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Different Methods for Dimensional Analysis

The example problem that will be used here is the same discussed in class on Thursday 2/21/08. We are looking for the dimensionless groups related to the problem of finding fluid forces on a surface ship. In lecture, we determined the force, F , would be a function of fluid density, ρ , the fluid dynamic viscosity, μ , gravity, g , the ship or fluid speed, U , and a length scale of the ship, L , which we can write as

$$F = f(\rho, \mu, g, U, L) \quad (1)$$

After we have defined the relevant variables, we should define the units of each. I will use M , L , and T to represent mass, length and time, respectively.

Variable	Units
Force, F	MLT^{-2}
Density, ρ	ML^{-3}
Dynamic Viscosity, μ	$ML^{-1}T^{-1}$
Gravity, g	LT^{-2}
Velocity, U	LT^{-1}
Length, L	L

Method 1 – Buckingham Pi Theorem

Obviously, for equation 1, the units on the left hand side (units of force) will have to equal the units on the right hand side. The Buckingham Pi theorem essentially gives us a method for rearranging this type of equation into a different equation relating dimensionless groups, called pi groups, which can be written as

$$\Pi_1 = \phi(\Pi_2, \dots, \Pi_{k-r}) \quad (2)$$

where k is the number of variables in the problem (in our case $k = 6$), and r is the minimum number of basic dimensions required to describe the variables (in our case $r = 3$; mass, length and time). Therefore, we need $k - r = 6 - 3 = 3$ dimensionless groups (or pi groups) to describe our problem. In other words, we have reduced our problem from one that has 6 variables to one that has only 3 variables.

The Buckingham Pi theorem is easy to carry out if you follow these steps:

1. List all variables involved in the problem. We already did this.
2. Express the variables in terms of the basic dimensions. We did this in our table.
3. Determine the required number of pi groups. We already showed that this is 6.

4. Select a number of repeating variables, where the required number is equal to the number of basic dimensions (in our case $r = 3$). The rules for choosing the repeating variables are as follows:
 - a. Do not choose the dependent variable (in our case this is force).
 - b. All of the basic dimensions must be contained within the set of repeating variables.
 - c. Each repeating variable must be dimensionally independent from the other repeating variable. This means that the dimensions of one variable cannot be reproduced by the products of powers of the other repeating variables. As an example unrelated to our problem, the units of gravity can be produced by forming the product U^2L^{-1} .
5. Find the first pi group by multiplying one of the nonrepeating variables raised to the power 1 by the repeating variables each raised to a power that makes the combination dimensionless. This will require solving a set of linear equations.
6. Repeat step 5 for the remaining nonrepeating variables to get other pi groups.
7. Check all pi groups to make sure they are dimensionless.
8. Now, we can express an equation relating the pi groups,

$$\Pi_1 = \phi(\Pi_2, \dots, \Pi_{k-r})$$

Of course, we don't know how these pi groups are related; people often do extensive experimentation and/or simulation to find these relationships. Also note that determining the relevant variables is difficult, because it often requires intuition and physical insight into a problem.

Okay, let's apply the Buckingham Pi theorem to our problem of forces on a surface ship. I will follow the outline above to solve the problem.

1. We determined the relevant variables in class and decided that we wanted the dependent variable to be force, giving us an equation of the form

$$F = f(\rho, \mu, g, U, L)$$

2. Express the variables in terms of the basic dimensions. We did this in our table.
3. Determine the number of pi groups. Well, we have $k = 6$ variables, and $r = 3$ basic dimensions. Therefore, we need $k - r = 6 - 3 = 3$ pi groups.
4. Since $r = 3$, we need a set of 3 repeating variables. I will choose density, length and velocity. The basic dimensions (M , L , and T) are all contained within the set of repeating variables. Also notice that we cannot combine any 2 variables in any way to form the dimensions of the third repeating variable.
5. I will show how to find 2 of the 3 pi groups and leave the 3rd as an exercise. First, we'll find the group corresponding to force being the nonrepeating variable, which is given by

$$\Pi_1 = F\rho^a U^b L^c = [MLT^{-2}][ML^{-3}]^a [LT^{-1}]^b [L]^c$$

Now, we need to find the exponents a , b , and c such that the net exponent on each of the basic dimensions is zero (i.e., the group has no dimensions). The method for this is to write a linear corresponding to each basic dimension.

