PROFESSOR: All right, let's get started. Today is all about Lagrange method. We will talk a lot about what we really mean by generalized coordinates and generalized forces and then do a number of application examples. There's a set of notes on Stellar on the Lagrange method.

It's about 10 pages long and I highly recommend you read them. They're not somehow up with the notes associated with lecture notes or [INAUDIBLE] way down at the bottom. So you have to scroll all the way down in the Stellar website to find them. Our second quiz is November 8. That's a week from next Tuesday. OK. Pretty much same format as the first one. OK.

So let's talk about how to use Lagrange equations. So I defined what's called the Lagrangian last time. t minus v. The kinetic energy minus the potential energy of the entire system. Total kinetic and total potential energy expressions. Then we have some quantities. qj's.

These are defined as the generalized forces. Generalized coordinates, I should say. And the capital Q sub j's are the generalized forces. And the Lagrange equation says that d by dt the time derivative of the partial of l with respect to the qj dots, the velocities, minus the partial derivative of l with respect to the generalized displacements equals the generalized forces.

And for a typical system, you'll have a number of degrees of freedom, like say three. And if you have three degrees of freedom, you need three equations of motion. And so the j's will go from one to three in that case. So the j's here refer to an [? index ?] that gives you the number of equations that you need. So you do this calculation for coordinate one, again for coordinate two, again for coordinate three, and you get then three equations of motion. OK

So this is a little obscure. Let's just plug in. For l equals t minus v. And just put it in here and see what happens. You get d by dt of the partial of t with respect to qj dot minus d by dt of the partial of v with respect to qj dot plus-- I'll organize it this way. Minus the partial of t with respect to qj plus the partial of v with respect to qj equals capital Qj.

Now when we first talked about potential energy a few days ago, we said that for mechanical systems, the potential energy is not a function of time or-- anybody remember? Velocity. So if the potential energy is not a function of time nor velocity, what will happen to this term? This goes away. So this is 0 for mechanical systems. If you start getting into electrons moving and magnetic fields, then you start have a potential energies involving velocities. But for mechanical systems, this term's 0. And I think the bookkeeping.

So this is the form of a Lagrange equations that I write down when I'm doing problems. I don't write this. Mathematicians like elegance. And this comes down to this is beautifully elegant simple looking formula. But I'm an engineer and I like it to be efficient and practical useful. This is the practical useful form of Lagrange equations. So you just use what you need.

Kinetic energy here, kinetic energy there, potential energy there. And I number these. There's a lot of bookkeeping in Lagrange. So I call it term one, term two, term three, and term four. Because you have to grind through this quite a few times. And so when you do, basically you take one of the results of 1 plus 2 plus 3 equals 4. And you do that j times to get the equations you're after. OK.

So now we need to talk a little bit about what we mean by generalized coordinates. qj. What's this word generalized mean? Generalized just means it doesn't have to be Cartesian. Not necessarily Cartesian as in xyz. You got a lot of liberty and how you choose coordinates. Not necessarily Cartesian. Not even inertial.

They do have to satisfy certain requirements. The coordinates, they must be what we call independent. They must be complete. So it must be independent and complete and the system must be holonomic. I'll get to that in a minute. So you need to understand what it means to be independent, complete, and holonomic.

So what do we mean by independent? So if you have a multiple degree of freedom system and you fix all but one of the coordinates, say the system can't move in all but one of its coordinates. That last degree of freedom still has to have a complete range of motion. So if you have a double pendulum and you grab the first mask, the second mask can still move. It takes two angles to define your double pendulum.

So independent. When you fix all but one coordinate, still have a continuous range of movement essentially in the free coordinate. And that's independent. And we'll do this by example mostly. And complete. The complete really means it's capable of locating all parts of the system at all times.

So let's look at a system here. It's a double pendulum. It's a simple one just made out of two particles and strings. I didn't bring one today. And I need to pick some coordinates to describe this. And we'll use some Cartesian coordinates. Here's an x and a y. And here's particle one.

And I could choose to describe this system xy coordinates. And I'll specify the location in the system with coordinates x1 and y1. Two values to specify the location of that. And down here I'm going to pick two more values, x2 and y2, just to describe the-- so x1 is a different coordinate from x2. x1 is the exposition of particle one. x2 is the x position of particle two. y1 and y2.

