2.004 Dynamics and Control II
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.004 *Dynamics and Control II*
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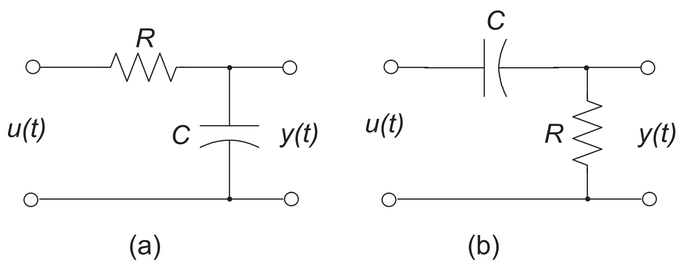
Solution of Problem Set 10

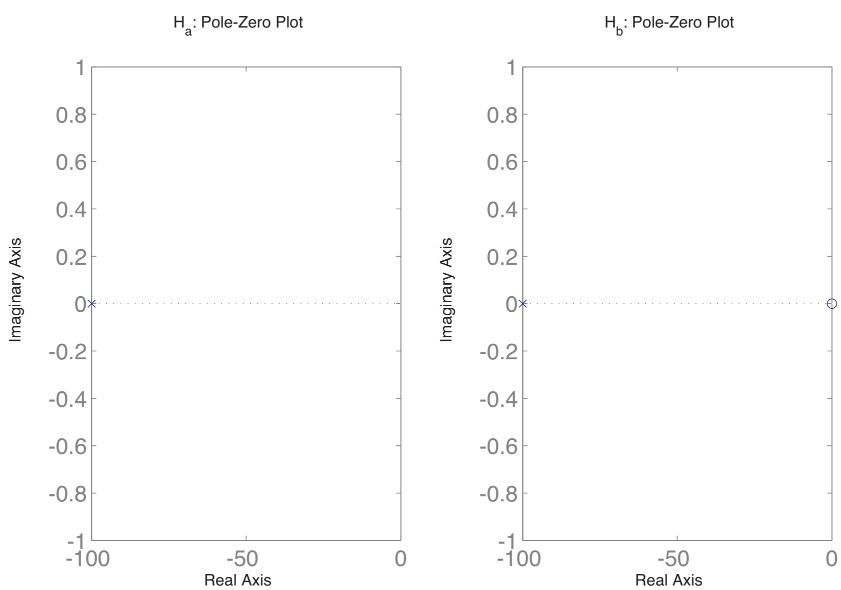
Problem 1:



(a) Transfer functions can be simply calculated by utilizing the voltage divider rule:

$$H_a(s) = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{1}{RCs + 1} \qquad H_b(s) = \frac{R}{R + \frac{1}{Cs}} = \frac{RCs}{RCs + 1}$$

(b)