$$\begin{aligned} M: 1 + a &= 0 \\ L: 1 - 3a + b + c &= 0 \\ T: -2 - b &= 0 \end{aligned}$$

solving yields $a = -1$, $b = -2$, and $c = -2$, which gives the dimensionless group

$$\Pi_1 = \frac{F}{\rho U^2 L^2}$$

which, if we divide by 2, is a force coefficient. Next, we'll find the group corresponding to gravity being the nonrepeating variable, which is given by

$$\Pi_2 = g\rho^d U^e L^f = [LT^{-2}][ML^{-3}]^d [LT^{-1}]^e [L]^f$$

$$\begin{aligned} M: d &= 0 \\ L: 1 - 3d + e + f &= 0 \\ T: -2 - e &= 0 \end{aligned}$$

solving yields $d = 0$, $e = -2$, and $f = 1$ which gives the dimensionless group

$$\Pi_2 = \frac{gL}{U^2}$$

which is a form of the Froude number. See if you can derive the 3rd pi group which is the Reynolds number

$$\Pi_3 = \frac{\rho UL}{\mu}$$

Notice that $1/\Pi$ is a valid dimensionless group, but it is not independent of Π ; so we can choose one or the other.

Reference:

Munson, Bruce R., Young, Donald F., and Theodore H. Okiishi, *Fundamentals of Fluid Mechanics*, 4thed. (Hoboken: John Wiley & Sons, Inc., 2002), Chapter 7.

Method 2 – Table Method Presented by Professor Techet in Lecture 4

This method is illustrated in slides 19-26 from Lecture 4. Please review these notes for the steps involved in this method. The same dimensionless groups will be found using this method.

Reference:

Smits, AJ., *A Physical Introduction to Fluid Mechanics*, (Hoboken: John Wiley & Sons, Inc., 2000), Chapter 8.

Method 3 – Intuitive Method

This method requires a deeper initial understanding of the physical problem, something that usually comes with engineering training, but something that will definitely wow your boss, research advisor, etc. We start with the same physical variables, and we again are looking for force to be the dependent variable. As a reminder, we start with the equation

$$F = f(\rho, \mu, g, U, L)$$

The number of dimensionless groups is still determined by subtracting r from k . We need $k - r = 6 - 3 = 3$ dimensionless groups. Since force is of interest, I want to non-dimensionalize (sometimes called normalizing) force by meaningful physical quantities. From knowledge of fluid mechanics, we tend to normalize pressures by the dynamic pressure, $\frac{1}{2}\rho U^2$. Since we want to normalize force, not pressure, we can multiply the dynamic pressure by length squared to make a quantity that has units of force. This gives a non-dimensional group given by

$$\text{Group 1} = \frac{F}{\frac{1}{2}\rho U^2 L^2} \equiv C_F$$

For the second group, I would like to make a ratio of “inertial effects” to “viscous effects;” these are two “effects” that play into the physics of the problem. When we think of inertia, we think of velocity and mass, so we expect U and ρ to come into play. Of course, viscosity is given by μ . Multiplying ρU by L gives a quantity with the same units as viscosity. So we can make a second non-dimensional group given by the ratio

$$\text{Group 2} = \frac{\rho UL}{\mu} = \frac{\text{Inertial Effects}}{\text{Viscous Effects}} \equiv \text{Reynold's Number} = \text{Re}$$

Finally, I am looking for a ratio of “inertial effects” to “gravitational effects;” these are two more “effects” that play into the physics of the problem. This time, multiplying g by L gives a quantity with the same units as U^2 , thus the final dimensionless group is given by

$$\text{Group 3} = \frac{U^2}{gL} = \frac{\text{Inertial Effects}}{\text{Gravitational Effects}} \equiv \text{Froude Number} = \text{Fr}$$

We then seek the manner in which these three groups are related; in other words, we seek to find the *form* of the function

$$C_F = f(\text{Re}, \text{Fr})$$

which is a very common problem in fluid mechanics.