So how many coordinates do I have to describe the system? How many have I used? Four, right? How many degrees of freedom do you think this problem has? Two. So there's something already a little out of whack here. But the point is these aren't independent, you'll find. You just do a test. You'll find that these aren't independent. If I fix x1 and x2, systems doesn't move. If I say this is going to be one and this has got to be three, this system is now frozen.

So this system of core coordinates is not independent. What did we say?

Independent. When you fix all but one coordinate, you still have continuous range of movement of the final one. I could fix only just two of these and I've frozen the system. I don't even have to go to the extent of fixing three. I'm assuming the strings are of fixed length. You can't change the string length.

So this is not a very good choice of coordinates. And we had a hint that it might not be, because it's more than we ought to use. We only really need two. So and then if we choose these angles, v1 and v2, let's do the test with that. Are those independent? So those are the coordinates of the system. If you fix v1, is there still free and continuous movement of v2 of the system? Sure.

And if you fixed v2, it means you can require this angle stay rigid like that, and move v2, well, the whole system will still move. So v1 and v2 are a system which satisfies the independence requirement. Complete. They're both systems that are complete. They're both capable locating all points at all times. But only the pair v1 and v2 in this example are both independent and complete.

Now, the third requirement is a thing called holonomicity. And what it means to be holonomic is that the system, the number of degrees of freedom required is equal to the number of coordinates required to completely describe the motion. Now, every example we've ever done so far in this class satisfies that. We picked v1 and v2 and that's all the coordinates that we need to completely to describe the motion.

Let me see if I can figure out a counter example. I didn't write down this definition. So holonomic. And if the answer to this question is no, you cannot use Lagrange equations. So let's see if I can show you an example of a system in which you need more coordinates than you have degrees of freedom.

I've got a ball. This is an xy plane. And I'm not going to allow it to translate in z. And I'm not going to allow it to rotate about the z-axis. So those are two constraints. So this is one rigid body. In general how many degrees of freedom does it have? Six. I'm going to constrain it so no z motion. Five. No z rotation. Four.

It's not going to allow it to slip. This is x and that's y. I'm not going to allow it to slip in the x. So now I've got another. Now I'm down to three. And I'm not going to allow it to slip in the y. Two. So by our calculus of how many degrees of freedom you need, we're down to two. We should be able to completely describe the motion of this system with two coordinates. OK.

So I've put this piece of tape on the top. And it's pointing diagonally. That way. And I'm going to roll this ball like this until it shows up again. So it's right on top, just the way it started. Now start off same way again. I'm going to roll first this way. And then I'm going to roll this way to the same place.

Where's the stripe? It's in the back. So I've gone to the same position but I've ended up with the ball not in the same orientation as it was. I went by two different paths. And the ball comes up over here rather than up there where it started. OK

So to actually describe where the ball is at any place out here, having gotten there by rolling around, without slipping and without z rotation, how many coordinates do you think it'll take to actually specify where that stripe is at any arbitrary place that it's gotten to on the plane? Name them.

AUDIENCE: [INAUDIBLE].

PROFESSOR: So. In order to actually fully describe it, you've got to say where it is x and y and you actually have to say some kind of theta and phi rotations that it's gone through so that you know where this is. So this system is not holonomic. And it has to be holonomic in order to use Lagrange equations. So when you go to do Lagrange problems, you need to test for your coordinates. Complete, independent, and holonomic. And you get pretty good at it.

So here's my Lagrange equations. And I have itemized these four calculations you have to do. Call them one, two, three, and four. And what I'm going to write out is just to get you to adopt a systematic approach to doing Lagrange. Left hand side. To the left hand side of your equations of motion is everything with t and v. The right hand side has these generalized forces that you have to deal with. And generalized forces are the non conservative forces in the system. So this is going to get a little bit cookbook, but it's, I think, appropriate for the moment here.

So step one. Determine the number of degrees of freedom that you need. And choose your delta j's. Not deltas, excuse me. qj's. Choose your coordinates. You find the number of degrees of freedom and choose the coordinates you're going to use, basically. Verify complete, independent, holonomic. Three. Compute t and v for every rigid body in the system.

