PROFESSOR: Today we are going to talk about the vibration of continuous systems. Not covered on the quiz, but it's a really important part of real-world vibration and the most-- one of the easiest ones to demonstrate, I've shown you this one before, is the taut string. But I want to show you something unusual about-- something you may not know about strings. Wait until it calms down here a little bit.

OK, so this is your guitar string or a piano string. It's under tension. We've already seen that it exhibits natural frequencies in mode shape, so there's the first mode. Looks like half a sine wave. Has a particular frequency associated with it.

Get it to stop doing that-- but if I excite it at twice the frequency-- I don't know if I can do this. There we go. That turns out to be exactly twice the frequency of the first. The mode shaped one full sine wave. The mode shapes for a taut string are sine n pi x over L.

But strings can do something else kind of neat. And that is if I hit this thing-- I'm going to wait till it calms down here. If I give this thing just a pulse, what do you expect to see? Are you going to see vibration? Tell me what you see.

What do you see happening? Something running back and forth. Right? What you're seeing is wave propagation. It's not really vibration. Vibration we see of its modes and standing waves and things like that. Right?

So the taut string satisfies an equation of motion that's called the wave equation. We're going to talk quite a bit about that this morning. And the wave equation has its name give something away. The wave equation describes continuous systems of a particular kind that support travelling waves.

And so the string will both support-- I can give it a little pluck. I'll try to just place it in a particular shape and let go. There it is. And that little pluck just goes back and forth back and forth at a particular speed.

So is there a relationship between the speed at which things can travel in a string and the natural frequencies of the string? Well, we'll get into that today. And I'm going to start by just showing you a little something that comes from my research and-- let's see. Let me do this. I think this will work.

Hear that? As I go slower, does frequency go up or down? It's kind of slow, and I'm going to speed up. Right? Goes up as the speed goes up.

So that's the result of the phenomenon called flow-induced vibration. And I'll give you a very brief intro to flow-induced vibration. You have a cylinder sitting still, flow coming by it-- water or air.

The cylinder is diameter D, velocity U, for the flow. What happens in the wake of that cylinder, vortices are formed. And just like if you're paddling a canoe or something and stick a paddle in the water, you'll see vortices shed off the side.

First you get one that's positive and then one that's negative And so one full cycle of this is from here to here. There's a frequency to this shedding. And the shedding frequency, FS, in hertz, can be predicted by a simple dimensionless parameter called the Strouhal number, St U over D. And that's approximately 0.2 U/D for stationary cylinders. You can predict the frequency at which these vortices are shed.

Now, associated with the shedding of vortices is a lift force. I'll call it some FL cosine omega t, which is 2 pi FS, times t. So at this frequency of vortex shedding there is a transverse force.

There's actually an inline force also, which I'll call FD for drag. And it goes like cosine 2 omega s times t. It's twice the frequency of that. And so you'll get some inline oscillatory excitation and what we call cross-flow oscillatory excitation. And this is the cause of lots of things that the people who work on it call flow-induced vibration.

Now, an amazing thing happens is if this cylinder is elastically mounted or is flexible, and that force starts to act on it, it will begin to vibrate. And the amazing thing, as it begins to vibrate, it correlates the shedding of these vortices all along the cylinder. So it's like soldiers marching in step going across the bridge. If everybody's walking randomly, then the bridge doesn't respond too much. But if everybody marches together, you can put a pretty good excitation into it.

Well, the motion of the cylinder itself organizes these vortex shedding all along the cylinder, so they're all marching in step. And that means the force is all correlated on the length. And you can get some pretty substantial response. So that's the subject called flow-induced vibration. And with that, I'm going to show you a few slides.

Let's dim the lights a little bit, if you could, to see this. There's some pictures I just want you to see better. All right. So I do flow-induced vibration. I've been doing this-- working on this for all my professional career.

And it's applied, primarily, to big, flexible cylinders in the ocean. Particularly associated with the things that the US Navy does. Long cables and things and also the offshore oil industry. Next slide. Can we dim the lights? Can we dim the lights? I want you to be able to see.

This is a tension leg platform. It's one of the structures that's used in the offshore industry to produce oil. And one of these might be moored in 3,000 feet of water, 1,000 meters of water. Might weigh 20,000 tons.

And what's connecting it-- what holds in place-- are steel cylinders a half a meter in diameter, 3,000 feet long, going vertically down from each of those three pontoon legs sticking out. And they're under a lot of tension. And in fact, it pulls the thing down into the water so the buoyancy of the whole thing puts tension on these cylinders. But now, what happens if an ocean current comes by those cylinders? Vortex shedding, and the cylinders vibrate. And if they vibrate, over time they will fatigue and fail. OK. Next slide.

