2.004 Dynamics and Control II
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.004 *Dynamics and Control II*
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Lecture 28[1]

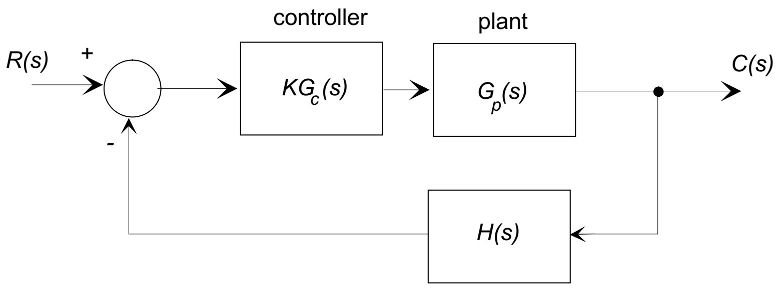
Reading:

- Nise: Chapter 8

1 Root Locus Development (contd. from Lecture 27)

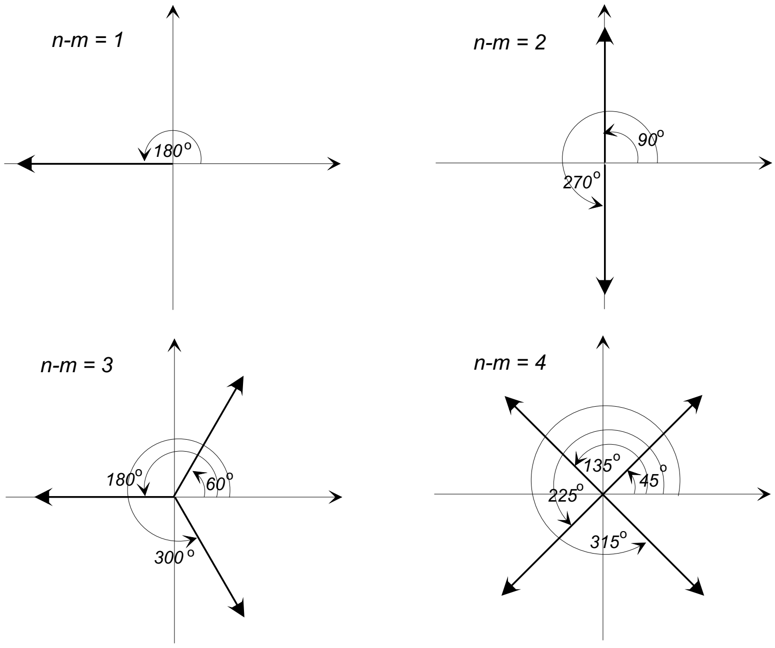
1.1 Behavior of the Root Locus as the Gain K Becomes Large (contd.)

For a closed-loop system



with open-loop transfer function $G(s) = KG_c(s)G_p(s)H(s)$, we saw in Lecture 27 that the asymptotic angles are summarized in the following table:

$n - m$	Asymptote Angles
1	180°
2	90°, 270°
3	60°, 180°, 300°
4	45°, 135°, 225°, 315°



We now refine this property a little further. Let

$$G(s) = K\frac{s^m + b_{m-1}s^{m-1} + b_{m-1}s^{m-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-1}s^{n-2} + \ldots + a_0} = K\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}.$$

For any polynomial of degree k, the coefficient of the term in s^{k-1} is the *sum of the roots of the polynomial*, so that

$$\begin{aligned} b_{m-1} &= -(z_1 + z_2 + \ldots + z_m) \\ a_{n-1} &= -(p_1 + p_2 + \ldots + p_n). \end{aligned}$$

Further for large s, assume that we can ignore all but the first two terms in each polynomial, and write:

$$G(s) \approx K\frac{s^m + b_{m-1}s^{m-1}}{s^n + a_{n-1}s^{n-1}} = K\frac{s^m - (z_1 + z_2 + \ldots + z_m)s^{m-1}}{s^n - (p_1 + p_2 + \ldots + p_n)s^{n-1}}.$$

For large K, m closed-loop poles approach m of the open-loop zeros (which are also closed-loop zeros). and if we assume pole/zero cancellation we can do polynomial division and write

$$G(s) \approx K\frac{1}{s^{n-m} + (a_{n-1} - b_{m-1})s^{n-m-1}}$$

We can then use the relationship that for large x

$$x^k + cx^{k-1} \approx (x + c/k)^k$$

to write

$$G(s) \approx K\frac{1}{(s-\sigma_a)^{n-m}}$$

where

$$\sigma_a = -\frac{a_{n-1} - b_{m-1}}{n-m} = \frac{(p_1 + p_2 + \ldots + p_n) - (z_1 + z_2 + \ldots + z_m)}{n-m}$$