Compute your kinetic and potential energies. One, two, three for each qj. So for every coordinate you have, you have to go through these computations. One, two, three, four, for every coordinate. And this is your left hand side. And if you don't have any external forces and your non conservative external forces, then 1 plus 2 plus 3 equals 0. But if you have non conservative forces, then you have to compute the right hand side.

So the right hand side. So for each qj, each generalized coordinate, you need to find the generalized force that potentially goes with it. And you do this by computing the virtual work delta w. I'll put the little nc up here to remind you these are for the non conservative forces. The delta w associated with the virtual displacement delta qj.

So for every generalized coordinate you have, you're going to try out this little delta of motion in that coordinate and figure out how much virtual work you've done. So delta wj is going to be qj delta k. So this is the thing you're looking for. And it's going to be a function of all those external non conservative forces acting through a little virtual displacement, a little bit of work will be done. Mostly I'm going to teach you how to do this by example.

So let's quickly do a really simple trivial system. Our mass spring dashpot system, single agree freedom mkb. It's going to take one coordinate to describe the motion. X happens to be Cartesian. There'll be one generalized coordinate. So qj equals q1 equals qx in this case. It's our x-coordinate. Actually I should just call it x. That's our generalized coordinates for this problem. Is it complete? Yeah. Is it independent? Yes. Is it holonomic? No problem.

We need t 1/2 mx dot squared. We need v. And we have 1/2 kx squared for the spring minus mgx for the gravitational potential energy. And now we can start. And we have some external non conservative forces. What are they? fi non conservative.

And I think I'm going to put an excitation up here too, some f of t. So what are the non conservative forces? Pardon?

AUDIENCE: k x dot.

PROFESSOR: It's not k. My mistake. You're correct. My brain is getting ahead of my writing here. That's normally b and this would be k. I'm not trying to really mess you up there. So would be bx dot, right. And is there anything else? Are there any other non conservative forces, things that could put energy into or out of the system?

AUDIENCE: [INAUDIBLE].

PROFESSOR: So the damper can certainly extract energy.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah, the force f. That external, it might be something that's making it vibrate or whatever. But it's an external force, and it could do work on the system. And it's not a potential. It's not a spring and it's not gravity. It's coming up and somebody's shaking it or something like that.

So f is also non conservative. So the non conservative forces in this thing are f in the i direction and minus vx dot in the i direction. And we could, in our normal approach using Newton, we draw a free body diagram and we identify a bx dot on it and an f on it. But we'd also have our kx on it.

That would be what our free body diagram would look like. That's a conservative force. Oops, and we need an mg. So we have two conservative forces, kx and mg, and we have two non conservative forces, bx dot and f. So in this case, some of the non conservative forces is that f in the i direction minus bx in the x dot in the i direction.

So let's do our calculus here. So 1 d by dt of the partial of the t, which is 1/2 mx dot squared with respect to x dot. So that gives me the derivative of x dot squared with respect to x dot gives me 2 mx dot. So this is d by dt of mx dot. But that's mx double dot. And as you might expect when you're trying to drive equation of motion, you're probably going to end up with an mx double dot in the result. And it always comes out of these d by dt expressions. OK, so that's term one.

Term two in this problem. Minus t with respect to x in this case. Is t a function of x? It's 1/2 mx dot squared. So is t a function of displacement x? It's a function of velocity in the x direction, but is it a function of displacement? No. So this term is 0.

Three. Our third term. Partial of v with respect to x. Well, where's v 1/2 kx squared. The derivative of this is kx minus mg. And we sum those. So we get mx double dot plus kx minus mg equals. And on the right hand side this is 4.

Now we need to do four for the right hand side. And four is really the summation of the fi's, the individual forces, dotted with dr. These are both vectors. dr is the movement. Little bit of work and it's going to be a delta quantity, like delta x. And f are the applied forces. And you need to sum these up. So this dr in general is going to be a function of the delta j's.

The virtual displacements in all the possible degrees of freedom of the system. We do them one at a time. This case we'll only have one, so it's trivial. But this could be delta one, two, three, four. And each one of them might do some work when f moves through it. But work is f dot the displacement. So it's the component of the force in the direction of the movement, the dot product, that gives you this little bit of virtual work.

OK, so in this problem, this is going to be equal to-- we actually have an f of t of some function of time in the i direction minus bx dot in the i direction dotted width delta x, which is our virtual displacement in our single generalized coordinate. And this whole thing is going to be equal to Qx delta x. So you figure out the virtual work that's done.

So if you do this dot product, this is also in the i hat direction. So i dot i, i dot i. You just get ones. Because the forces are in the same direction as the displacement. You're going to get an ft. f of t delta x is one of the little bits of virtual work. And you'll get a minus bx dot delta x. And that together, those two pieces added together, are the generalized force times delta x.

This total here gives you delta w non conservative for in this case coordinate x. So we're trying to solve for what goes on the right hand side. We need the qx. You notice what'll happen, it'll cancel out the delta x is the result. And in this case, what you're left with is Qx equals f of t minus bx dot.

So this is number four. So Qx. Delta x is the bit of virtual work that's done. What goes into our equation of motion is the Qx part. And we got it by computing the virtual work done by the applied external non conservative forces as we imagine them going through delta x.

And we're done. You have the complete equation of motion for a single degree freedom system. You could rearrange it a little bit. mx double dot plus bx dot plus kx equals mg plus f of t if you well. So it's the same thing you would have gotten from using Newton. In a trivial kind of example, but it helps to find each of the steps, things that we said were required.

OK, so we're going to go from there to a much harder problem. So any issues or questions about definitions, procedure? So we start getting into multiple degrees of freedom. You need is set up a careful bookkeeping. So I just do this myself.

The top of the page, I identify my coordinates, write down t, write down v. Then I say, OK, coordinate one. One, two, three, four. Equation. Then I started the coordinate two. Calculus for one, two, three, and four and so forth until you get to the end. OK, questions? Yeah.

AUDIENCE: On the [INAUDIBLE] what's that thing after the [INAUDIBLE] It's like an open parentheses [INAUDIBLE].

PROFESSOR: Oh, these are functions of the delta j's. This dr, where it comes from, the work that's being done in a virtual displacement around a dynamic equilibrium position for the system is a little movement of the system. dr. And we express it. It's expressed in terms of a virtual displacement of the generalized coordinates of the system. So where the dr comes from is going to be delta. In this case, it's only delta x.

And in the next problem, we're going to do the force in the problem is not in exactly the same direction as the delta x's and delta theta's and so forth. So when you do the dot product, only that complement of the force that's in the direction of the virtual displacement does work. And you account for that.

So let's look into a more difficult problem. So the problem is this. I tried to fix assessed before I came to class. I didn't really quite have the parts and pieces I needed. But this a piece of steel pipe here. It's a sleeve on the outside of this rod. And I've got a spring that's on the outside connected to this piece. And so it can do this. And it's also, though, a pendulum.

So the system I really want to look at is this system. So this swings back and forth, the thing slides up and down. So this has multiple sources of kinetic energy, multiple forms of potential energy. And for the purpose of the problem, I'm going to say that there's a force that's always horizontal acting on this mass pushing this system back and forth. Some f cosine omega t, always horizontal. And I want to drive the equations of motion of the system.

So is it a planar motion problem? How many rigid bodies involved? There's two rigid bodies. Each could have possibly six degrees of freedom. But when you say it's a planar motion, you're actually immediately confining each rigid body to three. Each rigid body can move x and y and rotate in z. So when you [? spread ?] out and say this is planar motion, you've just said each rigid body has max three. So this is a maximum of six possible. Where the other three disappeared to is no z deflection and no rotation in the x or y.

OK, so we have a possible maximum six. How many degrees of freedom does this problem have? How many coordinates will we need to completely describe the motion of the system? So think about that. Talk to a neighbor. Decide on the coordinates that we need to use for this system while I'm drawing it.

OK. What did you decide? How many? Two. All right, what would you recommend? What would you choose? Pardon?

AUDIENCE: The angle and how far down it is.

PROFESSOR: An angle and a deflection of what I'm calling m2 here. So this is m2. The rod is m1. And he's suggesting an angle theta and a deflection which I'll call x1. And I've attached to this bar, the rod I'm calling it, a rotating coordinate system x1 y1. About point A. So A x1 y1's my rotating coordinate system attached to this rod. OK.