There's a picture of a real one. That's a bigger one called Marco Polo. It's on a launch ship that'll take it out to the site that it is. And the ship will lower and it will slide off. So these are big. Next slide.

This is a diagram of the Gulf of Mexico. South America is at the bottom. The Yucatan Peninsula is sticking up there right in the middle of the bottom. This is a picture of satellite imagery of currents in the Gulf of Mexico. And there's a current that flows up off of South America into the Gulf of Mexico, goes around in a loop, and then comes out. You can see Florida sticking down in there on the right.

That current comes out of the Gulf, goes around the tip of Florida, and goes up the Atlantic Coast, and is known as the Gulf Stream. But it starts as a big current that comes into the Gulf of Mexic. And, every now and then, that current pinches off an eddy.

And that's what that red circle is in the middle. And it's an eddy that's many, many kilometers in diameter with surface currents on the order of a meter per second or more. And those are the biggest threat for causing flow-induced vibration failures of long members from hanging off of offshore structures. Next.

So I've been doing research in this area for a long time. This is a picture taken in the summer of 1981. It is a piece of steel pipe about 2 inches in diameter and 75 feet long. It's under 750 pounds of tension, and it's pinned at each end.

It behaves almost exactly like my rubber cord here. It has natural frequencies, and it will vibrate if a current comes by it. So this is actually a sandbar. And at low tide, we'd do all the work putting it up.

Then, as the tide comes in, the flow is perpendicular to the cylinder, and vortices start shedding. And as the pipe begins to move, they get organized all along the length. And a typical response mode was when the vortex shedding frequency, therefore the lift force frequency, coincided with the natural frequency. Then you'd expect it to give quite a bit of response.

The diagram on the left is if you cut the cylinder and looked down its axis, this is the trajectory that you'd see the cylinder make. It would sit there and just make big figure eights. So up and down vertical is its vertical motion. Flow's coming from, say, left to right.

Its vertical motion is up and down. In-line motion's like this. And exactly such a phase it just makes big beautiful figure eights. That's the kind of motion you'd see. OK?

So then, very much what I was talking about a minute ago, very much behavior dominated by vibration. Vibration in the third mode, cross flow, was a typical one. And fifth mode, inline, was typical. But as cylinders go in the ocean, that one's kind of short. Third mode vibration is sort of low.

So as years have gone by and oil is being produced in deeper and deeper and deeper water, the cylinders we're putting out there get longer and longer and longer and longer. And the modes that are excited by currents coming by get quite high.

So this is an experiment we did. It was roughly a 1/10 scale model. Model is almost 2 inches in diameter, 500 feet long. Scale that up by a factor of 10, you're up around 20 inches in diameter and 5,000 feet long, which is exactly the size of the drilling riser that BP had hung off the drilling ship when the blowout occurred. It's a piece of steel pipe, 21 inches in diameter, 3/4 of an inch wall thickness, 5,000 feet long, under a lot of tension. And when ocean currents come by, it behaves just like this string.

And so we're out-- this is a 1/10 scale model. So we put a big weight on the bottom of the cylinder, put it behind a boat, and towed it in the Gulf Stream. Next picture. So there's the boat. It's an oceanographic vessel. It's actually a catamaran. Next.

This is a spool that had our test cylinder on it. There's a reddish object down on the bottom which is-- that's a 750-pound piece of railroad wheel, and it's the weight on the bottom. And so you'd spool this thing off, lower it down, and then do your tests. Next.

Top, we measured tension inclination. And then we also had-- it's a pin joint at the top, so it would vibrate freely. Inside, though, was fiber optics. Next.

We had eight optical fibers. And in those optical fibers were what we call optical strain gauges. So we had 280 optical strain gauges instrumented up and down that pipe so we could measure its vibration. And so you're looking at a cross section of the pipe. There were two optical fibers in each quadrant, and each one of those fibers had 35 sensors on it. Next.

This is typical experimental case. This is the surface. This is 500 feet down. This is the current profile. So the flow velocity is about 2 feet per second near the surface, up to 4 feet per second down on the bottom.

And this is the region where most of the excitation was coming from that would drive the flow-induced vibration. This is measured RMS strain caused by the bending vibration in the cylinders. And peak-- the maximum strain-- is about right there. Next.

