

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering
2.004 Dynamics and Control II
Fall 2007

Problem Set #2

Solution

Posted: Friday, Sept. 21, '07

1. In class, we showed in two different ways that the torque constant of a DC motor equals the back-EMF constant, $K_m = K_v$. Verify from the definitions of these constants, $K_m i = T$ and $K_v \omega = v_e$, respectively, that the units associated with these constants are consistent as well.

Answer: From the definition of K_m and K_v ,

$$[K_m] = \frac{[N \cdot m]}{[A]} = \frac{[J]}{[A]},$$

and

$$[K_v] = \frac{[V]}{[1/s]} = \frac{[W/A]}{[1/s]} = \frac{[W \cdot s]}{[A]} = \frac{[J]}{[A]}.$$

Therefore they have the same units.

2. Rework Problem 2 of Problem Set #1 (the motor-shaft system of Lecture 2 with non-zero initial condition) by using the Laplace transform. You should arrive at the same step response as you did in Problem Set #1.

Answer: The equation of motion is

$$\frac{d\omega}{dt} + \frac{b}{J}\omega = \frac{T_0 u(t)}{J}.$$

Using the Laplace transform, we can write as

$$s\Omega(s) - \omega_0 + \frac{b}{J}\Omega(s) = \frac{T_0}{Js}.$$

and it can be solve as

$$\begin{aligned}\Omega(s) &= \left(\frac{T_0}{sJ} + \omega_0 \right) \frac{1}{s + b/J} \\ &= \frac{\omega_0}{s + b/J} + \frac{T_0}{sJ(s + b/J)} \\ &= \frac{\omega_0}{s + b/J} + \frac{T_0}{b} \left(\frac{1}{s} - \frac{1}{s + b/J} \right).\end{aligned}$$

By taking the inverse Laplace transform, we end up with

$$\omega(t) = \omega_0 e^{-t/\tau} + \frac{T_0}{b} (1 - e^{-t/\tau}),$$

where $\tau = J/b$. The result is identical to the solution we derived in the lecture.

3. Obtain the inverse Laplace transform of the following frequency-domain expressions:

a) $F_1(s) = -\frac{(4s - 10)}{s(s + 2)(s + 5)}$;

Answer: Using partial fraction expansion, it can be re-written as

$$F_1(s) = \frac{1}{s} - \frac{3}{s + 2} + \frac{2}{s + 5}.$$

Apply the inverse Laplace transform, then we end up with

$$f_1(t) = (1 - 3e^{-2t} + 2e^{-5t})u(t)$$

b) $F_2(s) = \frac{4}{s^2(s^2 + 4)}$;

Answer: Since $F_2(s) = \frac{1}{s^2} + \frac{-1}{s^2+4}$, its inverse Laplace transform is

$$f_2(t) = \left(t + \frac{1}{2} \sin 2t \right) u(t)$$

c) $F_3(s) = \frac{s^3 - 3s^2 + s + 2}{s} H(s)$, where $H(s)$ is a well-behaved function of s ,

whose inverse Laplace transform is $h(t)$.

Answer: It can be re-written as

$$F_3(s) = \left(s^2 - 3s + 1 + \frac{2}{s} \right) H(s).$$

Therefore its inverse Laplace transform is

$$f_3(t) = \frac{d^2 h(t)}{dt^2} - 3 \frac{dh(t)}{dt} + h(t) + 2 \int_{0^-}^t h(\tau) d\tau.$$

4. Obtain the transfer function of problem 4.a in the Problem Set #1 by Laplace transforming the equations of motion that you derived previously. (If you don't have copy of your solution, you can find the equations on motions in the solutions to Problem Set #1 that have been posted on **Stellar**.) The input to the system is the force $f(t)$ and the output the rotation angle $\theta(t)$.

Answer: The equations of motion are

$$\begin{aligned} M\ddot{x}(t) + 2f_v\dot{x}(t) + Kx(t) - f_v r\dot{\theta}(t) &= f(t), \\ J\ddot{\theta}(t) + r^2 f_v\dot{\theta}(t) - r f_v\dot{x}(t) &= 0. \end{aligned}$$

By taking the Laplace transform, they can be written as

$$\begin{aligned} Ms^2X(s) + 2f_v sX(s) + KX(s) - f_v r s\Theta(s) &= F(s), \\ Js^2\Theta(s) + r^2 f_v s\Theta(s) - r f_v sX(s) &= 0. \end{aligned} \quad (1)$$