So I'm going to locate the position of this by some value x1 measured from point A. And locate the position of the rod itself by an angle theta. Good. Is it complete? So if you freeze one, do you still have-- the complete. [INAUDIBLE] describe the motion at any possible position. Those two things. Yes. Is it independent? If you freeze x, can theta still move? If you freeze theta, can the x still move? OK, is it holonomic? Right. You need two, we got two and they're independent and complete. Good.

Now the harder work starts. So I'm going to give us the mass of the rod, the mass moment of inertia the rod about the z-axis but with respect to A. The length of the rod is l1. The sleeve mass m2 izz with respect to its g. So it has a g. There's also and I'd better call it g2. That's the g of the sleeve. There's also a g1. A center of mass for the rod and a center of mass for the sleeve. Those are properties we'll need to know and I'll give them to you. OK.

So we need to come up with expressions for potential energy and kinetic energy. So this problem, the potential energy's a little messy. Because you have to pick references. You have to account for the unstretched length of the spring. So call l 0 is the unstretched spring length. We know that also. So I propose that the potential energy look like 1/2, for the spring, anyway, 1/2 the amount that it stretches in a movement x1. The amount that it stretches then should be whatever that x1 position is.

And that x1 position, and I drew it slightly incorrectly. I'm going to use x1 to locate the center of mass, which is always a good practice. So here's the center of mass. So my x1 goes to the center of mass. That's x1. So that's the total distance. And from that, we need to subtract l0, the unstretched length of the spring.

And we need to subtract 1/2 the length of the body because that's that extra bit here. So this is the amount that the string is actually stretched when the coordinate is x1. And you've got to square. And that'll be the potential energy stored in the spring.

Then we got to do the same thing for the potential energy. We have two sources of potential energy due to gravity. And they are? Two objects, right? Two potential energy. So why don't you take a minute and tell me the potential energy associated with the rod. So the rod has a center of mass.

It's a pendulum basically. So it's the same as all the pendulum problems you've ever seen. And I would recommend that we use as our reference position its equilibrium position hanging straight down. And I'll tell you in advance, I'm going to use the unstretched spring position this time. Just stay with that. That's where it's going to start from. That's my reference for potential energy.

But does the unstretched spring position have anything to do with the potential energy of the rod? No. OK. So its reference position is just hanging straight down. So figure it out. What's the potential energy expression for just the rod part? Think about that.

So I'm going to remind you about something about potential energy. Potential energy, one of the requirements about it is the change in potential energy from one position to another is path independent. So you don't actually ever have to do the integral of minus mg dot dr.

You don't have to do the integral. You just have to account for the change in height between its starting position and its some other position. Spend a minute or two, think about that. Work it out. You got a question? OK. Can you talk? Talk to a neighbor, check your ideas. So you have a suggestion for me? Ladies?

AUDIENCE: [INAUDIBLE]. l1/2 [INAUDIBLE].

PROFESSOR: OK. Anybody want to make an improvement on that? Or they like it? Improvement?

AUDIENCE: [INAUDIBLE].

PROFESSOR: 1 minus cosine theta. So let's put that up and let's figure out if we need that. We have a bid for cosine theta and 1 minus cosine theta. So you need to have a potential energy at the reference and you need to have a potential energy at the final point. And the difference between the two is a change in potential energy here.

So what's the reference potential energy is mg l1 over 2 when it hangs straight down. And then when it moves up to this other position, this is the l1 over 2 times this is a delta h. This is the change in height that it goes through. So you need the 1 minus. Do we have the sines right? Yeah. OK.

So now we need another term. And I'll write this one down. This one's a little messier. We need a potential energy term due to gravity for the sleeve. And that's going to mimic this. You're going to have a term here plus m2g.

And it's reference, I'm just going to do it as a reference amount minus the final amount. The reference will be at the initial location of its center of mass, which is l0 plus l2 over 2 minus m2 g x1 cosine theta. Because this one is a little messy because you've got this thing. It can move up and down the sleeve.

And if that moves, you've lost your reference. So you can't do this as a concise little term like this. You have to separate out the reference and then this is the final. And the l0 plus l2, this quantity here is the starting height. This x1 cosine theta is the finishing height. And the difference between the two gives you the change in the potential energy. So this is your potential energy expression. This plus this plus these. All right.