Typical response spectrum. Basically, the frequency content at three different locations. Down deep, in the middle, near the top. This is frequency. So this would be the peak that describes the principal cross-flow vibration at the vortex shedding frequency. Next.

This is position, bottom to top. This is time, and these are strain records from all of those strain sensors. There's a strain sensor about every 2 meters along here. But what you're seeing is-- this is evidence. The red is the amplitude and red-- let's say red is positive strain and blue is negative strain.

And so at any location on the pipe where it's vibrating, it's going to go from red to blue, red to blue, red to blue. But it's showing you that they're highly correlated all along the length, that there's a red streak all lined up, but it's not parallel to the pipe. It's inclined.

This is showing you wave propagation. The behavior of the pipe is completely dominated by wave propagation, not by standing wave vibration. So totally different than that short pipe in 1981.

The wave equation. Let's imagine we have a long pipe or a string like that, and it can carry waves traveling along it. The position at any location on here-- here's a coordinate x.

We describe the motion at a point by a coordinate w of x and t. So it's a function of where it is and time. What describes the motion of something which obeys the wave equation is the following equation. Partial squared w with respect to x squared equals 1/c squared partial squared w with respect to t squared.

That's what's known as the one-dimensional wave equation. And the one-dimensional wave equation governs an incredibly broad category of physical phenomena. Light behaves according to the wave equation. Sound propagating across the room to you is governed by the wave equation.

Longitudinal vibration of rods, torsional vibration of rods-- all governed by the wave equation. So it's worthwhile to know a little bit about the wave equation. And what I showed you this morning, it has this kind of duality to it. You can have things that vibrate with standing waves and mode shapes, but the same system can support waves that travel along it. So let's figure out why that is.

So I'm going to do the derivation for you of the wave equation for a string, just so you know where it comes from because then that general derivation applies to all these different things. So imagine you've got now-- we're interested in eventually getting to vibration. So I'm going to make this a finite length string. And it has this position we'll describe as a w of x and t.

It has a tension, T, a mass per unit length, m. So this is like kilograms per meter is the mass per unit length of this thing which can vibrate. So tension. Mass per unit length. L, the length of it.

What other parameters do we need? That'll do for the moment. Now-- so let's draw it again without. In some displaced position and what's exciting it may be my vortex shedding, and so I'm going to draw that excitation here. And that we'll describe as F of x and t, some force per unit length. So this has units of newtons per meter.

Now, in that little-- there may also be drag forces, the fluid damping. So I'm going to cut out a little piece of this cylinder and do a force balance on that piece of cylinder. So basically, F equals ma. We're just applying Newton to this piece of cylinder.

And I'll draw it right here. A little section of it is curved. Here's horizontal. There's horizontal. We need to evaluate all the forces on it.

So the tension on this end-- so like that. And the tension on this end is some different angle. This we'll call theta 1. This we'll call theta 2.

And along here are my excitation forces, F of x and t. There may be some resistance-- drag forces, damping. That'll be a damping constant, R of x, which is force per unit length per unit velocity, times-- the force on this would have to be multiplied by the velocity, so the derivative of this displacement with respect to time. That's the force along here, and it can vary with position.

Have we accounted for everything? Ah, well, this is position x, and this is at x plus dx. So this little element is dx in length. And this is all for small motions.

And if you assume small motions, then you can say theta 1 is approximately equal to sine theta 1. That's also approximately equal to tan theta 1. And that's equal to the derivative of w with respect to x, just the slope. We're going to take advantage of that.

Theta 2, same thing. It's approximately equal to tan theta 2 here, and sin and all those things. But that, then-- the slope has changed a little bit when you go through dx. And this is equal to the slope on the left-hand side plus the rate of change of the slope times dx.

So the slope on the left, this is now the slope on the right-hand side. And so now, all that's left to do is to write a force balance for that little piece on the element dx. So if positive, upward. We have a T sine theta. But because sine theta is approximately tan theta is equal to dw dx, then there's an upward force on the right-hand side, which is T.

And this turns into partial squared w with respect to x squared dx. So on the right-hand side-- positive upwards-- you have T times the partial of w with respect to x, plus partial square w with respect to x squared dx. That's the upward force on the right-hand side. On the left-hand side, we have a downward force, minus T partial of w with respect to x.

And you notice that this one's going to cancel that one. We have minus R of x partial w with respect to t-- that's the velocity-- dx long. Because that's force per unit length. And have we missed anything?

So that's the sum of the external forces on this little slice. And that has to be equal to-- what did Newton say? The mass, which is the mass per unit length, times dx, is the total mass, times the acceleration, partial squared w with respect to t squared.