The closed-loop characteristic equation for large s and large K may be approximated

$$1 + KG(s) \approx 1 + \frac{K}{(s-\sigma_a)^{n-m}} = 0$$

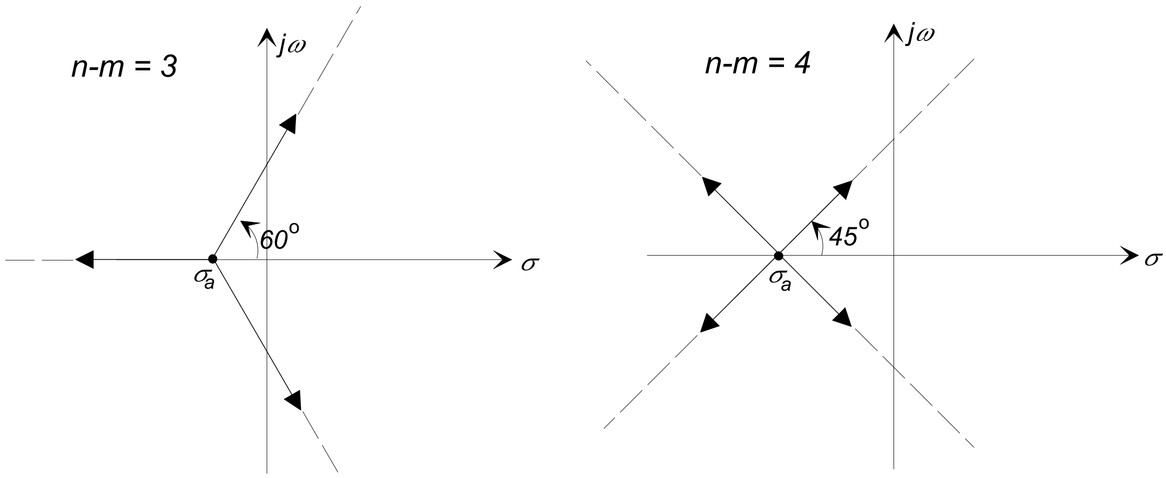
and the asymptotes are given by

$$(s - \sigma_a) = K^{1/(n-m)}(-1)^{1/(n-m)}$$

which is similar to the result obtained in Lecture 27, namely that the roots lie on a circle of radius $K^{1/n-m}$, but this time with the difference that the asymptotes radiate from the point $s = \sigma_a$. The point $s = \sigma_a$ is defined as the *centroid* of the asymptotes and

$$\sigma_a = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m},$$

and σ_a is always real because poles and zeros are either real of complex conjugates. The following figure shows the displacement of the origin of the asymptotes:



The root locus sketching rule, for constructing the locus of a system with m open-loop zeros, and n open-loop poles, for large values of gain K is therefore:

- **As the value of $K \to \infty$, m of the closed-loop poles approach the m open loop zeros.**
- **As the gain K becomes large, $n - m$ branches of the root locus diverge away from a point $s = \sigma_a$ on the real axis and approach $n - m$ radial asymptotes, at angles $\theta_k = (2k+1)\pi/(n-m)$, for $k = 0 \ldots (n-m-1)$.**

■ Example 1

Sketch the root locus plot for the open-loop system

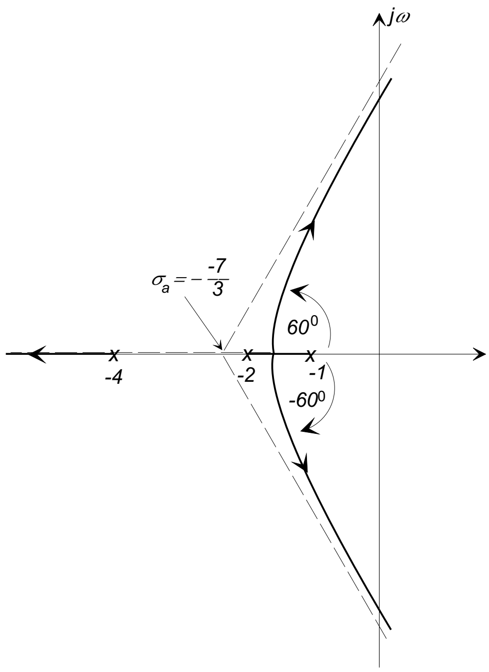
$$G(s) = K\frac{1}{(s+1)(s+2)(s+4)}$$

and find the gain K at which it becomes unstable. We proceed as follows:

1. Determine and plot the open-loop poles and zeros.
2. Determine and plot the regions of the real axis that lie on the root locus.
3. Determine the number of asymptotes. There are no finite zeros, therefore $n - m = 3$.
4. Determine the asymptote angles and centroid, then sketch the asymptotes. For three asymptotes the angles are (see the above table) 60°, 180°, 300°. The centroid is

$$\begin{aligned}
\sigma_a &= ((\text{sum of the poles}) - (\text{sum of the zeros}))/(n-m) \\
&= \frac{1}{3}((-1-2-4)-(0)) = -\frac{7}{3}
\end{aligned}$$

These steps were used to produce the following sketch:



The closed-loop characteristic equation is:

$$(s+1)(s+2)(s+4) + K = s^3 + 7s^2 + 14s + 8 + K = 0$$

and at the point of marginal stability (when $s = j\omega$)

$$-j\omega^3 - 7\omega^2 + j14\omega + 8 + K = 0 + j0$$

Equating the real and imaginary parts

$$\begin{aligned}
-7\omega^2 + 8 + K &= 0 \\
-\omega^3 + 14\omega &= 0
\end{aligned}$$

giving

$$\begin{aligned} \omega &= 0, \sqrt{14} \\ K &= -8, 90 \end{aligned}$$

Since the roor locus is defined only for $K > 0$ we conclude that the system will become unstable for $K > 90$, and the locus will cross the imaginary axis at $s = \pm j\sqrt{(14)}$ rad/s.

