MITOCW | 10. Equations of Motion, Torque, Angular Momentum of Rigid Bodies

PROFESSOR: OK. I've been giving out the money cards for a few of the lectures, and two or three questions came up in those that I haven't addressed so far. I'm calling them loose ends. And I'm going to pick up a couple of those today, I think they'll help you consolidate the knowledge around the quiz. So I'm going to tie up a little loose ends there. And then the lecture topic I started last time, which is making this transition from thinking about angular momentum of particles to using the full angular momentum equations for rigid bodies, where we talk about mass moments of inertia and products of inertia. And that's where we'll pick up, there again today. Because that's where we're going for the next few lectures. OK.

Let's pick up with a first example here. This is on the topic of basically finding equations of motion. And there's been a little confusion with people, who have asked me what do you mean find? Do you mean solve, et cetera. And so I'm going to go through, and just a real quick example, skipping some of the steps because my purpose is emphasizing the steps, not working out all the details.

So finding equations of motion. Where does it begin? One of the really important steps is this, determine the number of independent coordinates you need. Because when you've done that, that tells you, basically-- it really starts finding the number of degrees of freedom. Should have put this in a different order. Degrees of freedom tells you the number of independent coordinates you need. This is 1, 2, and then 3, that leads you to the number of equations of motion that you need. So this is really an important step.

Secondly, draw a free body diagram. And third, apply summation of forces. External vector equations gives you mass times acceleration, and summation of torques gives you DHDT plus this V cross P term. So this is just kind of the step by step. So let's apply it briefly.

We've talked a lot about things on hills, so here's a cart. It's got wheels attached by a cord to a second mass that's sliding. m1, m2, doesn't stretch the cord in between. Let's think of these things as rigid bodies. So how many degrees of freedom? How many possible degrees of freedom? For the maximum possible, you have how many rigid bodies? 2. How many degrees of freedom for rigid bodies possible? 6 each.

So we're at 6 times m plus 3 times n minus the number of constraints is the number of independent degrees of freedom. This is the number of rigid bodies, number of particles, so we have 6 times 2, 3 times 0 minus constraints, so this one comes out 12 minus the constraints. You have to figure out the constraints quickly. We're not going to allow rotation in any of the three directions on either. They're on carts, they're big, they're sliding, they're not rolling or any of that. So no rotation for 3, no rotation for 3 more. That's minus 6. So c equals minus 6, or c equals 6 for rotation.

And then what else can we say? There's no-- I'll designate this the y direction so we can talk about directions here. And I'll designate this x in general. No acceleration at all in the y direction, right. Can't move in the y. So that gives us 1, 2 for each mass, plus 2 more, that's 8 constraints that we've come up with. Now a 9th constraint is the fact that these two are tied together. And so if you had just temporarily assigned a coordinate here, x1, and another one here, x2, we know for a fact that x1 has got to be equal to x2, and that gives you yet one more constraint. So that's 9.

So we might just stop there. We say OK, that's a total of 9, so the number of degrees of freedom, 12 minus 9 is 3, and that implies that you need three equations of motion. Some confusion comes, you know, if something's not-- let me rephrase that. We haven't talked anything about the z direction. I haven't described any constraints in the z direction. If this is me in a car and I'm dragging a sled down a hill, or I'm in a vehicle and I'm dragging a sled down a hill.

I don't know if you've ever been in a vehicle with a trailer on an icy road in the winter time, that's a dicey maneuver, going down a hill trying to put brakes on. So this thing could conceivably move in this direction. And you can either constrain it to be that way to make the problem simple, or you can just say it's possible, so we have three equations of motion, of which two, for now, the summation of the forces, external in the-- since I've got x and y, z must be this way-- in the y direction. And we'll make it in z direction, this is coordinate system 1, so this would be z1 or z1. The summation of forces, z1, is m1 z1 double dot. But I'm going to set that equal to 0, I just know there's no forces.