So what about t? We got to be able write it. Kinetic energy is generally easier. Got to account for all the parts and pieces. So we have to chunks. And we're going to have rotational kinetic energy associated with the rod, rotational kinetic energy associated with the sleeve. But also some translational kinetic energy associated with the sleeve. And I'll write these terms down. Make the problem go a little faster here.

1/2 izz about A. That's the rod. Plus 1/2 izz for the sleeve about g. We'll discuss why the difference here. And that's theta dot squared. Now for the kinetic energy that comes from translation of the center of mass. Because I'm broken up.

Let me start over. This system is pinned about A. And the rod is just simply pinned at A. And the last lecture I put up these different conditions and simplifications. You can account for a something about a fixed pin by computing maximum inertia about A. It's basically a parallel axis theorem argument. Times 1/2 times that times theta dot squared.

So this gives you all the kinetic energy in one go with the rod. But for the sliding mass, because its position is changing, you can't do that. You have to account for the two components of kinetic energy separately. This accounts for rotation about g. Even though g is moving. That accounts for that energy.

Because it's only a function of theta dot. It's not a function of that position x. This term is going to account for the kinetic energy associated with the movement of the center of mass. So we need a vg2 in the inertial frame dot vg2. These be in vectors. And does that get everything? I think that does.

So vgo is it certainly has a component that is its speed sliding up and down the rod, right? And that's in the i hat direction. But it has another component due to what? Can you tell me what it is? Its contribution to its speed due to its rotation.

AUDIENCE: [INAUDIBLE].

PROFESSOR: It's got a theta dot. Yep. It needs an r, right?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah. So this would be an x1 plus. No actually, I made x1 go right to the-- so just x1 theta dot in one direction. Yeah, so j hat here. Actually that's the moving coordinate system unit vector in the y direction. And so we do the dot product. You get this times itself. i dot i and j dot j.

This quantity here is 1/2 m2 x dot squared plus x, this next one I guess. x1 squared of theta dot squared. That's the kinetic energy of accounting for the velocity of the center of mass. So now we have our entire kinetic energy expression.

So now we have how many coordinates? Two, right? How many times do we have to turn the crank and go through the Lagrangian? Got to go through it twice. So let's apply Lagrange here. And we'll just do number one first. So and let's see. Which one do I have on my paper first? I guess we'll do the x1 equation. This is delta x1. So this generalized coordinate x1. And we need to do term one, which is in d by dt of partial of t with respect to x1 dot. OK.

So we look at this and say, well, is this a function? Is this term a function of x? Nothing. You get nothing from there. Is this term a function of x? Yeah, it's down here. We only have to take the derivative of this. We have to do that job. So the derivative of this with respect to x dot, you get a 2x dot here.

Do you get anything from here when you do this with respect to x dot? You only get a contribution from here. The two cancels that. And so this should look like m2 x1 dot but d by dt. Do this once in two steps here so you see what happens. You get an m2 x1 double dot out of that.

So we've gotten the first piece of this. We've got a couple to go. But you know a lot about Newton's laws and you know a lot about calculating equations of motion now using sum of torques and all that stuff, right? So this is just something moving, has a circular motion, has translational motion.

What other accelerations had better appear in this equation of motion? And which equation are we getting? There's going to be two equations and it has physical significance to it. What equation does this begin to look like? Just physically, what movement is being accounted for here?

Looks like translation in the x direction. It's this thing sliding. It's this part of the motion sliding up and down. You're writing an equation of motion and mx double dot has units of what? Torque? Force. So it's a force equation. This is just f equals ma is what this is going to show us.

Remember, the direct method has to give you the same answer as Lagrange. So we're getting a force equation. It's describing mx double dot. What other acceleration terms do you expect to appear in this from what you know? Yeah.

AUDIENCE: [INAUDIBLE].

PROFESSOR: A centripetal term. Do you believe there ought to be a centripetal term in this answer? Why? Because it's got circular motion involved. For sure. Any others? Is there any Coriolis in this? In this direction. Which direction are we working in? Is there Coriolis acceleration in the x direction?

By the way, these equations, do we have any ijk's in here? These are pure scalar equations. No unit vectors involved. This equation only described motion in the x. So will there be a Coriolis force in this acceleration in this problem? Will there be an Eulerian acceleration in this equation of motion? The reason I'm going through this with you, I want you to start developing your own intuition about whether or not when you get it at the end it's got everything it ought to have and doesn't have stuff it shouldn't have. OK.

So your forecasting, then we better get a centripetal term. Well, let's see what happens. So that was number one. Number two here is our dt by minus the derivative with respect to x, in this case. So we go here. x1 we've been calling it. Is this a function of x? This piece? Nope, it's x dot.

How about this one? Right. Take this derivative, you get 2x. So this fellow is going to give us minus m2 x1 theta dot squared. What's that look like? There it is. There's your centripetal term you're expecting to get.

OK. And step three is plus partial of v with respect to x. In this case with respect to x. And where's our potential energy expression? Well, it's up here. And where the x dependency is in it. There is no x in that term and no x in that term. But we have x's in both of these other terms.

So when we run through this, I'll write down what we come up with. We get certainly a spring term. k x1 minus l0 minus l2 over 2. So that's the spring piece when you take the derivative. The two cancels the 1/2 and the derivative of parts inside just gives you 1. So that's the first piece of the potential energy expression.

And the second piece is only going to come from here. The derivative of this with respect to x1 is just m2g cosine theta minus. And you add those bits together, you end up with m2 x1 double dot minus m2 x1 theta dot squared plus k x1 minus l0 minus l2 over 2 minus m2g cosine theta. So those are the three terms, 1 plus 2 plus 3, that go on the left hand side. And they're going to equal my qx that I find. I still have to find what the generalized force is in the x direction.

So all that's left to do for this problem is to find q sub x, the generalized force that goes on the right hand side. So now let's draw a little diagram here of my system. And at the end of the sleeve. So here's my sleeve. I've applied this force. This is f of t. And maybe it's some f not cosine omega t. It's an oscillatory force, external force. Make it vibrate.

And I need to know the virtual work done making that force go through a displacement in what direction? So this equation is the x1 equation, right? And so the virtual displacement I'm talking about is delta x1. And the amount of work that it does is delta x1 times the component of this force that's in its direction.

So I'm going to take this force and break it up into two components. And if this is my theta, this is also theta. So this will be f0 product. And I'll leave out the cosine omega t here. It's a function of time. But this side then is cosine theta i.

No, hey, I got this wrong. I drew this wrong, I'm sorry. This is theta. This is going to be sine. This side is sine theta in the i direction. And this piece is f0 of t cosine theta in the j. So I break it up in two parts. And the virtual work associated with x1 is the thing I'm looking for, qx, dotted with delta x1. And that is f of t here, the vector dotted with dr, my little displacement.

But in this case, this then all works out to be f0 cosine omega t. And it has sine theta i plus cos theta j components dotted width delta x in the i. So you're only going to get i dot j gives you 0, i dot i gets you 1. So you're going to get one piece out of this. This says in the qx equals f0 cosine omega t sine theta. And start with you have a delta x here and a delta x here and that gives you the delta virtual work.

Personally when I do these problems, I have to think in terms of that little virtual deflection. I've actually figure out what's the virtual work done. And then at the end I take this out and this is the qx that I'm looking for. So my final equation of motion says, this equals f0 cosine omega t sine theta. And that's your equation of motion in the x1 direction.

So when you finish one of these, you need to ask yourself, does this make sense? Does this jive with my understanding of Newtonian physics? Better have a linear acceleration term, because that's what it's doing. You have another acceleration term in the same direction due to centripetal. A spring force for sure. And a component of gravity in the direction of motion, up and down the slide, equal to any external forces in that direction. So it makes pretty good sense. OK.

Now, also another test you can do is does it satisfy the laws of statics? That's another check you could perform. Does this thing at static equilibrium tell the truth? A static equilibrium all time derivative is 0. So this would be 0, this would be 0. You know its static equilibrium hangs down, so cosine theta is 1. Static you don't have any time dependent forces. That's 0.

So the static part of this says that k x1 minus l0 minus l2 over 2 equals m2g cosine. And that's cosine theta is 1, so it's m2g. And you could figure out then this must be k times something. This is the x. This is the amount of the spring stretches, the static stretch of the spring, so the spring has an equal and opposite force to the weight of the thing m2g. So that's another check you can do when doing the problems.