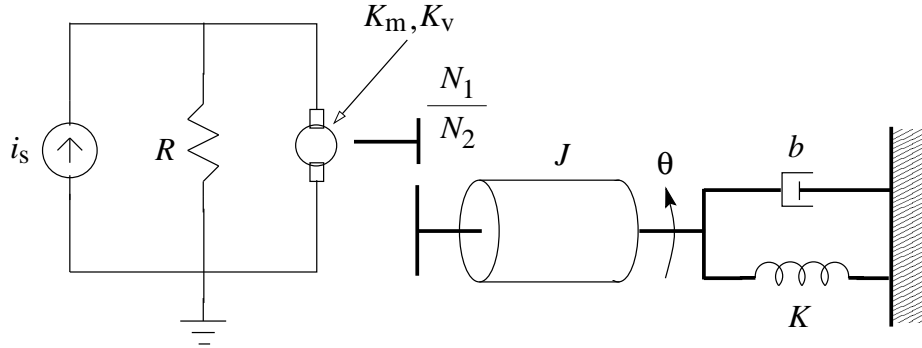
Combining above two equations, we have

$$\{JM s^3 + f_v (r^2 M + 2J) s^2 + (f_v^2 r^2 + KJ) s + Kr^2 f_v\} \Theta(s) = f_v r F(s),$$

and the transfer function is

$$\frac{\Theta(s)}{F(s)} = \frac{f_v r}{JM s^3 + f_v (r^2 M + 2J) s^2 + (f_v^2 r^2 + KJ) s + Kr^2 f_v}$$

5. On the next page is a diagram of a DC motor connected in parallel to a current source i_s . The torque and back-EMF constants of the motor are K_m , K_v , respectively, the motor resistance is R , also modeled as connected in parallel, the motor inertia is J_m (not shown), and the motor inductance is negligible. The motor load is an inertia J with compliance K and viscous friction coefficient b , and it is attached to the motor via a gear pair with gear ratio N_1/N_2 . The system input is the current i_s and the output is the rotation angle θ of the inertia. Derive the transfer function of this system.



Answer:

Let's define $\phi(t)$ as the rotation angle of the motor. From the relation of the gear pair, we know $\phi N_1 = \theta N_2$ and $\dot{\phi} N_1 = \dot{\theta} N_2$. T is the torque generated by the motor and it is scaled by N_2/N_1 at the inertia by the gear pair. Also the inertia of the motor is scaled by N_2^2/N_1^2 by the gear pair. (For the detail, please refer the lecture note 1 or section 2.7 of Nise)

From KCL at the node attached to the resistor,

$$i_s(t) - \frac{v_e}{R} - \frac{T}{K_m} = 0.$$

From torque balance at the inertia,

$$\left(J + \frac{N_2^2}{N_1^2} J_m \right) \ddot{\theta} = T \frac{N_2}{N_1} - b\dot{\theta} - K\theta.$$

(Converted torque from the motor drives the effective inertia (inertia + converted inertia of the motor shaft), viscous friction and compliance. For those who are still wondering, the equation of motion is derived in the last part of the solution.)

Using the Laplace transform and $v_e = K_v \dot{\phi}(t) = K_v \frac{N_2}{N_1} \dot{\theta}(t)$, the above two equations can be written as

$$\begin{aligned} I_s(s) - \frac{K_v}{R} \frac{N_2}{N_1} \Theta(s) s - \frac{T(s)}{K_m} &= 0, \\ \left\{ \left(J + \frac{N_2^2}{N_1^2} J_m \right) s^2 + bs + K \right\} \Theta(s) &= T(s) \frac{N_2}{N_1}. \end{aligned}$$

Solving the first equation with respect to T and plugging it into the second equation, we get the transfer function defined by

$$\frac{\Theta(s)}{I_s(s)} = \frac{(N_2/N_1)K_m}{\left(J + \frac{N_2^2}{N_1^2} J_m \right) s^2 + \left(b + \frac{N_2^2}{N_1^2} \frac{K_v K_m}{R} \right) s + K}.$$

* * *

To derive the equation of motion of the inertia, here T_m is torque generated from the motor (T of the above derivation), the torque rotating the first gear is T_1 and the torque rotating the second gear is T_2 . The relation between T_1 and T_2 is $T_2 = (N_2/N_1)T_1$. Therefore, the torque balance at the first and second gear is

$$\begin{aligned} (Js^2 + bs + K)\Theta(s) &= T_2, \\ J_m s^2 \Phi + T_1 &= T_m. \end{aligned}$$

From the second equation

$$T_1 = T_m - J_m s^2 \Phi$$

And using the relation of $T_2 = (N_2/N_1)T_1$, we end up with

$$T_2 = \frac{N_2}{N_1} \left(T_m - J_m s^2 \frac{N_2}{N_1} \Theta \right).$$

Plugging it into the first equation, finally we've got the following equation, which is the same as the above derivation.

$$\left\{ \left(J + \frac{N_2^2}{N_1^2} J_m \right) s^2 + bs + K \right\} \Theta = \frac{N_2}{N_1} T_m.$$