So this becomes a trivial equation of motion. And I add another one, summation of forces on the second mass. This is on m2 in the z2 direction, is m2 z2 double dot, and we set that equal to 0 also. So what it boils down to, I had 3 degrees of freedom, 2 trivial equations of motion, leaving me with just 1 equation of motion that's going to be meaningful. Yeah.

AUDIENCE: If x1 is equal to x2, would that mean that the rod has to be entirely along the x-axis? So that would mean that--

PROFESSOR: So he asked if x1 equals x2, does that mean they both have to be along the x-axis? I'm assuming that. So I really am assuming this thing's going down the hill. I'm making a point about this z direction thing because it's just a subtlety that you have to decide on when you're figuring out how to actually analyze the situation. If you really were thinking about what happens when a vehicle is going down a steep, icy hill towing a trailer, maybe put the brakes on, maybe not. You could probably say well, unless I really have a disaster, we're not going to get rollovers and things like that. But you could imagine that it can get out of the x direction, right. It could start sliding into z. It's probably not going to go anywhere in the y, that's still a good assumption.

So this is about modeling, and how complicated do you make the modeling. You actually have to make quite a few modeling decisions when you go to do this, and we tend, in class and examples, tend to really oversimplify problems so that we can do them. So I've boiled this down to where I'm going to end up with one significant equation of motion. And that one's going to say that the summation of the forces in the x direction-- and I'm just going to write mx double dot here, not putting down m1s or m2s because we're going to have to go to free body diagrams to figure out how to apply this.

Free body diagrams. First mass. So this is m1 here. m1g, a normal force, no doubt a friction force, I'll call that f1, and a tension in the cord. And I'm assuming that tension's going to always be there. So if, again, in a situation where the trailer starts overtaking the car and the rope goes slack, we're not going to consider that one today. So there's your free body diagram for the first one and we're going to need to break mg into a couple of components. We're going to need the slope here, and that translates into an angle here. So there's our first free body diagram.

And our second free body diagram, second mass, tension, normal force, another m2g. And this one's on wheels, and we're going to consider this one frictionless. So we don't have any friction force holding it back. So the reason I've kind of-- this seemed like a ridiculous simple problem, but the point I want to make is not the solution, not the particular problem, but an issue that crops up. How many unknowns are there in this problem.

AUDIENCE: 2.

PROFESSOR: Well, you know n1 immediately, or n2, or the friction force, or the tension, or, for that matter, the x double dot we're looking for. There's actually-- you start off this problem with five unknowns. But you're only looking to derive one equation of motion. So the fact that we're looking for one equation of motion doesn't say you don't have to deal with several intermediate equations to get there. That's just part of the work.

So this is five unknowns to start with, n1, n2, f1, t, and x double dot. So if you'll find out that the summation of the forces on y and the y1 on the first mass, this one, this gives you n1. Summation of the forces in the y direction on m2, this one gives you m2 directly. From this flows directly, you know that the friction force is mu n1. See, that's a third equation. So this gives you one equation, this gives you another equation, this gives you a third equation. You have 5 unknowns, you need 5 equations. So there's 3 of them right off the top.

So this leaves-- you solve for the those, this leaves t and x double dot to solve for. So now the sum of the forces on mass 1 in the x direction says m1 x double dot. And the sum of the forces on the second mass in the x direction gives you m2 x double dot. And I explicitly haven't said this yet, what I've done is, one of my requirements is x1 equals x2, and I'm just going to call them x. Both of these are exactly the same, both masses have to move with the same motion, is the assumption.

So that means that x1 double dot equals x2 double dot equals x double dot, and that's what I'm assuming when I'm writing down these two equations. I can write those two equations, one from each free body diagram. I've already eliminated three of the unknowns. And now because I have two equations, each have t in them. Essentially, you eliminate t and solve for x double dot. And if you do that in this problem, you get your equation of motion.

You eliminate t, solve for x1 double dot. Look at this, and you just look at things that doesn't make sense. This says the total mass times the acceleration is a system, it's one system. Mass times the acceleration of the center of gravity of the system, if you will, has got to be equal to the sums of the forces on it. Well, it's got m1 plus m2g sine theta pulling it down the hill, and it has minus m mu m1g cosine theta dragging it back up the hill. And that's the entire equation of motion, it makes sense.

But the equation of motion, the thing you're looking for, is the one that ends up with this acceleration term in it. If you have multiple degrees of freedom, multiple coordinates-- if you have, let's say, three significant equations of motion that result, there won't necessarily be one in terms of each coordinate. They'll have the coordinates mixed in them. Like we did that that two mass system with springs the other day, each equation of motion had x1 and x2. They don't necessarily separate. They're coupled through their coordinates. This one, it's one equation of motion for the system, therefore you have only one coordinate. But that's not generally true of multiple degree of freedom systems.

OK, that's your method, though, for a simple problem. I want to do a little more difficult problem that involves rotation. And this is the problem, I'm sure you've done this problem in physics. It's a classic problem that people do. A disk, a pulley, really, supporting two masses rotates about this point, which I'll call a here. So at some theta there's no slip, so theta is going to be related to movement, x. And I'm going to assign this one a coordinate, x1 going down. This one a coordinate, x2 going up. A little foreknowledge here, because you've worked the problem. So I want to solve for the motion of this system.

Now again, we need to know the number of degrees of freedom. So it's the maximum possible, which is our 6m plus 3n minus the constraints. And let's think about how we want to model this again. This time I'm just going to model these as particles, doesn't matter how big they are. My problem, really, they only go up and down. So I'm going to model them as particles. This is 6 times 0 plus 3 times 2 minus constraints. So 6 minus the constraints. So the issue is really how many constraints.

Well, we're going to require x1 equal x2. Cord's taught, doesn't stretch. If this thing goes down, that has to go up exactly an equal amount, and that's one constraint. In that, leaving us 6 minus 1 is 5, leaving us with a lot of degrees of freedom here. Kind of back to the issue I was making before, are there any constraints in the-- I'll call it x, y, z directions here. Are there any constraints in the y or z directions on either of those masses?

No, I haven't shown any, no tracks, no guides, no anything. So technically, there are no additional constraints in this problem. But if there's no forces in the y, x, I guess y this way, and no forces in the z, I'm going to end up with two trivial equations of motion for this one, 1 in the y, 1 in the z, for this one, 1 in the y, 1 in the z.

So back to this issue of, there's a difference between constraints and trivial equations of motion. We're going have 4 to reveal equations of motion. So really, again, I'm going to come down to one significant equation of motion. So I have 1 constraint, 4 trivial EOMs, and 1 significant equation of motion. OK.

Now, because I want to talk about rotation, we need to pick a coordinate. Now I can pick-- I can either let x1 equal x2 and just let it be the sum x, a single coordinate, or theta, the rotation. I'm actually going to use x for a second. But there's 2 obvious ways to approach this problem. One is to draw a free body diagrams of each of these masses, sum the forces on each, and how many-- if I do that, how many unknowns do I end up with?

We can draw the free-- here's the free body diagram for mass 1. What's it got on it? Well, m1g. What else is acting on it?

AUDIENCE: [? Tension. ?]

PROFESSOR: And here's the second one, m2g, and tension acting on it. So now we can sum, sum of the forces equals mass times the acceleration of each one, and the external forces are going to involve t. So you're going to end up with how many unknowns?

How many unknowns? I can write two equations for the sum of the forces in the x direction. x double dot is certainly an unknown. What else?

AUDIENCE: t.

PROFESSOR: t. So I end up with this other [? end. ?] So that means I'm going to have to write two equations, I'm going to have to eliminate t, going to go through the same thing there. So I don't want to bother with that. Is there another way to do this problem?