So this cancels this term. And then you notice I'm left with everything as just something dx, something dx, something dx. Get rid of the dx's, and I can write-- oh, I left out something.

I left out my distributed force, F of x and t dx. It's positive as it's drawn. It's over here also. So this, and I cancel out that dx. So I put them all together now and assemble them. I can write down the equation that governs this motion.

So T partial square w with respect to x squared minus r of x times velocity plus f of x and t equals m partial square w with respect to t squared. And that just says that the sum of the forces on the object equals its mass times its acceleration.

Now, if we're interested in natural frequencies and mode shapes, when we've been doing one and two degree of freedom systems, and we want to get the natural frequencies in mode shapes, we temporarily let the damping be 0 and the force be 0, right? So we want to do the same thing now. We're interested in how do you find the omega n's and what I call the psi n's.

Because now the mode shapes are functions. And so this is a natural frequency and the mode shape for mode n. We know there's lots of modes. So we let r of x and f of x and t be 0. And when we do that, this term goes away. This term goes away. I'm just left with T partial squared w with respect to x squared equals this. And I'm going to divide through by t.

So I get partial squared w with respect to x squared equals 1 over T over m partial squared w with respect to t squared. And this T/m quantity turns out to be the speed of wave propagation in the medium. And that is the wave equation.

So we've just found the wave equation for the string just by applying Newton's law to a little section of string. You can do that for the vibrate. You're going to do the same thing, cut out a little section of a beam, do the force balance on it, set it equal to the mass times acceleration.

And for a beam, you'll get a fourth order differential equation. And it's not the wave equation. It still vibrates, but it's not governed by what we call the wave equation.

OK, so this is the one dimensional wave equation. This quantity T/m is the phase velocity. It's called phase velocity. You know, that's a good one to remember. For a simple string, the speed of phenomena running down the string is the square root of the tension divided by the mass per unit length.

And if you had a long string, I put that little pluck in it, and you can see that pluck running back and forth on it. That's the speed it's going at. Basically, it's called-- well, so if I have my string, and I put a little bump on it, and that bump goes zipping along, your eye will see this thing propagating at c.

So to get natural frequencies in mode shapes, we basically need to solve this equation. And it's quite straightforward to do. And a technique known as separation of variables works, which means that all you're doing is saying, I believe that I'm going to be able to write the solution as some function of x only times some function of time only, product of two terms. And that in fact-- because we're interested in vibration. You can tell me what the function of time is.

You're going to tell me half the solution just from observation. What is it? Just the time dependent part. It's the same as anything else that vibrates. So a single degree of freedom system, what is the time dependent function that we substitute in to find the natural frequency?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Say again?

AUDIENCE: e to the i omega t.

PROFESSOR: e to the i omega t would be just fine. Cosine omega t works. Sine omega t works. But e to the i omega t is pretty easy to use. Because it's so simple to take the derivatives. So we can guess that this is going to be some W of x times Ae to the i omega t. And plug it in. Plug it into our wave equation over here.

So I'll make sure I write it consistently. So we plug this into the first term. It's two derivatives with respect to x. So this is just-- and the time-dependent part just stays outside. And on the right-hand side, when we plug it in here, 1 over c squared, two derivatives with respect to time, it's going to give me minus omega squared, so minus omega squared over c squared.

And then it gives me back W of x Ae to the i omega t. And now I can get rid of the Ae to the i omega t's. And I'm left with just an equation involving x only. And it's an ordinary differential equation in w of x.

So it turns into d2W dx squared plus omega squared over c squared W equals 0. And you've seen this equation before. Does this not look like, have some similarity to, Mx double dot plus kx equals 0? They're basically the same equation. This one's a function of x. That one's a function of time.

And we know the solution to this one is some x of t is some amplitude e to the i omega t. So therefore, we can guess that the solution to this one is W of x is going to be-- I'll write it as some B. Now I need a function of x. But it can be just like this-- e to the i, and I'll say kx. I know that's going to be a solution.

So let's plug it in. If I plug that in, I get minus k squared Be to the ikx plus omega squared over c squared Be to the ikx equals 0. Well, now I get rid of these. And what I found out is that k squared is omega squared over c squared.

And this has a name-- k. It's called the wave number. And it also happens to be 2 pi over lambda. We'll come back to that. Lambda is the wavelength. You have sinusoidal waves running through the medium. 2 pi over lambda is the same as omega over c. And this is called the wave number-- really important quantity if you're trying to understand wave propagation in systems.

And actually, this one, this definition applies to all wave bearing systems, whether or not they obey the wave equation. It'll apply to waves traveling down a beam as well. So the definition of wave number is frequency divided by speed, or 2 pi over the wavelength.

Well, let's see. We can't go much further with just the wave equation itself. In order to get the natural frequencies, we have to invoke other information that we know in the problem. In particular, we know that in order to get natural frequencies, we had to create conditions where this could vibrate. In particular, I fix that end, and I fix this end, and I put some tension on it. And now it'll vibrate. But it clearly has something to do with its ends and its length.

And so this is a boundary value problem. And we have to invoke the boundary conditions to actually finish finding the natural frequencies and mode shapes. Apply the boundary conditions-- so I assumed here that my W of x is going to look something like that.

In order to get a little more information out of this, I'm going to write now W of x in an alternative form that's equally valid. And I'll call it B1 cosine kx plus a B2 sine kx. And I could relate that to e to the ikx, B to the ikx, by real and imaginary parts, and so forth. This is a real part. I'm saying in general it could have a cosine part and also a sine part.

But now I know my boundary conditions are W at x equals 0. W of 0 is what? What's the displacement at x equals 0?

AUDIENCE: [INAUDIBLE]

PROFESSOR: 0. That's the pin. That's the end where it's fixed at. And we started out here with a second order partial differential equation. And a second order equation requires two boundary conditions. A fourth order equation for the beam will require four boundary conditions.

We only have to find two. One of them is it has no motion on the left. So you plug in 0 for x. Cosine of 0 is 1. Sine of 0 is 0. So we find out that this is B1 times 1. But it has to be 0 as the boundary condition. So that implies B1 is 0. There's no cosines in this answer. And W at L is 0. And so that says B2 sine kL equals 0.

And that's true. That's only true if kL equals n pi. So now I've found out that there's, just for vibration of a finite length string, only particular values of k that work. So that says that there are special values of k which I'll call k sub n which are equal to n pi over L.

And from that, we now have our mode shapes. Because we can say, ah, well, there's special solution for this W of x that applies only when we satisfy the boundary conditions. And that will be some undetermined amplitude. B2 came from the sine term. And those are our mode shapes.

And now the natural frequencies-- once you know mode shapes, natural frequencies actually become pretty trivial to find. In this case, if we know that's the mode shape, then how do we get the natural frequencies? Well, we know that-- what's the definition of k?

Therefore, the particular values of k that were allowed solutions here are going to correspond to particular values of omega n. And therefore, omega n squared is just kn squared c squared. And that's n pi over L squared T/m. That's omega n squared. So the natural frequencies of a string are n pi over L root T/m.

And this is in radians per second. And I like to work in hertz sometimes. So the natural frequencies in hertz-- omega n over 2 pi. And that becomes n over 2L root T/m. So the first natural frequency, f1, is 1 over 2L root T/m.

Now, let's draw. What's the mode shape for the first mode? Well, it's half a sine wave, vibrates like that. It's full wavelength. I didn't leave myself quite enough room. That's half a wavelength of a sine wave. So the full wavelength would be like that. This is of length L. And so is this piece over here.

So the lambda is 2L for this particular problem. Let's see, how do I want to pose this question? So how long does it take for a wave or disturbance to travel the length of this finite string? How long does it take it to go down there and back? How would you calculate that?

Distance equals rate times time. What's the distance? 2L. What's the speed? c. So the length of time ought to be 2L over c, right? So the time required-- and 2L divided by T over m. But f1 is T/m divided by 2L. Hmm.

So the period-- so there's a direct connection between propagation speed, frequencies, wavelengths. They're very closely related. So the natural frequency of the first mode of this string, that frequency, is exactly 1 over the length of time it takes for a disturbance to travel down and back.

So with that depth of understanding of how the wave equation behaves, you can guess the behavior of lots of other things that behave like that, like my rod here. I'll do a little demo with it in a second.

So for example, the longitudinal vibration, stress waves running up and down this thing, obey the wave equation. So if I take this thing and drop it on the floor, it'll bounce off the floor. How long does it take to bounce off the floor?

So what do you think actually-- what physics has to happen? What's required to make this thing bounce off the floor? So we're going to consider the floor infinitely rigid. It hits the floor. It actually stays there for some finite length of time, and then it leaves.

So physically, when I was holding up my string, if I smacked the end, what happened? A pulse took off, ran down the end, reflected, came back. And that was one round trip. What do you suppose happens here? I put a pulse into the end. Is it a tension or compression, the strain that's felt?

AUDIENCE: Compression

PROFESSOR: Compression. So a little compression pulse is put into the end. That compression pulse then, when it first hits, the compression and the speed of propagation is finite. So that compression wave starts traveling up here. Behind the compression wave, this rod has come to a stop.

In front of the compression wave, the rod doesn't know it hit the ground yet. It's still moving down. So that compression wave travels up, and it is decelerating each little slice of mass as it passes through. It brings it to a stop. And so the compression reaches the top end. The cylinder has come to a stop.

The end is free. It can't take any strain. So an equal and opposite tension wave has to start to make the sum of them go to 0 at the end. The boundary condition at the end is no strain. So it reflects as a tension wave.

Now you have a tension wave going down. And what it does is it accelerates every atom as it goes by, as it goes past it. So everything is stopped now. Now it starts down, and this thing starts rebounding from-- the top rebounds from the floor before the bottom does.

The top starts going up. All of it-- more and more goes up. And one hits the bottom. The tension wave hits the floor, and it jumps off. So how long does it take? Right?

And what do you guess the natural frequency of a free-free rod is? Now, it has a funny mode shape. The mode shape is not half a sine wave like this. The displacement of the rod, it has free ends. The ends are moving a lot. But I'll give you a clue.

[ROD RINGING]

I can hold it in the center and not damp it. What do you think the mode shape looks like? Half a wavelength long, ends are free-- cosine, maximum displacement, goes to zero, maximum negative displacement. So it's half a wavelength long, but it's a cosine half a wavelength. And the full wavelength is 2L.

So this has mode shapes. The mode shapes-- I've applied different boundary conditions. These are free-free boundary conditions. The mode shapes are cosine n pi x over L. But they have to obey a certain other law that we know about, conservation of momentum. Because I've got gravity to deal with, I have to hang on to this thing.

But I've picked a place to hang onto it that you can hear it. I'm not affecting the motion. There's no motion where I'm holding it. So if I were out in space, I could do this--

[ROD RINGING]

--and just let it hang there in space, right? And it would sit there and ring. What is happening to the center of mass of this system as it vibrates?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Stationary. So half of the mass of this thing's got to be moving that way. And half of the mass has to be moving that way so that the total center of mass doesn't move. Well, cosine mode shape, positive here, negative there, perfectly symmetric, center of mass doesn't move.

So there's all sorts of neat little problems that you can solve just by knowing the wave equation and figuring out boundary conditions. How many of you stand in the shower at home and sing, and every now and then, you hit a note, man, you just sound great, right? And it's just all this reverberation. How many of you have done that? OK, right, what's going on?

AUDIENCE: [INAUDIBLE] Natural frequency?

PROFESSOR: You've hit a-- somebody said natural frequency. Of what?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Huh?

AUDIENCE: [INAUDIBLE]

PROFESSOR: You've hit the natural frequency of the shower stall itself. If the shower stall is a meter across, pressure waves-- and you plot pressure inside of the shower, the lowest mode if you're plotting pressure. Well, let's plot actually molecular movement.

What's the boundary condition at the wall, the molecules at the wall? They can't move, right? 0. So the molecular motion at resonance in the shower stall, the molecules, the pressures making them move back and forth, looks like back to the string again.

This is L. The first natural frequency of sound waves bouncing off the walls in the stall is 1 over 2L root times c, whatever c is. And c is the speed of sound in air, which is 340 meters per second.

So 340 meters per second divided by 2L-- so if it's 1 meter across here, it's 340 divided by 2, 170 hertz. So that first note you can hit in the 1 meter across shower stall is about 170 hertz-- pretty low. But you can hit second mode. It'd be twice that, and so forth.

OK, what about an organ pipe? This is an organ pipe, wood. It's got a stoppered end. Actually, let's do it without the stopper. Now it's an open organ pipe.

[ORGAN NOTE]

Basic wave equation-- how would you model its boundary conditions? So you can talk about maybe particle molecular motion. This is, now again, just sound waves, so air particles. And this is now longitudinal. Things are moving inside. So what's the boundary condition at this end, free or fixed? Free. And here it's quite open, so the boundary condition on here is free.

So for the molecular motion in a free-free organ pipe, you have to get back to that half a wavelength cosine thing. And if you wanted to plot pressure instead, you can write the wave equation in terms of pressure. Pressure is-- this is pressure relief here and pressure relief there. So in fact, if is displacement of the molecules, pressure would plot like that. You'd have what's called a pressure relief boundary condition.

But again, it's a half wavelength long. What do you think the first natural frequency of this organ pipe is? The period would be 2L over c. The frequency would be c over 2L. So the frequency for the organ pipe open end f1 is c over 2L.

[ORGAN NOTE]

Check your intuition. I'm going to close the end-- still an organ pipe. Is the frequency now going to be higher or lower? Take a vote. How many think the frequency is going to go up? Raise your hands, commit. All right, down. We've got a lot of uncertainty. All right, let's do the experiment.

[ORGAN NOTE]

[LOWER ORGAN NOTE]

How come? I find that actually kind of counterintuitive. Until I learned this, I would have guessed the opposite way. What's going on with pressure in a closed pipe? Well, here at the orifice where the sound is actually generated, it's the pressure. If we wanted to plot pressure at the opening, that's a pressure relief place. So it's 0. But at the other end where the stopper is, it's maximum. How many wavelengths is that? A quarter.

And so the length of time it takes for the thing to go through one complete period is going to be 4L over c, half the frequency of the open pipe. OK, so the wave equation is really quite powerful, governs lots of things.

I've got 10, 15 minutes left here. I don't want you to go away thinking that the whole world behaves like the wave equation. Because there are some important other physical systems that we care about.

And I'm going to show you just one. And that's the vibration of the beam. So here's the cantilever beam. The whole table is moving. And you can see it up on the screen. OK, so its first mode vibration, tip moves maximum. It kind of looks like a quarter wavelength. It roughly is, but not exactly.

So let's draw a cantilever. And most of you have had 2.001. So if you put a load P out here-- bends, goes through a displacement delta. So you know that delta equals Pl cubed over 3EI, right? And what's this I?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Area moment of inertia. Now that you've been doing dynamics all term, we talk about mass moments of inertia. There's also area moments of inertia. So this is the area moment of inertia of a beam.

In this case, our beam is a little rectangular cross section. And the neutral axis is here, a little variable y at displacement. I is the integral of y squared dA. And dA is just a little slice of area here, dA.

And the integral of y squared dA is your cross sectional area moment of inertia in the direction of bending. So that is I. You can also write it as a kappa squared A. And we ran into this in dynamics. We called it the radius of gyration. You had the same thing with area moments of inertia, the radius of gyration. This is going to be really helpful in a second.

So if you solve the force balance for a beam like I did for the string, take a little slice, do force balance for transverse motions-- I'm not going to grind it out. And temporarily neglect external forces and damping. I want to get to the natural frequencies and mode shapes.

So the free vibration, no damping, equation looks like EI partial 4 w with respect to x to the fourth plus rho A partial squared w with respect to t squared equals 0. And now this is density, mass density. And the A, this A, is the area of the cross section.

So it's just some bh with thickness times the width. So rho times A is a mass per unit length. And so mass per unit length times dx would be the little mass associated with the element times the acceleration should be the forces on the element. So that's the fourth order partial differential equation that describes the vibration of a beam. And you have to apply the boundary conditions.

And for the string, it was just B1 cosine B2 sine. For the beam, it's B1 cosine plus B2 sine plus C2 cosh plus D2 sinh x. And then you have to apply four boundary conditions and solve for B1, B2, and so forth, all four of those. I won't do it. But that's how you do it. Separation-- and separation of variables works again. So we solve this, apply the boundary conditions.

What are the boundary conditions? Just so you understand what I mean by the boundary conditions, what are they for a free-free beam, zero motion at the wall? No strain at the end, no bending moment at the end, no sheer force at the end-- so there's no second derivative, no third derivative. And at the wall, the slope is 0, the first derivative.

No slope comes into the wall, but the slope is 0 there. So those are the different kind of boundaries. So if you have a free-free beam, you have no bending at either end and no strain at either end. Fixed-fixed beam-- no displacement, zero slopes at both at ends, and all different combinations. And every different combination gives you different natural frequencies.

So you apply the boundary conditions, and for a beam, you find out that for all beams omega n can be written as some beta n, a parameter, squared, I'll call it, times the square root of EI over rho A. And this thing varies according to the boundary conditions.

Now that's what you get shown in every textbook in the world. And I have a very hard time visualizing this, getting physical intuition by that. So something you never see in a textbook but I often do is let's replace I with kappa squared A.

And you get a square root of E over rho and a square root of I over A. But I is kappa squared A. The A's cancel. It's the square root of kappa squared. So this, you know what E over rho is? E over rho, square root of E over rho, is the sound speed in a solid material. So the speed of stress waves traveling up and down this thing is the square root of E over rho.

[ROD RINGING]

This is aluminum. It's about 4,000 meters a second. So if you know just the properties of the material, you have that. And that says then omega n for beams is some beta n squared a parameter times kappa CL. And this thing, this is often written CL. It's the longitudinal sound speed.

This is sound speed for waves traveling through the medium. So this tells you if you make the beam twice as thick, what do you do to its natural frequencies? Doubles-- instantly you know that. So bending properties depend a lot on the radius of gyration. And I'll give you a few natural frequencies for different boundary conditions just so you see what they behave like.

So a pin-pin beam looks like that. So you put a plank across the stream, rocks on both sides, you've got a pin-pin beam, basically. It's set there in rock. So some length L has properties EI. So the natural frequencies for a pin-pin beam, the beta n's, are just n pi over L.

And so your natural frequencies-- omega n looks like n pi over L quantity squared kappa CL. And for the cantilever, the natural frequencies look like omega n pi squared over 4L squared, is the beta n. And I'll write it this way again-- EI over rho A. You can always go back and do that. Or you can call it kappa CL. This is also kappa CL.

But then there are some numbers you've got to use here-- 1.194 squared. That's the first mode. Second mode-- 2.988 squared. And then after that-- 5 squared, 7 squared, 9 squared. So this is the natural frequency of a cantilever. Pi squared over

4L squared times 1.194 squared kappa CL, that's this natural frequency.

And one final case, because I can show it to you-- the free-free case. So that's a beam bending that vibrates like that. And I happen to know on a beam for the first mode-- this is the first mode of a beam. Where these nodes are, where there's no motion, I should be able to hold it there and not damp it. And that turns out to be at about the quarter points. So whack it like that.

[ROD RINGING]

And do it again.

[ROD RINGING]

All right, so I want you to hold it about right there. Nope, you can't hold it like that, though-- just got to balance it. Because you've got to be right where the node is.

[ROD RINGING]

You can hear that little bit lower tone. That's that free-free bending mode. And it's just sitting. You can feel it vibrating a little bit but not much. When you're right in the right spot, you're right on the mode shape. You can almost see it if you hit it hard enough.

So that's the free-free beam. And the free-free beam has natural frequencies omega n, again, pi squared over 4L squared kappa CL 3.0112 squared, 5 squared, 7 squared, 9 squared, so as you go up in n. So those are the natural frequencies of a free-free beam.

Oh, one last fact about beams-- so this is now a steel beam under no tension. It can support its own weight, long though. So can a beam support waves traveling down the beam, transverse waves traveling down the beam? What do you think? Well, if it can support this, it can probably support waves, right?

So waves will propagate in a beam even though this is fourth order partial differential equation. But how fast do they go? That's the question. So this is a beam. And I want to know about waves traveling down it. And I'm not going to go through-- this would take another hour or so to show you where this comes from.

But here's my beam. Here's a disturbance traveling along it with some speed that I'm going to call CT. It's transverse wave speed. It's the speed you'd see a crest of a wave moving at running down that beam.

CT for a beam-- square root of omega kappa CL. And CL, again, is the square root of E over rho. That's the speed of sound in the material. That just turns up in here. So what does this tell you about the frequency dependence of the speed? Does the speed change with frequency?

Omega kappa CL-- it's proportional to frequency. High frequency waves go faster than low frequency waves in a beam. I didn't emphasize it when we were talking about it. But the wave equation, what was c for the string? For the wave equation, the speed of wave propagation was square root of T/m. Was it frequency dependent? Always traveled at the same speed. And so there's an important consequence.

So for anything that obeys the wave equation, the speed of propagation is a constant and independent to frequency. So I can make any initial shape that I make in this thing and let it go. Its initial disturbance, that little shape will stay that shape and run up and down the thing forever. And that shape-- you could imagine a little pluck like this to start with. You could imagine doing a Fourier series to approximate that. It would be made up of a bunch of different Fourier components.

And yet for something that bears the wave equation, that little pluck will just stay the shape of that pluck and run around forever. But not so in a beam. If you did that in a beam, if you come up and put an impulse into a beam, all that energy would start out together. But in very brief time, the high frequency information would get out in front of the low frequency information.

And if you were way down this beam, and somebody up a mile away whacks one end, and you're down further along, you'll see high frequency waves past you, and then lower frequency, and finally really slow ones coming by, the really long waves. So that's called dispersion.

So beam waves are dispersive. Things that obey the wave equation are non-dispersive. The energy all travels at the same speed independent of frequency. All right, so that's it for the term. I'll see you guys on next Wednesday.