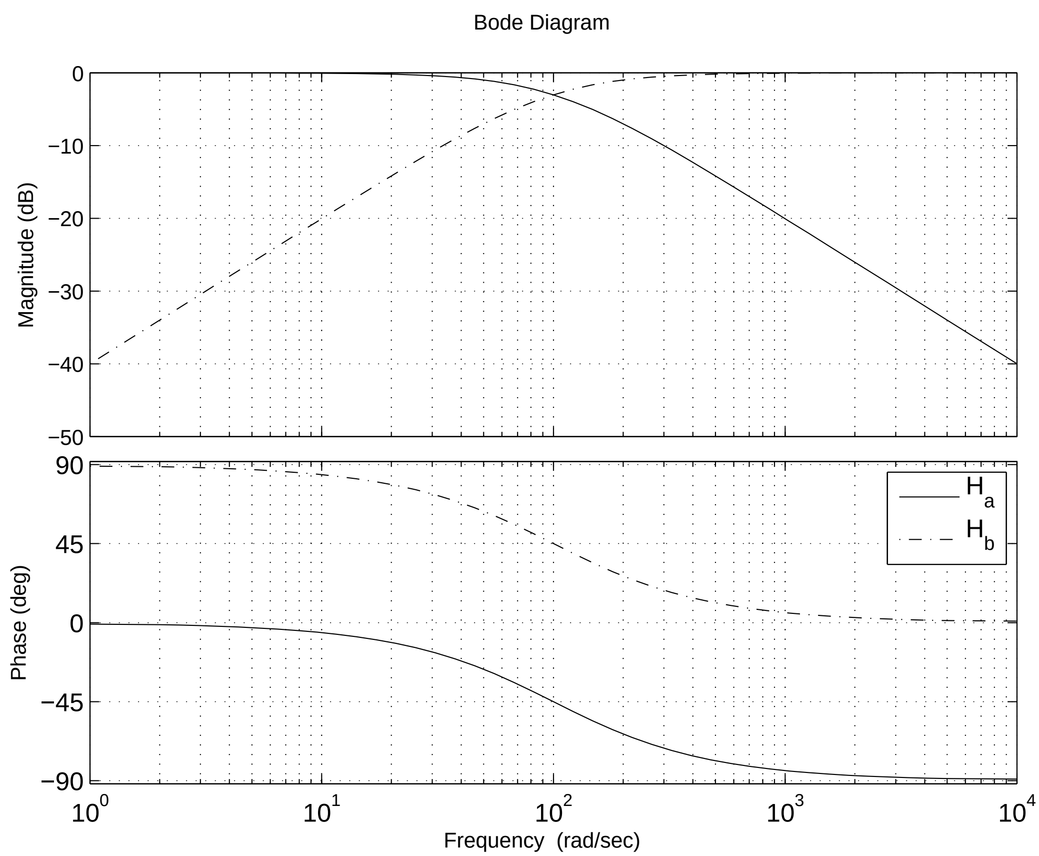
(c) The frequency response is:

$$H_a(j\omega) = H_a(s)|_{s=j\omega} = \frac{1}{jRC\omega + 1} \qquad H_b(j\omega) = H_b(s)|_{s=j\omega} = \frac{jRC\omega}{jRC\omega + 1}$$

giving

$$
\begin{aligned}
|H_a(j\omega)| &= \left|\frac{1}{jRC\omega+1}\right| = \frac{1}{\sqrt{(RC\omega)^2+1}} \\
\angle H_a(j\omega) &= \angle(1) - \angle(jRC\omega+1) = \tan^{-1}(RC\omega)
\end{aligned}
$$

$$
\begin{aligned}
|H_b(j\omega)| &= \left|\frac{jRC\omega}{jRC\omega+1}\right| = \frac{RC\omega}{\sqrt{(RC\omega)^2+1}} \\
\angle H_b(j\omega) &= \angle(jRC\omega) - \angle(jRC\omega+1) = \pi/2 - \tan^{-1}(RC\omega)
\end{aligned}
$$



(d) The circuit a is a low-pass system because high frequencies are attenuated. On the other hand circuit b is a high-pass one because as the frequency approaches zero $|H(j\omega)|$ tends to zero.

Problem 2:

$$H(s) = \frac{T(s)}{Q_s} = \frac{1/(mc_p)}{s + (hA/(mc_p))}$$

(a) From the transfer function:

$$\frac{dT}{dt} + \frac{hA}{mc_p}T = \frac{1}{mc_p}Q_s$$

(b) The steady-state temperature T_{ss} is found by letting all derivatives approach zero, that is if the solar heat flow is a constant:

$$T_{ss} = \frac{1}{hA} Q_o$$

(c) From the transfer function:

$$H(j\omega) = \frac{T(j\omega)}{Q_s(j\omega)} = \frac{1/(mc_p)}{j\omega + (hA/(mc_p))}$$

Then

$$|H(j\omega)| = \frac{1/(mc_p)}{\sqrt{\omega^2 + (hA/(mc_p))^2}}, \qquad \angle H(j\omega) = -\tan^{-1}\frac{\omega mc_p}{hA}$$

(d) Assume that $Q_s(t) = Q_o \sin(\omega_o t - \pi/2) + Q_{avg}$ and note that $\omega_o = 2\pi$ rad/year= $2\pi/365$ rad/day $= 2\pi/(365 \times 86400)$ rad/sec. The solution is composed from a constant input, part b, and a sinusoidal input from part c:

$$T(t) = \frac{Q_o/(mc_p)}{\sqrt{(\omega_o)^2 + (hA/(mc_p))^2}} \sin\left(\omega_o t - \pi/2 - \tan^{-1}\frac{\omega_o mc_p}{hA}\right) + \frac{1}{hA} Q_{avg}$$

The annual fluctuation of the pond temperature $\Delta(T)$ is equal to two times sinusoidal amplitude:

$$\Delta(T) = T_{max} - T_{min} = 2Q_o \left|H(j\omega)\right|_{\omega=\omega_o} = \frac{Q_o/(mc_p)}{\sqrt{(\omega_o)^2 + (hA/(mc_p))^2}}$$

(e) $T(t)$ is a maximum when $\omega_o t - \pi/2 - \tan^{-1}(\omega_o mc_p/hA) = \pi/2$ or

$$\begin{aligned} t_{max} &= (\pi/2 - (-\pi/2 - \tan^{-1}\omega_o mc_p/hA))/2\pi \text{ years} \\ &= 365\left(\frac{1}{2} + \frac{1}{2\pi}\tan^{-1}\frac{\omega_o mc_p}{hA}\right) \text{ days from the start of the year.} \end{aligned}$$

<u>**Problem 3:**</u>

(a) The slope of the high frequency asymptote is $-(n_p - n_z) * 20$ dB/decade where n_p is the number of system poles and n_z is the number of system zeros: (a) -20dB/decade, (b) -40 dB/decade, (c) -40 dB/decade, (d) -20dB/decade.

(b) The asymptotic high frequency phase response is $(n_z - n_p) * \pi/2$ rad: (a) $-\pi/2$ rad, (b) $-\pi$ rad, (c) $-\pi$ rad, (d) $-\pi/2$ rad.

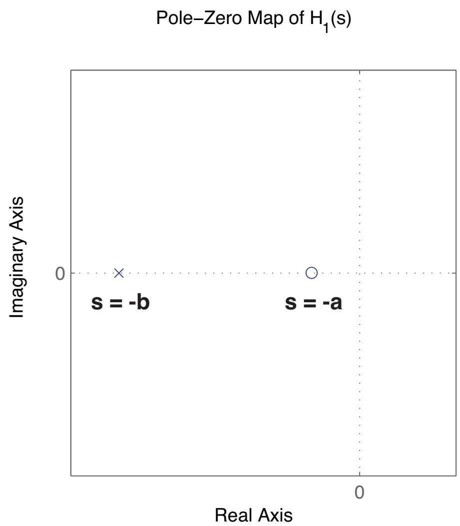
(c) The low frequency asymptotic behavior is determined by poles or zeros at the origin: (a) 0 dB/decade slope: the low frequency response tends to a constant value, (b) - /decade slope: the low frequency response tends to infinity, (c) 20 dB/decade slope: the low frequency response tends to zero, (d) 0 dB/decade slope: while in principle the low frequency response tends to a constant value, this is a marginally stable system.

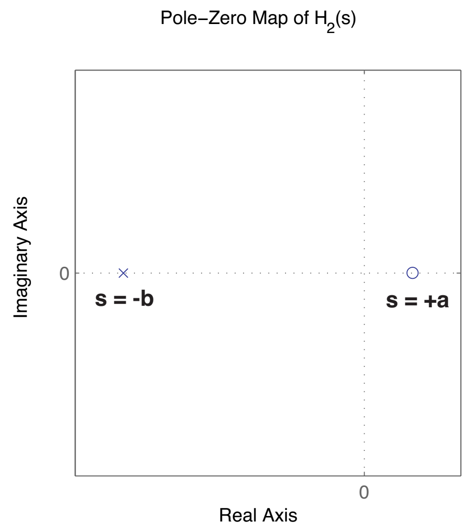
(d) The low frequency phase shift is determined by the contribution from each pole and zero: left-half plane poles/zeros do not contribute, right half plane zeros contribute π rad, zeros and poles at the origin contribute $\pm\pi/2$ rad. (a) 0 rad, (b) $-\pi/2$ rad, (c) $+3\pi/2$ rad, (d) 0 rad.