This is a problem where you can use angular momentum and not have to deal with t at all. So let's set that problem up. You know that the sum of the torques about that point a, with respect to point a, it's going to be derivative, and since we're dealing with particles here, of the angular momentum with respect to-- I'll just call it lowercase h for particles. Plus this velocity of a with respect to an inertial frame across a linear momentum with respect to an inertial frame. That's the full equation for sum of torques. What's velocity of a with respect to o in this problem? 0. Fortunately, this is one of those problems where you can get rid of this difficult second term. So it's just torques as the time derivative of the angular momentum.

So we need an expression, then, for both the sum of the torques with respect to a. Let's see, what would that be? So now the external torques with respect to-- I'll finish my-- oops, come here you-- free body diagram. So now what I really want is a free body diagram of the whole system. So here's the whole system created as one thing. You have a force down, m1g, another force down, m2g.

Up here you have some normal force up, that's the support of the pin. You have tensions in these, but now this equation applies to the system. The ts are internal to the system, they are irrelevant. So I'm talking about this whole thing treated as a system, and I'm going to compute the moments about point a, which is right there, where that axle is. Does n create a moment at the axle? Nope, but the m1 and m2 times g create moments. Sure, OK.

I'm going to have positive out of the board, the positive moment, positive angular direction. So the torques applied to this system are r cross t, so you're going to end up with-- I want to summarize these. An m1g, and I didn't write the radius on this problem, but at some radius, capital R. So the torques that are m1g are positive minus m2gR, k hat direction, and that must be equal to the time derivative of the angular momentum about a.

Now we need an expression for the angular momentum with respect to a. Angular momentum is, in general, this is a r cross linear momentum, right. So R for mass 1 with respect to a cross the momentum of that second mass with respect to an inertial frame. And a and the inertial frame are the same thing, a sticks in the inertial frame. But angular momentum is always with respect to the inertial frame. Plus the second piece, which is R of m2 with respect to a crossed with p for mass 2 with respect to some inertial frame.

I'm just going to give you the results for this. m1 plus m2 R x dot k. So x dot is this velocity, R cross and mass times velocity is momentum, so the perpendicular radius to that is the radius, R. So it's Rx dot times m, shouldn't surprise you, in the k hat direction. That's the total angular momentum that comes from these two particles with respect to a.

And taking their time derivative. These are constant, that's a constant, this is not, this is a constant, but it doesn't change direction, so this one is pretty simple. Now I can set equal the sum of the external torques, that, to the time derivative of the angular momentum, just to fulfill this expression. And in so doing, I end up with a solution for x double dot. m1 minus m2, m1 plus m2, times g. Turns out the R goes away. So one equation, never had to mess with tension. This is a pretty nice, direct way of solving this problem.

If you solve for g here, and you measure x double dot, this actually gives you an experimental way of determining acceleration of gravity. It's actually what this thing was used for a long time ago, before they had a lot of the measurement techniques and things that we do today. This is a way of determining the acceleration of gravity. So these two masses are quite close together. This number is pretty small, you can, however accurate your timing device is.

Now, just to mention it, I neglected something in this. I assumed something and I didn't even say it. What was it? What would screw up this measurement? I'm trying to measure the acceleration of gravity, if I built this apparatus, would I get a very good measurement?

AUDIENCE: The pulley would have to be massless.

PROFESSOR: Yeah, the pulley would have to be massless. I've made an assumption about that, right. So how would you fix up this equation to account for the pulley?

AUDIENCE: You'd have to take into account its moment of inertia.

PROFESSOR: Yeah, you'd put in something. And where would that go into the problem? How would you account its inertia, moment of inertia in the problem?

AUDIENCE: ha.

PROFESSOR: Yeah, you'd just put it into ha. So this expression for h would end up with one more term, it's going to look like-- well, when you take the time derivative, you're going to end up with another piece over here. Some i about a theta double dot. You're going to have to relate theta double dot to x double dot, which you can, because x equals R theta. x double dot is R theta double dot. You could fix that and you'd have an equation of motion. But that means we need to know about i about a, that's where we're going to at the end of this lecture and for the next several lectures. OK. That's that example, and I've got two more brief ones that I wanted to talk about. Any last questions? Yeah.

AUDIENCE: Can you explain again why you didn't take the tensions into account for your sum of torques?

PROFESSOR: OK, so why did I not take the tensions into account? So I can write the equation of motion for this thing as a complete system. One, the masses and the pulley are all the same thing, the summation of the external torques on that, they're going to amount up to taking into account the time rate of change of the angular momentum of the system. Now, if I didn't understand that, I could have blindly gone ahead and put the ts in there, right, they would've been exactly equal and opposite with respect to a and it would have cancelled out. So either way, if you're not sure about that assumption, you could just put them in and they would appear in the torque equation, but as a minus tR and a plus tR and they'd cancel.

I want to move on to a third example, and this is the third item that I want to clear up, loose ends I'm calling them. The muddy cards are really useful. I get questions in those that spark something. And this is a question that came up two or three times in the muddy cards and I haven't addressed it, and that is, we were working with rotor problems. And remember this problem. You have the rotor, it had an arm, I did it this way to make some things obvious. But this is the z direction, it's rotating about that axis. I've got a point mass up here. r hat, so this is R-- actually, I'm going to make it a capital R so it's easier to distinguish from the r hat. And this is z. This thing's rotating, it's got bearings here to keep it going.

And we talked about torques, so this is my point A. I want to write the sum of the torques about A, time derivative of the angular momentum. We've done this problem before, so I'm just putting up a couple of points for review to clear up some possible misconceptions. That's this term, so what about point A now in this problem? What's the velocity at point A? 0, so again we can get rid of this guy.

I'm going to come back to this and do an example one of these days where this isn't 0, where it's really handy to be able to do a problem where that's not 0. OK. This is true. I need a free body diagram of our little mass, so here's my free body diagram. And it has possibly a force in the z direction. That comes from the rod, there's rods that's supporting this thing, right. There's possibly a force in the z direction. There's a force in the r hat direction, in the R direction. There's a force in theta direction going into the board. And there's mg.

All sorts of forces on this thing. And the question was asked, when we did this problem before and did the time derivatives of the angular momentum, we found that we got-- there's three terms and I'll write them down here for you. I'm just saying in advance what we're going to do. When you solve this problem, you find out that it takes to torque to accelerate this shaft and spin. That the driving one, that's what makes it happen, makes it accelerate.

We had two more terms that were torques at this point, that is what it takes to support this system. It's trying to bend out, it's trying to bend back, those are torques that show up here. And we actually get them when we work through this. But we don't get something that tells us about the moment the torque created this point caused by gravity.

The question was, why don't we get the torque about this point caused by gravity. There's clearly mg down, there's clearly a moment arm. So mgR is the torque about this point. And if you were doing the statics problem in 2.001, there'd be a torque around this point caused by the weight of this thing just sitting there, not even spinning. And what we're doing here gives you no help with that. But just for the quick review of this problem, more in the line of helping you think about the quiz.

This then is R, we'll call this point B and this is point A, remember this is RB with respect to A cross p with respect to o. And that's where our angular momentum comes from. In this problem that is r hat plus z k hat cross m times the velocity, which is R omega z. And that must be in the theta hat direction. When you multiply these out, omega z is theta dot. They're kind of interchangeable in this problem. So when you multiply this out, you get two terms. mR squared theta dot k hat minus mRz theta dot r hat. Two terms from this.