OK, I'll write up the final one. We have one more question to go. Got to do all the derivatives with respect to theta. So you take a minute to decide how many acceleration terms and what acceleration terms do you expect to see come out of this second equation of motion.

Because now we're talking about which motion? Swinging motion. And what's its direction? In Newtonian sense, it would have a vector direction. It's in what we call j here. OK, so you're about to get the j equation. What terms do you expect to find in it? Talk to your neighbors and sort this out. And basically tell me what the answer's going to be.

What do you think? What are we going to get?

AUDIENCE: We were debating about whether or not it was going to be like speeding up in the theta [INAUDIBLE].

PROFESSOR: So it is a pendulum, just a weird pendulum. So does the theta change speed? Sure. When he gets up the top of the swing at 0. All the way down, it's maximum speed. So what term does that imply that you're going to get?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Well, maybe, maybe not. Yeah?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Going to get an Eulerian, which means you've got a theta double dot term. You're expecting a theta double dot term to show up. OK. What else? Will you get a Coriolis term? Do you expect a Coriolis term? Something that looks like x dot theta dot.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah. The thing is sliding up and down the sleeve. It has a non 0 value of x dot. Any time you got things moving radially while something is swinging in a circle, you will get Coriolis forces. It means the angular momentum of that thing is changing and it takes forces to make that happen.

So here's what this answer looks like. That's the one term. The two piece gives you 0. It's not a function of x in the three piece. The potential energy pace gives you m2g x1 sine theta plus m1 g l1 over 2 sine theta. And the fourth piece, the q theta, well, that's just going to be the virtual work done.

There's a tricky bit to this one. Now there's virtual work, but which direction? So we have an f dot dr. The only f we have is this. What's the dr? What direction is it? This is the theta coordinate. What direction does that give you displacements? f dot dr's a displacement, not an angle. To get the work done, you got to move a force through a distance.

So the distance, first of all, is in what direction when theta moves? j. Little j hat, right? And now if you get a virtual deflection of delta theta, what's the virtual displacement? You had a virtual change in [? angle ?] delta to put theta. But is that the virtual displacement? What's the displacement of this point here in the j direction, given a virtual displacement delta theta? Think that out.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Can't quite hear you.

AUDIENCE: [INAUDIBLE].

PROFESSOR: x1 delta theta will give you the motion to displacement at the center of mass in that direction. x1 comes from here to here. So x1 delta theta will give you a little displacement in that direction. But is that the displacement we care about? We need the displacement here. So you're close.

So we're going to get some force dot a displacement dr. And that's going to be our force, this guy, with its i and j components. i and j terms. But this term out here is x1 plus l2 over 2 to get to the end. And it's in the j direction. So it's a length times A. And you need the delta. This quantity. And you need a delta theta. Delta theta.

This is the term. This is the dr for the system. An angle, a virtual deflection in angle times the moment arm gives you a distance. It's in the j hat direction dotted with the same force breaking the force up into its i and j components. It had a sine theta i cos theta j.

So this is going to give me a f cosine omega t cos theta j dot j. f0 cosine omega t cos theta x1 plus l2 over 2 delta theta is the delta w. That's the work and the virtual. The generalized force q theta is this part of it. So this plus this plus this equals that on the right hand side. So this is part four. And look at it. Yeah?

AUDIENCE: [INAUDIBLE].

PROFESSOR: So f of t, I didn't want to write it all out. This thing breaks into an i and a j piece, which is written over there. This is the sine theta i cos theta j term. Which I brought back from over there. And we dot it with the dr that we care about, which is this length times that angle in the j direction. So j dot. We only pick up the j piece of this. And that gives us this cosine theta term.

OK. Let's look quickly. This is a rotational thing. It has units of is it force? Is this a force equation? i theta double dot has units of what? Torque. This is a torque equation. This is the total mass moment of inertia izz with respect to A for this system such that the Eulerian acceleration.

The torque it takes to make that happen is the sum of the mass moment of inertia of the rod plus the mass moment of inertia of g plus m2 x1 squared, which looks a lot like the parallel axis theorem. This is izz A for the moving mass. There's your Coriolis term. And here's your potential terms and there's your external force.

OK. Talk more about these things in recitation.