Problem 4:

(a) No, the stability of a system is not affected by its zeros.

(b) For the systems to be stable, we only consider $b > 0$ cases. A typical pole zero plot for the case when $a > 0,\ b > 0$ is shown below:





Both systems have the same magnitude plot ($|H_1(j\omega)| = |H_2(j\omega)|$), while they differ on the phase plot:

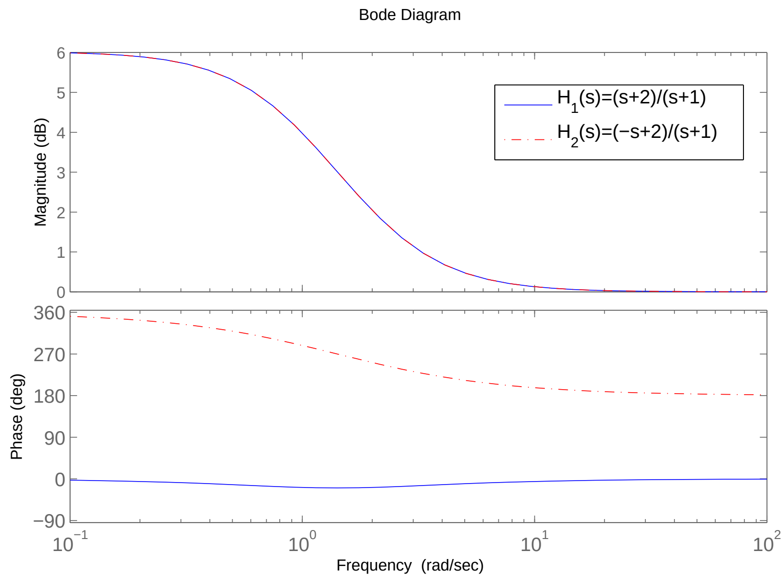
$$H_1(s) = \frac{s+a}{s+b}, \qquad H_2(s) = -\frac{s-a}{s+b} = \frac{-s+a}{s+b}$$

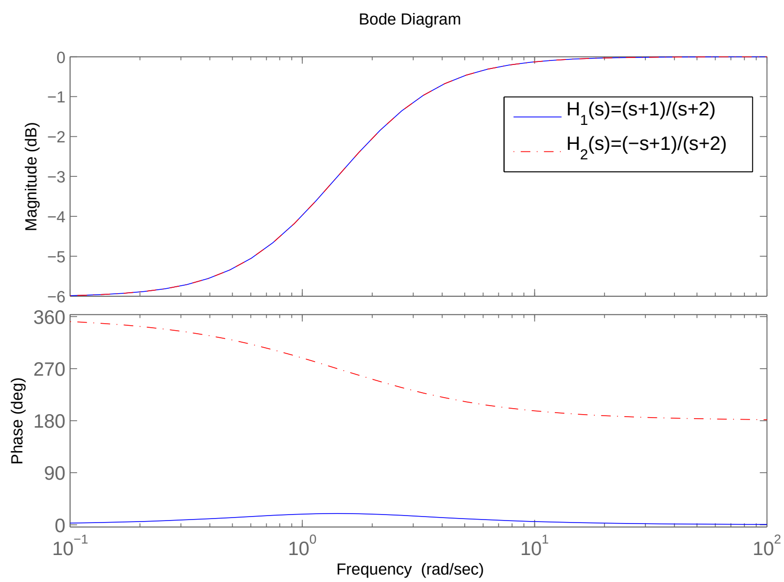
$$|H_1(j\omega)| = |H_2(j\omega)| = \frac{\sqrt{\omega^2+a^2}}{\sqrt{\omega^2+b^2}}$$

$$\angle H_1(j\omega) \;=\; \tan^{-1}\frac{\omega}{a} - \tan^{-1}\frac{\omega}{b}, \quad \text{if a} > 0,\ \text{b} > 0$$

$$\angle H_2(j\omega) = 2\pi - \tan^{-1}\frac{\omega}{a} - \tan^{-1}\frac{\omega}{b}, \ \text{if a} > 0,\ \text{b} > 0$$

The exact form of the Bode plots depends on the relative location of the pole and zero. Here for example, we examine two cases:
1. $\{a = 2,\ b = 1\}$.
2. $\{a = 1,\ b = 2\}$.

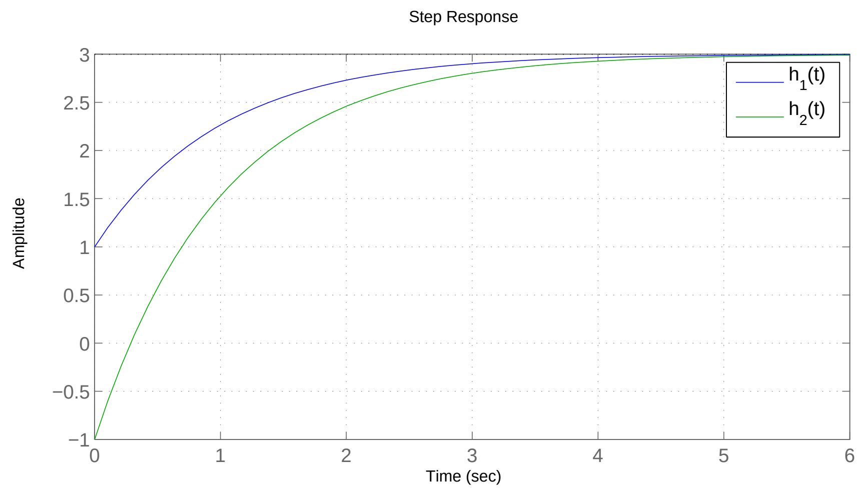




(c) The magnitude of the frequency response function is not affected by whether a zero is located in the left-half plane or its reflection about the imaginary axis. The phase response is significantly affect however. In general the phase-shift associated with a right-half plane zero is greater than that of the corresponding left-half plane position - this can be easily demonstrated using the geometric interpretation from the pole-zero plot. Hence the name "non-minimum phase" system.

(d) Both of them have the same final value (equal to $H(0) = H(s)|_{s=0}$). On the contrary their initial value is different (equal to $H(\infty) = H(s)|_{s=\infty}$). While both systems reach the same final value for the same input; the first one initially moves on the same direction as the input, while the second one initially moves on the opposite direction of the input.

$$H_1(s) = \frac{s+3}{s+1}, \qquad H_2(s) = -\frac{s-3}{s+1} = \frac{-s+3}{s+1}$$



(e) If

$$H_2(s) = \frac{s-a}{s+a}$$

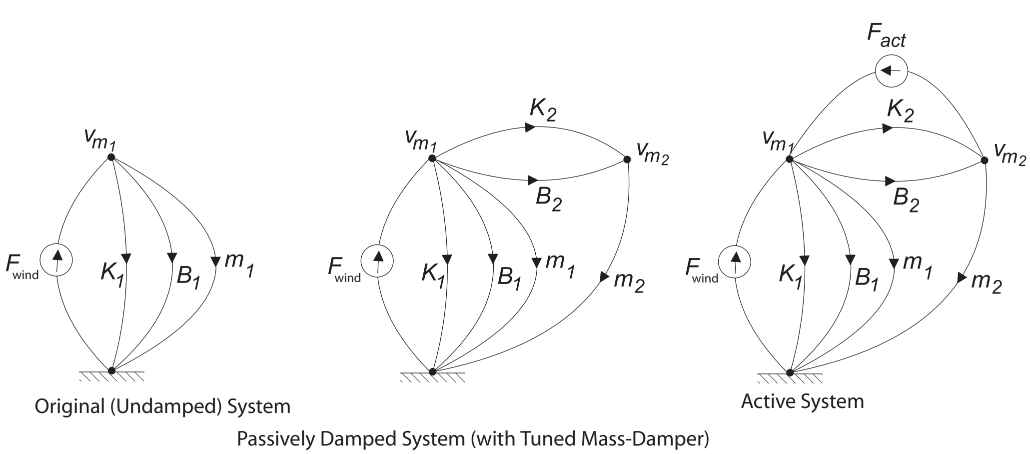
then

$$|H_2(j\omega)| = \frac{\sqrt{(j\omega)^2 + a^2}}{\sqrt{(j\omega)^2 + a^2}} = 1 \quad \text{and} \quad \angle H(j\omega) = \pi - 2tan^{-1}(j\omega/a)$$

The magnitude is independent of frequency, giving rise to the term "all-pass" filter. Note that the phase shift is a function of frequency.

Problem 5:

(a) For a comprehensive study, here we look at three transfer functions simultaneously. The first one corresponds to the original system, the second one corresponds to the passively damped system and the third one corresponds the active system. Those systems are shown in the below figure:



The transfer function of the original system can be computed as follows:

$$G_{original}(s) = \frac{V_{m_1}(s)}{F_{wind}(s)} = \frac{1}{Y_{m_1} + Y_{B_1} + Y_{K_1}} = \frac{1}{m_1 s + B_1 + K_1/s} = \frac{s}{m_1 s^2 + B_1 s + K_1}$$

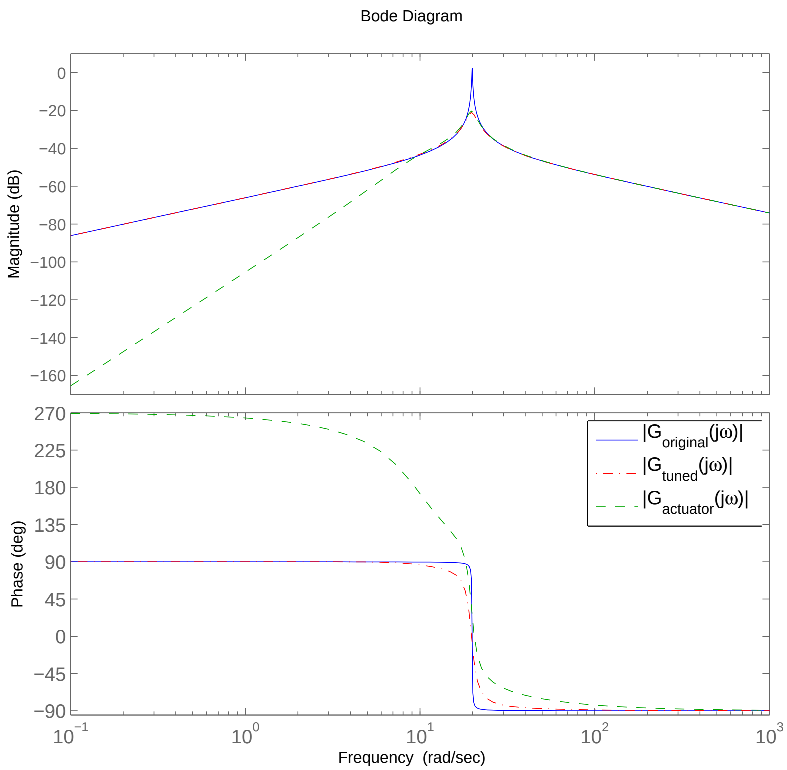
The second and third transfer functions are taken from Problem Set 7:

$$G_{tuned}(s) = \frac{V_{m_1}(s)}{F_{wind}(s)} = \frac{m_2 s^3 + B_2 s^2 + K_2 s}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$
$$G_{active}(s) = \frac{V_{m_1}(s)}{F_{act}(s)} = \frac{m_2 s^3}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