■ Example 2

Show the effect of PD control on the root-locus of the previous example.

Let

$$G_c(s) = K_p + K_d s = K(s + b),$$

where $b = K_p/K_d$ and $K = K_d$. The PD controller has added a zero at $s = -b$ to the system. The open-loop transfer function is now

$$G(s) = K\frac{s + b}{(s + 1)(s + 2)(s + 4)}$$

and we have $n - m = 2$.

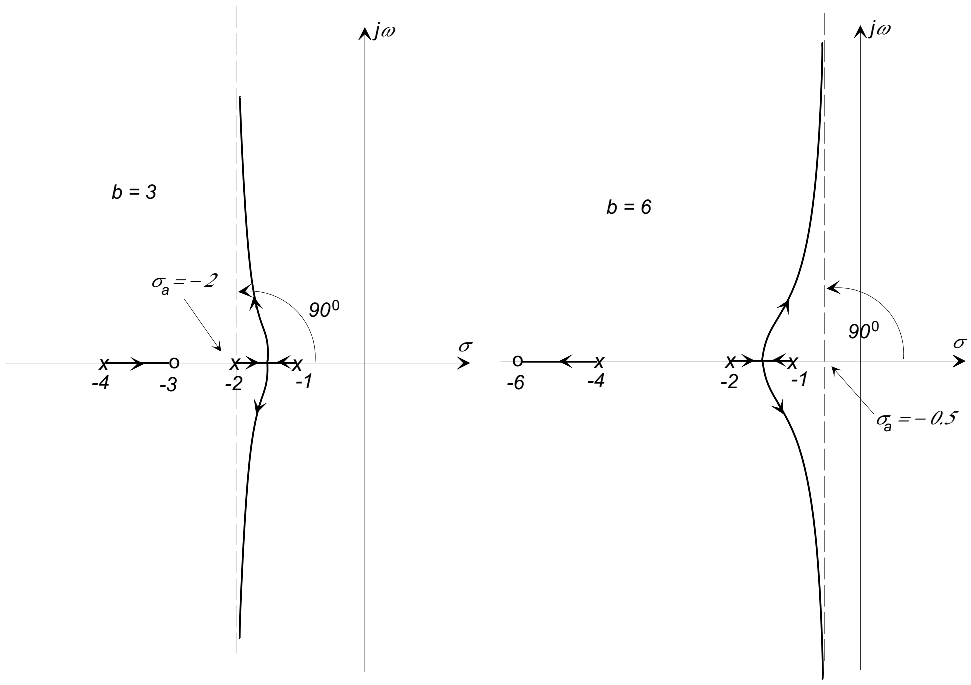
Assume for now that $b = 3$. There will be $n - m = 2$ asymptotes, at angles 90° and 270°. The centroid will be

$$\sigma_a \frac{1}{2}((-1 - 2 - 4) - (-3)) = -2$$

Now assume $b = 6$. The centroid will be

$$\sigma_a \frac{1}{2}((-1 - 2 - 4) - (-6)) = -0.5$$

These two cases are sketched below:



(a) P-D zero at s= -3

(b) P-D zero at s= -6

Notice that as the PD zero moves deeper into the l.h. plane, it moves the asymptote toward the imaginary axis, meaning that the dominant closed-loop poles become lightly damped. You can show for yourself that if $b > 7$, the asymptote origin $\sigma_a > 0$ and this system will become unstable as K is increased.

■ Example 3

Show some typical effects of PID control on the root-locus of the previous example.

With PID control

$$\begin{aligned} G_c(s) &= K_p + \frac{K_i}{s} + K_d s \\ &= K_d \frac{s^2 + (K_p/K_d)s + K_i/K_d}{s} \\ &= K \frac{(s - z_1)(s - z_2)}{s} \end{aligned}$$

The PID controller has added a pole at the origin, and two zeros, which we as designers can place in order to shape the root locus to meet a set of specifications on the dynamic response. The open-loop transfer function is now

$$G(s) = K \frac{(s - z_1)(s - z_2)}{s(s + 1)(s + 2)(s + 4)}$$

and we have $n - m = 2$. There will be $n - m = 2$ asymptotes, at angles 90° and 270°. The two root loci below show cases for

(a) two real zeros ($z_{1,2} = -4.5, -5$), and

(b) two complex conjugate zeros ($z_{1,2} = -2 \pm j2$).

In the first case the centroid will be

$$\sigma_a = \frac{1}{2}((0 - 1 - 2 - 4) - (-4.5 - 5)) = 1.25,$$

while in the second case the centroid will be

$$\sigma_a = \frac{1}{2}((0 - 1 - 2 - 4) - (-2 + j2 - 2 - j2)) = -0.5$$

These two cases are sketched below: