MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

2.004 Dynamics and Control II
Fall 2007

Problem Set #4

Solution Posted: Friday, Oct. 5, '07

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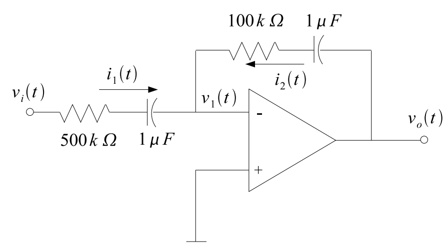
1. **(a)** Problem 21(a) from Nise textbook, Chapter 2 (page 113). **(b)** After you find the transfer function, locate the zeros and poles, draw them on the s–plane, and determine the type of the response of the system (undamped, underdamped, critically damped, or overdamped).

a. *Answer:*



We choose current $i_1(t)$ and $i_2(t)$ as shown in the figure.

$$I_1(s) = \frac{V_i - V_1}{5 \times 10^5 + \frac{10^6}{s}}$$

$$I_2(s) = \frac{V_o - V_1}{10^5 + \frac{10^6}{s}}$$

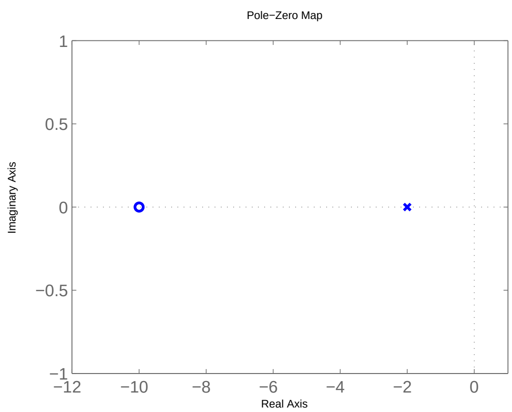
and

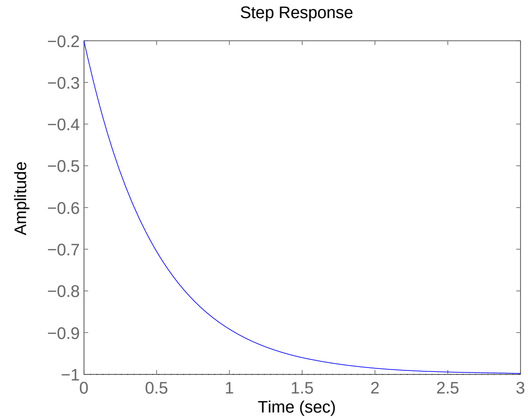
$$I_2(s) = -I_1(s).$$

Also $V_1 = 0$ because the positive terminal is connected to the ground. The negative terminal should have the same potential. Arranging the above equations, we obtain the transfer function given by

$$\frac{V_o(s)}{V_i(s)} = -\frac{1}{5}\frac{(s+10)}{(s+2)}.$$

b. *Answer:* Pole: -2, Zero: -10. It is a first order system since has only one pole. It also has one zero.

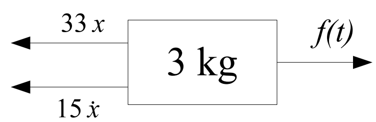




(Drawing the step response is not required)

2. Problem 25 from Nise textbook, Chapter 4 (page 236).

Answer:



From the free body diagram,

$$3\ddot{x} = f(t) - 33x - 15\dot{x}.$$

In the Laplace domain, the equation of motion is

$$(3s^2 + 15s + 33)X(s) = F(s).$$

The transfer function is
$$\frac{X(s)}{F(s)} = \frac{1/3}{s^2 + 5s + 11}.$$

(For the following items, refer to the lecture note 7 p.26)

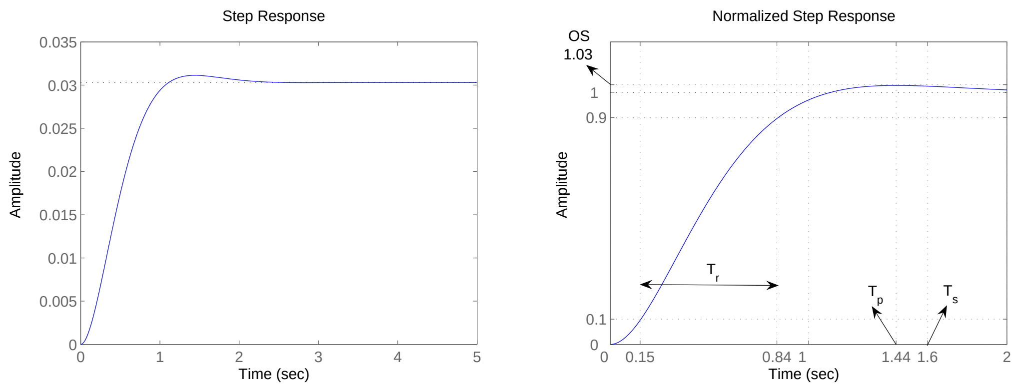
- The natural frequency $\omega_n = \sqrt{11}$ rad/s.
- The damping ratio $\zeta = 5/2/\omega_n = 0.7538$.
- $\%\text{OS} = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100 = 2.7\%$
- $T_s \approx \frac{4}{\zeta\omega_n} = 1.6 sec.$
- $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.4415 sec.$
- To compute the rise time approximately, we can use the normalized rise time method in section 4.5 of Nise textbook (pp.196). Since the damping

ratio $\zeta \approx 0.75$, the normalized rise time will be 2.2965 (s). If we divide it by ω_n, the rise time $T_r \approx 0.6925$ (s).

If you want to compute the exact rise time, you may use the exact step response. Since $\sigma_d = \zeta\omega_n = 2.5$ and $\omega_d = \omega_n\sqrt{1-\zeta^2} = 2.1794$, (by partial fraction expansion) the exact step response is

$$f(t) = 0.0303\left\{1 - e^{-2.5t}\cos(2.1794t) - 0.0681e^{-2.5t}\sin(2.1794t)\right\}u(t).$$

The step response and the normalized one (steady state = 1) are plotted.

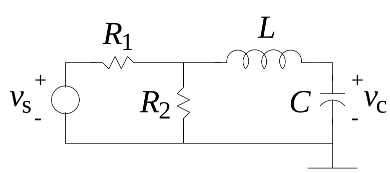


All specifications are marked and note that T_r ≈0.69 agrees with the approximated method very well.

3. Derive the transfer function of the electrical network shown below with the source voltage $v_s(t)$ as input and the voltage $v_c(t)$ across the capacitor as output, where $R_2 = 10\Omega$, $L = 1$ mH, $C = 10\mu$F and

i) $R_1 = 5\Omega$;

ii) $R_1 = 100\Omega$.



For each case,

a) locate the zeros and poles and draw them on the s–plane;

b) determine the type of the response (undamped, underdamped, critically damped, or overdamped);

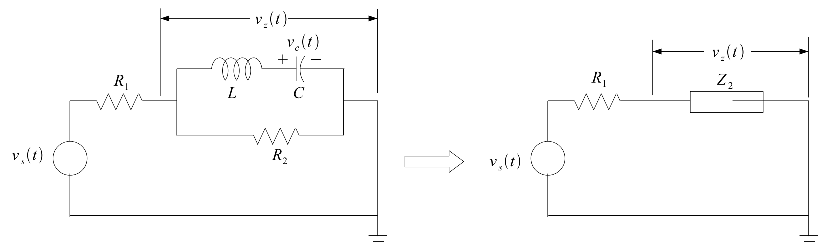
c) if you find that the system is *underdamped*, determine the natural frequency, the damped frequency, the settling time and the percent overshoot (%OS); if, on the other hand, you find that the system is *overdamped*, determine the two time constants;

d) determine the steady–state value directly from the transfer function;

e) plot the response to a step voltage of magnitude 5 V and verify that it matches your answers to the previous questions.

Answer:

From the voltage divider rule learned in lecture 11, we can think the following equivalent circuit.



The combination of R_2, C and L can be considered as impedance Z_1 and the voltage across Z_2 is denoted as $v_z(t)$. From the two figures, we can derive the following two relations in the Laplace domain.

$$V_c(s) = \frac{1/Cs}{Ls + 1/Cs} V_z$$
$$V_z = \frac{Z_2}{R_1 + Z_2} V_s$$

The impedance Z_2 is given by

$$\frac{1}{Z_2} = \frac{1}{R_2} + \frac{1}{Ls + 1/(Cs)}.$$

Rearranging the above equation, we can derive the transfer function.

$$\frac{V_c(s)}{V_s(s)} = \frac{R_2}{LC(R_1+R_2)s^2 + R_1R_2Cs + (R_1+R_2)} = \frac{R_2}{(R_1+R_2)LC} \times \frac{1}{s^2 + \frac{R_1R_2}{L(R_1+R_2)}s + \frac{1}{LC}}$$

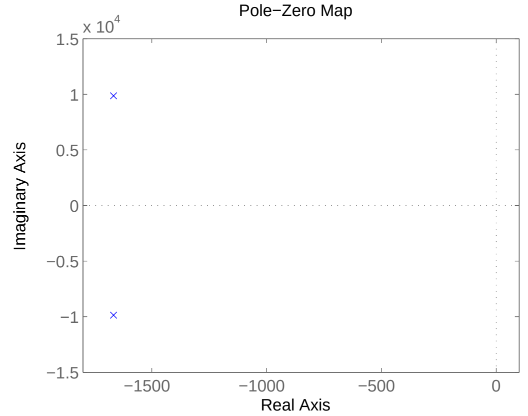
From KCL and KVL, you will end up with the same result.

i) $R_1 = 5\Omega$;

a) The transfer function is

$$T(s) = \frac{(2/3)\times 10^8}{s^2 + (1/3)\times 10^4 s + 10^8}.$$

Poles: $p_1 = \left(-5/30 + j\frac{\sqrt{35}}{6}\right) \times 10^4$ and $p_2 = \left(-5/30 - j\frac{\sqrt{35}}{6}\right) \times 10^4$



b) The system is underdamped because the two ples have imaginary component and their real part is negative (*i.e.*, on the left–hand plnae).

c) $\omega_n = 10^4 \;\; \text{rad/s}$,

$\zeta = 0.1667$.

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 9860 \;\; \text{rad/s}$.

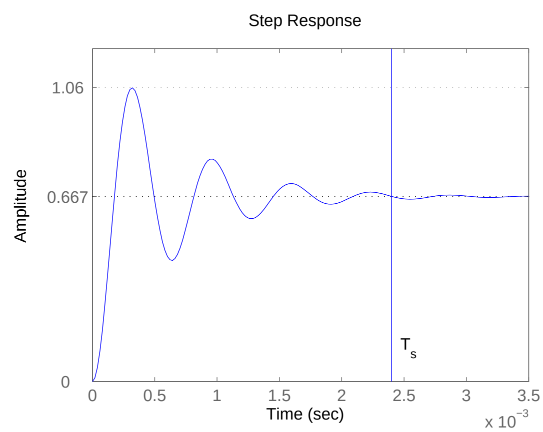
$T_s = \frac{4}{\zeta\omega_n} = 0.0024$ (s).

$\%\text{OS} = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100 \approx 58\%$.

d) From the final value theorem,

$$f(t)|_{t\to\infty} = \lim_{s\to 0} s\frac{1}{s}T(s) = \lim_{s\to 0} T(s) = \frac{2}{3}.$$

e) The step response.

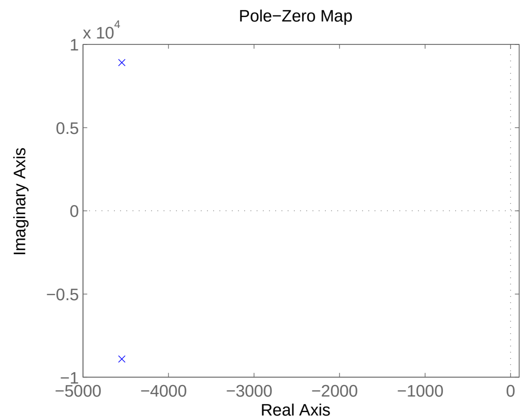


ii) $R_1 = 100\Omega$.

a) The transfer function is

$$T(s) = \frac{(1/11) \times 10^8}{s^2 + (1/11) \times 10^5 s + 10^8}.$$

Poles: $p_1 = \left(-10/22 + j\frac{\sqrt{96}}{11}\right) \times 10^4$ and $p_2 = \left(-10/22 - j\frac{\sqrt{96}}{11}\right) \times 10^4$



b) The system is underdamped because the two ples have imaginary component and their real part is negative (*i.e.*, on the left–hand plnae).

c) $\omega_n = 10^4$ rad/s.

$\zeta = 0.4545.$

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 8907.5$ rad/s.

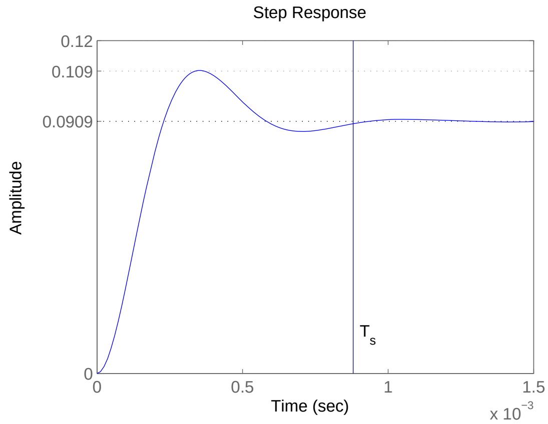
$T_s = \frac{4}{\zeta\omega_n} = 0.00088$ (s).

$\%\text{OS} = \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100 \approx 20\%.$

d) From the final value theorem,

$$f(t)|_{t\to\infty} = \lim_{s\to 0} s\frac{1}{s}T(s) = \lim_{s\to 0} T(s) = \frac{1}{11}.$$

e) The step response.



4. Consider again the system of a compliant mass driving a shaft that we modeled in PS01/4.a and derived the transfer function for in PS02/4. Substitute numerical values $M = 1$ kg, $J = 1$ kg $\cdot$ m^2, $f_v = 2$ kg/ sec, $r = 0.1$ m, and two cases of translational compliance

i) $K = 0.1$ N/ m;

ii) $K = 1.0$ N/ m.

For each case,

a) find the poles of the system (you can use the `roots` function in MATLAB , or any numerical analysis software of your choice);

b) determine if the system has dominant poles;

c) *sketch* by hand the step response of the system as accurately as you can from the transfer function but without solving for the exact solution. Do not use any numerical analysis tools for this part of the problem.

Answer:

The transfer function is

$$\frac{\Theta(s)}{F(s)} = \frac{f_v r}{JMs^3 + f_v\left(r^2M + 2J\right)s^2 + \left(f_v^2 r^2 + KJ\right)s + Kr^2 f_v}.$$

Plugging in the numerical values, we end up with

$$\frac{\Theta(s)}{F(s)} = \frac{0.2}{s^3 + 4.02s^2 + (0.04 + K)s + 0.02K}.$$

i) $K = 0.1$ N/m

$$T(s) = \frac{0.2}{s^3 + 4.02s^2 + 0.14s + 0.002}.$$

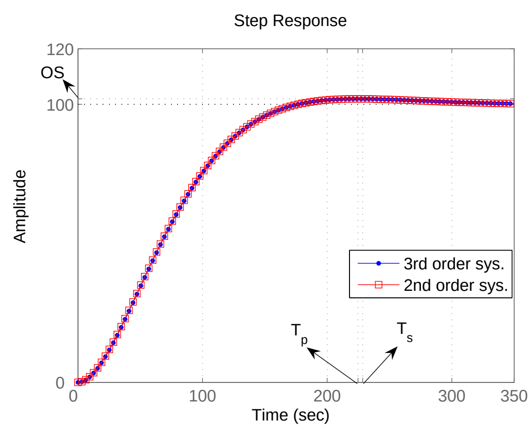
a) Poles: $p_1 = -3.9850$, $p_2 = -0.0175 + j\sqrt{0.014}$, $p_3 = -0.0174 - j\sqrt{0.014}$
Since two of the poles have imaginary parts, we can expect this three–pole system to have oscillatory response; this will become even clearer immediately next when we examine the system's dominant poles.

b) Dominant poles: p_2 and p_3. ($|\text{Re}\{p_2 \text{ or } p_3\}| << p_1$)

c) Neglecting the fast pole p_1, we can approximate the system transfer function as follows

$$T(s) = \frac{0.2}{(s-p_1)(s-p_2)(s-p_3)} \approx \frac{0.2/(-p_1)}{(s-p_2)(s-p_3)} = \frac{0.0502}{s^2-(p_2+p_3)s+p_2p_3}.$$

(we divide by $-p_1$ to ensure that the steady state value remains the same after approximating the three–pole system with its two dominant poles only.) From the denominator of the response, $\omega_n = 0.0224$ rad/s and $\zeta \approx 0.78$. This is an underdamped system. The estimated specifications are %OS$\approx$ 2%, $T_s = 228.5$ (s), $T_p = 224.7$ (s).



(The exact response of the third order system and the response from the two dominant poles alone are plotted together for your information.)

ii) $K = 1.0$ N/m

$$T(s) = \frac{0.2}{s^3 + 4.02s^2 + 1.04s + 0.02}.$$

a) Poles: $p_1 = -3.7436$, $p_2 = -0.255$, $p_3 = -0.0209$

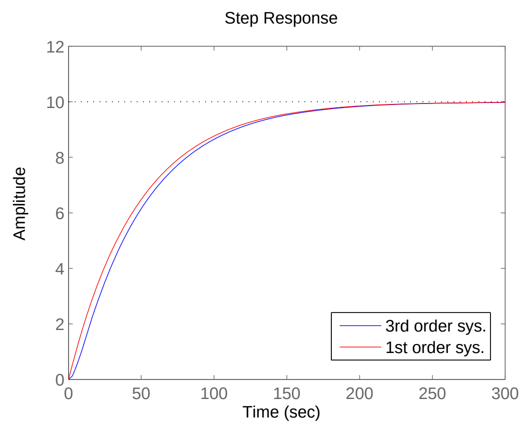
b) Dominant poles: $p_3 = -0.0209$. ($|10|p_3 < |p_2|$ and $10|p_2| < |p_1|$).

c) Its steady state value is 10. Neglecting the two fast poles p_1, p_2, we can approximate the system's transfer function as follows

$$\frac{K_0}{(s-p_1)(s-p_2)(s-p_3)} \approx \frac{K_0/(p_1p_2)}{(s-p_3)} = \frac{0.2091}{s+0.0209}.$$

(we divide by p_1p_2 to ensure that the steady state value remains the same after approximating the three–pole system with its single dominant pole.). The approximated system is a first order system, with step response

$$f(t) = 10(1 + K_0 e^{p_3 t})u(t).$$



(The exact response of the third order system and the response of the approximated first order system are plotted together for your information.)