And when you do the time derivative of the dhdt, you get three terms. mR squared theta double dot and the k. Now, why do you get three terms? Because this term has two variables in it that are functions of time, theta dot has a derivative, and r hat has a derivative, because it rotates. So one of the key bits of mathematics you have to learn in this course, I'm kind of giving you a little quiz review here, you need to know how to take the derivative of a rotating vector. And that's what we do here, gives us two terms minus mR z theta double dot r hat minus mRz theta dot squared theta hat.

So three terms in this time derivative of the angular momentum, and they have to be equal to the external torques. This is equal to the summation of the torques about A, the external torques. Well, you'll need a torque in the k direction. That's what it takes to accelerate the thing, make it go faster. This mass has a force on it to make it go faster, that's this f in the theta hat direction. And that rods have a push on that mass, the mass pushes back on the rod. So if in the theta direction it's like that, the mass pushes back on the rod, it twists the rod, or tries to. That's a torque about this in the r hat direction.

So there's centripetal acceleration, it takes force to cause centripetal acceleration. It's that force is inward. It's about a moment arm z, and so this gives you a torque about the point A in the theta hat direction. So these are three different terms, each one has a purpose. No work is done here, no work is done here, because there's no movement.

Now, but gravity, we started this question as why doesn't gravity pop out of this. Because this only tells you about the time rate of change of angular momentum. Gravity has nothing to do with angular momentum. r cross p is all that angular momentum is. The linear momentum of little clumps of mass times the radius from the point you're computing the angular momentum. Has nothing to do with g, never will. You'll never get the g related static moment out of this equation. It's there, though, and if you were designing the system, you'd have to take it into account.

So remember, I didn't bring it today, but I have my shaker. I've bolted it to the floor. Inside of that shaker is a little rotating mass. It has a little arm and eccentricity, it has some mass that I'm going to make m, it's rotating theta direction. And it rotates a constant speed. So it's some constant omega, theta double dot equals 0. So I just got my shaker bolted to the floors, it's putting a lot of vibration into the floor. And the question that someone came up with on a muddy card that was a really inside insightful question, why-- or they didn't say why-- they said, shouldn't the torque required to drive this thing somehow be affected by gravity? So does the torque that it takes to run this around and around depend on gravity, was the question that was asked.

Let's take a quick look at that. We just discovered that dhdt doesn't tell you anything about torque from gravity, right? Well, let's see what happens then. So the summation of the external torques-- I'll call this now point A, where it's rotating about. This point now doesn't move in this problem. It's an inertial point. Summation of the torques with respect to A is dh with respect to A, dt, and there's no additional terms because that velocity point is 0. And that's d by dt, the torque is just r cross p, so that is me theta dot. That's the velocity, that's the momentum. I've left out something. So r cross p, I need an e squared in here. me squared theta dot in the k hat direction.

I need to take the time derivative of that. That's a constant, that's a constant, that's a constant, it only comes from this term. And that gives me me squared theta double dot k hat direction, and that's got to be equal to the sum of the torques in the system, the external torques. And what are they? So torques about this point. So axial forces in this thing contribute no torques, transverse forces, external forces only come from the mg on this thing.

So my torques on this system, there is some mechanical torque being applied. That's what I'm looking for. I've got a motor driving this thing, so there's some t of t in there, some torque, minus mge cosine theta is the moment arm. So there's this force, there's this moment arm is e cosine theta. So this is the external torque caused by gravity, but all of this equals what? What's theta double dot? 0.

The external torque is mge cosine omega t, theta is omega t. And so indeed, as this thing goes around, when it's coming up, you've got to apply enough torque to lift it against gravity. When it clears the top, gravity is helping it, it's going down the other side. So in fact, if you plotted the torque as a function of time for this system, it's like this. It's just lifting that mass up and down. Then, of course, if there's any friction in this thing, et cetera, it's going to have to apply a little bit of torque for that, too. But indeed, this is an insightful question that someone asked, is that the gravity does have to enter into this thing. So there will be a torque that the motor has to supply to drive this thing in gravity. Yeah.

AUDIENCE: Should that expression also have-- the expression for torque also me squared theta double dot--

PROFESSOR: Ah, now theta double dot is? Yeah, see, it would. If this thing was spinning up and I was trying to account for the torque required to spin it up, then here is. Then I would include that, this would be an equation of motion that says all these things are true, and I can solve for torque again. And it will allow me to decide how fast I could spin it up. If I have a dinky little motor, it doesn't spin up very fast, if I had a really powerful motor that could really put it to it, spin it up quickly. OK.

So now I want to move on to the third topic, which is to kind of go back to where I left off last time, talking about we need to move on from particles to rigid bodies so we can do more interesting problems. So I want to pick up with the subject of angular momentum for rigid bodies.

Now last time I just barely scratched the surface of this. And lots of muddy cards said I don't get it. I didn't expect you to get it with it being half baked and the first time you've seen it. So we're going to continue and we won't finish today. So let's think about a general rigid body. Here's my inertial system, got a body out here that's rotating about some point, A. A could even be outside the body and have it rotate about it. And attached to A is a reference frame. Little x, little y, little z.

So my Axy frame. Now, I put up last time, there's two pages out of Williams which gives the equations for the moment and products of inertia in terms of summations of masses times particle locations. And in order to do that, Williams defines a coordinate system on this body, and that coordinate system is fixed to the body, rotates to the body, and Williams calls that coordinate system little oxyz. In his book, he calls the inertial frame big Oxyz. It's really hard to do that on a blackboard and how you'd be able to tell it apart, OK.

So I'm going to depart, and my frame in here is an Axyz frame. But A and o, if you're reading that handout, are the same thing. A and little o. It's a frame fixed to the body that's rotating with it. We can write angular momentum for rigid bodies as a vector hx having a component in the i direction, j direction, and these coordinates as the product of a matrix of constants. And these constants are these moments of inertia and products of inertia terms. And so forth.

I'll write out a couple more of these, iy. It's a symmetric matrix, and you multiply it by the components of the rotation that you are rotating this object, so here's a vector omega. This object is rotating about A, the direction, the axis of rotation is like that. And you can break this rotation rate into components in the xyz system. And that's what these are, these are the components of it. So you multiply out this matrix in a vector, you will get individual equations for the hx, hy, and hz components of the angular momentum of that object.

Now let's consider, let's just do a case where the spin is only about the z-axis. We do lots of these problems, the book has a whole chapter on it and they're called planar motion problems. We just typically pick the spin around the z as a convention. And if you have a case like that, then h here is i times 0, 0, omega z. Then you multiply that out, you get ixz omega z, iyz omega z, and izz omega z. Vector times the square matrix gives you back a vector. That's what you get back.

And if you want to write h as a vector, which we frequently do, h, now this is with respect to A, and we'll find that i here as also with respect to A. Have to be very careful in your construction of this matrix. It has to do with the point about which you are computing your angular momentum. OK, if you want to write this as a vector, then this becomes hx i hat plus hy j hat plus hz k hat. That's just where the unit vectors come in. When you want to express this as a vector, you take these three components, and these are hx, hy, hz.

This little double subscript, the first one tells you the component of h, this is hx, hy, hz. The second one tells you the axis of rotation about which the object is spinning to give you this piece of angular momentum. So ixz is hx spinning at rate omega z. Now, the direction of spin was? What's the unit vector in the direction of rotation for this problem? What's omega? We said we're going to start off with just-- direction, it's only spinning in z direction. So it's just spinning in z direction. But I multiply this thing out, I get three terms. And I get a term in the i, a j, and a k. Now these two terms, so this is i, x is z, omega is zi plus iyz omega zj plus izz omega zk. That's these three terms.

These two terms exist because I've assumed that these off diagonal terms are not 0. The problem we started with, we started with an example last time. Our bicycle wheel thing with the unbalanced masses on it, we use the Williams formulas to compute these different terms. If the off diagonal terms here are not 0, then when you write the angular momentum expression, you get parts of the angular momentum that are not in the direction of spin. That's a really important conclusion.

So the off diagonal terms lead to angular momentum not in the direction of spin. And when you take the time derivative, you end up with torques, and they're right back to this problem up here. If you have off diagonal terms in this matrix, when you spin it around one of its axes, it is dynamically unbalanced. If these are not 0, you spin it around one of the axes of the system for which these are defining, in which these are defined, you find out that you get unbalanced torques in the system.

So those two go together. Now, another way of saying that is any time you end up with the angular momentum vector not pointing in the same direction as the rotation, then the system is going to be dynamically unbalanced. Actually, I kind of want to keep Atwood's machine here.

So this was our unbalanced bicycle wheel problem we had talked about last time. I can simulate that with this. I basically have drawn it like this. So this is the problem. This thing is definitely unbalanced, it's trying to do this as it goes around. And last time we actually worked up, from the William formulas, what the moment of inertia matrix looked like. So now this xyz system are attached and rotating with that frame. So my axis of spin is-- this one's a little exaggerated. That drawing is like this. The x is like that.

So x is like this, z is like that, minus x minus z. So when this thing spins, that's the problem that's drawn there. And if I compute with those with Williams formulas, the various quantities-- so i with respect to A for this system. The first term, the ixx term, is summation miyi squared plus zi squared, and so forth. You get a bunch of terms, and I will write out one other one here. This term over in the corner is ixz, and that's minus summation of the mixizi and so forth.

And if we went through and worked up each of these things, i with respect to A for this problem, comes out mz1 squared 0 minus mx1z1 0 minus mx1z1. The middle term mx1 squared plus z1 squared, 0, 0, and mx1 squared. So that's what this mass moment of inertia matrix looks like for these two particles.

So now if I want to write the angular momentum of this system using this new notation, I would say that it's i computed with respect to A times my omega, and our case is 0, 0, omega z. And if we write that out, we do that, multiply that out, we end up with a minus mx1z1 omega z, 0, and mx1 squared omega z. These are our three components, hx, hy, hz.

And if you wanted to write it as a vector, then you'd add the unit vectors. So the hx and the i plus 0 for the hy plus hz in the k. So now if you went and took the time derivative of those terms, what do you get?

AUDIENCE: Torques.

PROFESSOR: Torques. And you'll get 3 terms. When we did the example a minute ago, what we're doing here is not very different from that. You're going to get the torque that it takes to accelerate it around the spin axis, but you're also going to get the torque two derivatives of this one, which gives you two terms. And these are the moments of torques about that center of the axle, in this case, trying to twist the system around.

Now, reach some closure here. We've got a good stopping point. Here's our system one last time. Here's the z-axis. The angular momentum that comes out of this, you have a component hz in the z direction, and you end up with a component-- it's got a minus in it-- in the x direction like this, so that the total h vector with respect to A looks like that. And it's not in the direction of spin, it's actually perpendicular to our bar here. And it's dynamically unbalanced.

So just to-- how do we make the transition from that to rigid bodies? The Williams formulas, that are these, say that if you want the mass moment of inertia of a body, all you have to do is sum up all the little mass bits at the correct distances off of axes, and you will get it. So when you have particles, you can just add them up. When you have a rigid body, those summations become integrals. And, for example, izz is the integral of-- how should I say this. x squared plus y squared dm, every little mass bit.

It looks like-- is there an exam some distance away? I see a lot of people vanishing. OK, so let me-- I'll tell you what. I'll just make it easy for it and let you go. Let me just say one thing to where we're going. For every rigid body, there is a different set of axes for which, when you go to make up this matrix, you can make a diagonal. And those are called the principal axes, and that's where we're going next, those play a really important role in what we want to do. OK.