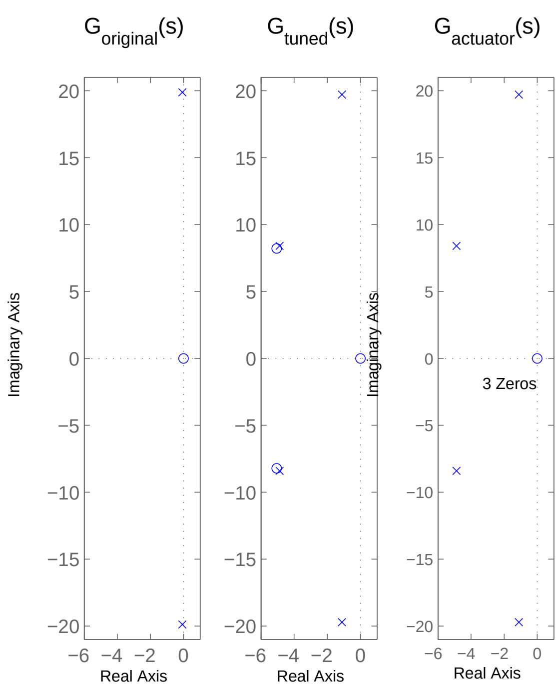
$$\begin{aligned}
\text{where} \quad a_4 &= m_1 m_2 \\
a_3 &= (m_1 + m_2) B_2 + m_2 B_1 \\
a_2 &= (m_1 + m_2) K_2 + m_2 K_1 + B_1 B_2 \\
a_1 &= K_1 B_2 + K_2 B_1 \\
a_0 &= K_1 K_2
\end{aligned}$$

The groups get slightly different estimated values, but typically: $m_1 = 5.11$ kg, $B_1 = 0.77$ N.s/m, $K_1 = 2020$ N/m, $m_2 = 0.87$ kg, $B_2 = 8.8$ N.s/m, $K_2 = 81$ N/m . Hence the transfer functions would be equal to:

$$G_{original}(s) = \frac{V_{m_1}(s)}{F_{wind}(s)} = \frac{s}{5.11\ s^2 + 0.77\ s + 2020}$$
$$G_{tuned}(s) = \frac{V_{m_1}(s)}{F_{wind}(s)} = \frac{0.87\ s^3 + 8.8\ s^2 + 81\ s}{4.446\ s^4 + 53.29\ s^3 + 2249\ s^2 + 17840\ s + 163620}$$
$$G_{active}(s) = \frac{V_{m_1}(s)}{F_{act}(s)} = \frac{0.87\ s^3}{4.446\ s^4 + 53.29\ s^3 + 2249\ s^2 + 17840\ s + 163620}$$


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
|G_original(jω)|
|G_tuned(jω)|
|G_actuator(jω)|

(b)


G_original(s)
G_tuned(s)
G_actuator(s)
Imaginary Axis
Real Axis
3 Zeros

(c) The original and tuned system both have the same asymptotic behaviors at low frequencies and high frequencies (+20 dB/decade slope for low frequencies and -20 dB/decade slope for high frequencies). Their slopes match because they have the same number of zeros at origin, as well as the same number of excessive poles ($n_p - n_z$). Furthermore, asymptotes lie on each other because both have the same approximation for low and high frequencies ($\frac{s}{K_1} = \frac{K_2 s}{K_2 K_1}$ and $\frac{s}{m_1 s^2} = \frac{m_2 s^3}{m_2 m_1 s^4}$). Moreover, both of them have a resonant peak located at $\omega_n \approx 20$ rad/s and very close to the imaginary axis. Careful examination of pole-zero map shows that while their ω_n is almost the same; they only differ in ζ value which contributes to the sharper peak of the original system. There is another pair of complex poles in the tuned system as well, but since it is very close to a zero complex pair, its net effect is very subtle.

The tuned and active systems both have the same denominator and exactly the same term for the largest power of s in the numerator. This means that their high frequency behaviors are the same. At high frequencies, both have a slope of -20 dB/decade and a phase of $-\pi/2$ rad. At low frequencies, G_{tuned} has a slope of +20 dB/decade and a phase of $\pi/2$ rad, while the G_{active} has a slope of +60 dB/decade and a phase of $3\pi/2$ rad. Furthermore, they have only one resonant peak which matches and is located at $\omega_n \approx 20$ rad/s and very close to imaginary axis. At the resonant peak, their magnitudes differ less than 1 dB while their phases differ about $\pi/4$ rad.

(d) The original system and tuned system have the same Bode plot for low and high frequencies. This means that tuned mass-damper is only effective for intermediate frequencies (around the peak $\omega_n \approx 20$ or ≈ 3 Hz), where we have a significant building sway reduction.

(e) The peak is decreased by 23 dB which means that maximum amplitude around 3 Hz is decreased by $10^{\frac{23}{20}} \approx 14.5$ times.