s}$$

The change in linear momentum is zero as before when viewed from the non-moving inertial reference frame.

ii. The kinetic energy computation before and after the collision is as follows.

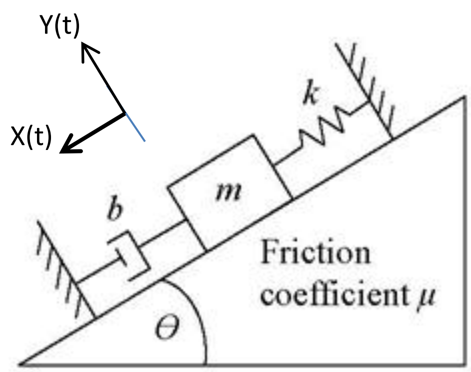
$$T_{before} = \frac{1}{2}m_1V_{1/train}^2 + \frac{1}{2}m_2V_{2/train}^2 = 0 + \frac{1}{2}2000kg(-50m/s)^2 = 2,500,000\frac{kg-m^2}{s^2}$$
$$T_{after} = \frac{1}{2}(m_1 + m_2)V_{G/train}^2 = \frac{1}{2}(3000kg)(-33.33\frac{m}{s})^2 = 1,666,667\frac{kg-m^2}{s^2}$$
$$\Delta T = T_{after} - T_{before} = -833,333\frac{kg-m^2}{s^2}$$

This is exactly the same energy loss as computed from the non-moving inertial frame.

The generalization that can be made is that the principles of conservation of energy and conservation of momentum hold for any inertial frame of reference, even two frames travelling at different constant velocities. Changes in linear momentum or kinetic energy will be the same regardless of differences in velocity, as long as both frames are inertial reference frames.

Problem Set 1, problem 8.

The problem statement asks us to obtain an equation of motion for the mass spring damper system shown below.



Begin by describing the motion:

The mass is constrained to move up and down the slope. It has one degree of freedom and therefore only one coordinate is needed to completely describe the motion. The coordinate x(t) has been selected as shown in the figure. It is measured from the unstretched spring position, so that at x=0.0 the force in the spring is zero. **It is very important that you make clear where your coordinate system is located.**

Select the physical laws that are to be applied:

In this case, Newton's second law will be applied to the mass. This requires identifying all of the external forces acting on the body. This is aided by drawing a free body diagram, as shown below. Since this is the first problem set in the course some problem solving procedures are described in depth in this solution so as to set an example for problem sets to follow. One such topic is the drawing free body diagrams.

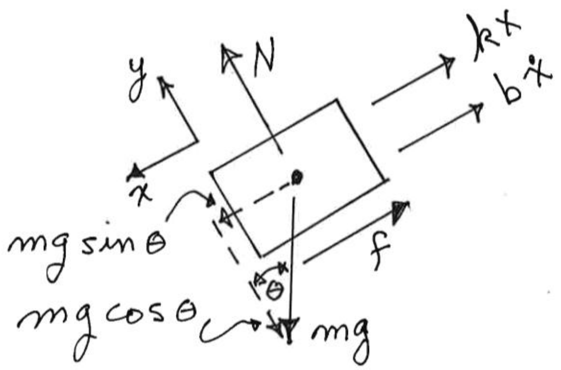
Drawing free body diagrams: The intention is to draw known forces in the direction in which they act. Determining the correct signs on some forces is sometimes confusing. In this case identifying the correct sign for springs, dampers and friction is a challenge. It is recommended that for such forces one always assume that the displacements and velocities of the mass are positive. In this case that would mean assuming that the mass moves in the positive x direction with positive velocity. A positive displacement from x=0 stretches the spring, resulting in a restoring force in the negative x direction. On the free body diagram indicate the spring force as kx with an arrow pointing in the negative x direction. The sign is accounted for by the direction of the arrow.

To deduce the direction of the damper force assume a positive velocity. The damper resists the motion by applying a force on the mass in the negative x direction. We draw it as an arrow pointing in the negative x direction.

In order to correctly indicate the direction of the friction force, it is important to note that the velocity is in the positive x direction. The friction force opposes the velocity and therefore must be in the negative x direction. Therefore, we draw an arrow pointing in the negative x direction to indicate the friction force with value ‘f’.

Next we assemble the equations of motion: In the plane of the diagram there are possible motions in the x and y directions. It is possible to write two equations of motion, one for each vector component in the x and y directions. From Newton‟s second law summing the forces in the y direction is equal to the mass times acceleration in the y direction, which must be 0.0, because the motion is constrained by the slope. This allows us to solve for the normal force, **N**. The expression for **N** will be useful in specifying the friction force, **f**.

$\vec{N} = mg\cos(\vartheta)\hat{j}, \quad \vec{f} = -\mu N\hat{i} = -\mu mg\cos(\vartheta)\hat{i}$



Next sum forces in the x direction and equate them to the mass times acceleration in the x direction. This leads to the following equation of motion. Note that the spring, damper and friction forces are all preceded by minus signs, because they are drawn on the fbd in the negative x direction, given the assumption that the displacement and velocity were assumed to be positive.

$\sum_i F_x = [-kx - \mu mg\cos(\vartheta)\frac{\dot{x}}{|\dot{x}|} - b\dot{x} + mg\sin(\vartheta)]\hat{i} = m\ddot{x}\hat{i}$, which can be rearranged as:

$m\ddot{x} + b\dot{x} + \mu mg\cos(\vartheta)\frac{\dot{x}}{|\dot{x}|} + kx = mg\sin(\vartheta)$, the equation of motion.

The undamped natural frequency is obtained by the solution to the free vibration of this system with no damping and no friction. It is simply $\omega_n = \sqrt{\frac{k}{m}}$. It is not a function of the slope. The constant force term on the right hand side of the equation tells you that the static equilibrium position is not at x=0 but at a location down the slope where the x component of the weight of the mass is balanced by the spring force. This static equilibrium position is a function of the slope.