2.016 Hydrodynamics

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Pressure effects

Fluid forces can arise due to flow stresses (pressure and viscous shear), gravity forces, fluid acceleration, or other body forces. For now, let us consider a fluid in static equilibrium – with no velocity gradients (thus no viscous stresses).

Forces are then due only to:

1. Pressure acting on the fluid volume
2. Gravity acting on the mass of the fluid
3. External body forces

Using standard conventions we will consider pressure to be positive for compression. Recall that we said pressure is isotropic: Consider a triangular volume of fluid with height, *dz*, length, *dx,* and unit width, *b*, into the page:

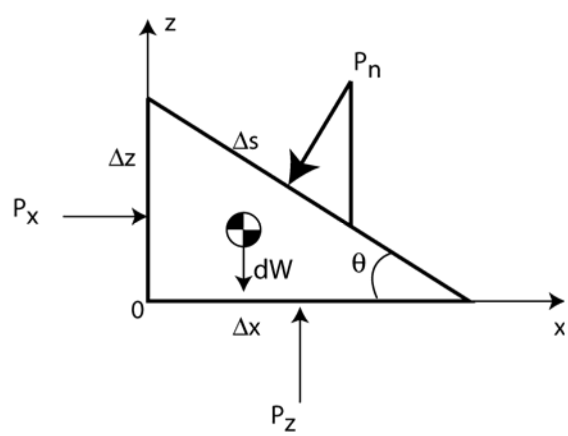

Figure 2.1: Elemental fluid volume

The fluid element can support no shear while at rest (by our definition of a fluid). Thus the sum of the forces on the triangle, in the *x*- and *z*- directions, MUST equal zero:

$$\sum F_x = p_x b(\Delta z) - p_n b(\Delta s)\sin\theta = 0 \tag{2.1}$$

$$\sum F_z = p_z b(\Delta x) - p_n b(\Delta s)\cos\theta - \frac{1}{2}\gamma b(\Delta x \Delta z) = 0 \tag{2.2}$$

We can simplify the above equations using simple geometry:

$$\Delta z = (\Delta s)\sin\theta \text{ and } \Delta x = (\Delta s)\cos\theta, \tag{2.3}$$

such that

$$\sum F_x = \left(p_x - p_n\right) = 0 \tag{2.4}$$

and

$$\sum F_z = p_z - p_n - 1/2\gamma(\Delta z) = 0\,. \tag{2.5}$$

Taking the limit as Δx, Δz goes to zero (i.e. the triangle goes to a point), we see that

$$p_x = p_z = p_n = p\,. \tag{2.6}$$

Since θ is arbitrary, pressure at a point in a fluid is independent of orientation and is thus isotropic. Pressure (or any stress for that matter) causes NO net force on a fluid element unless it varies spatially!

Take for example a small fluid element, δx, δy, δz:

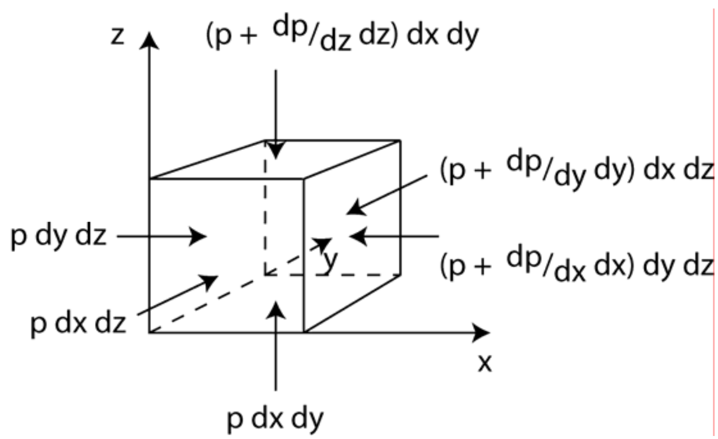

Figure 2.2: Fluid Element Volume (z is positive upwards by convention).

We will continue to assume that the only forces are due to gravity and pressure gradients. Let's start by looking at the pressure acting on the top and bottom of the element volume. Using Taylor's series expansion:

$$p_{top} = po + \frac{\delta z}{2}\frac{dp}{dz}\bigg|_0 + \frac{1}{2!}\left(\frac{\delta z}{2}\right)^2 \frac{d^2 p}{dz^2}\bigg|_0 - \ldots \tag{2.7}$$

Second term is positive b/c we are taking z positive upwards from the center of the element.

$$p_{bot} = p_o - \frac{\delta z}{2}\frac{dp}{dz}\bigg|_0 + \frac{1}{2!}\left(\frac{\delta z}{2}\right)^2 \frac{d^2 p}{dz^2}\bigg|_0 + \ldots \tag{2.8}$$

Taking the limit as δz goes to zero we are able to ignore the higher order terms and keep only up to second order. The resultant force due to pressure on the face is

$$F_{press.} = p\, dA = p\, dx\, dy\,. \tag{2.9}$$

$$F_{press} = (p_{bot} - p_{top})\,dx\,dy = \left.\frac{dp}{dz}\right|_o dx\,dy\,dz\,. \tag{2.10}$$

F_{press} acts in the positive z-direction opposite to gravity. In summing the pressures acting on the top and bottom we see that the second order terms cancel exactly and we are left with the resulting force due to pressure.

Acting in tandem to the pressure force is the force due to gravity. The fluid density and volume dictate the mass of the element (***F***=*m**g***).

$$F_{weight} = \int_{top}^{bot} \rho(z) g\ dxdydz = g\ dxdy \int_{top}^{bot} \rho(z) dz \tag{2.11}$$

$$= g\ dxdy \frac{1}{2}(\rho_{top} + \rho_{bot}) dz \tag{2.12}$$

Using Taylor's series expansion similar to the pressure terms above we get:

$$\rho_{top} + \rho_{bot} = 2\rho_o \tag{2.13}$$

For static equilibrium, pressure forces MUST balance gravitational/body forces:

$$F_{weight} + F_{press} = 0 \tag{2.14}$$

$$\left.\frac{dp}{dz}\right|_o (dxdydz) = -\rho g (dxdydz) \tag{2.15}$$

We can drop the subscript "o" from the pressure gradient term since we have taken the limit to a small element of fluid. This equation is valid at any point in the fluid. Canceling the elemental volume terms we are left with the hydrostatic equation.

Hydrostatic Equation:

$$\boxed{\frac{dp}{dz} = -\rho g} \tag{2.16}$$

The negative sign is valid here since the z-direction is taken positive pointing up, in the opposite direction to the gravitational force. If z-direction coincided with gravity then the negative sign would be dropped.

In this case, there are no acting forces or pressure changes in the x- or y-directions therefore dp/dz is representative of the pressure gradient:

$$\frac{dp}{dz} = \nabla p\,. \tag{2.17}$$

We can rewrite the hydrostatic equation as:

$$\vec{\nabla} p = -\rho \vec{g} \qquad (2.18)$$

If there were additional gradients in the x- and y-directions similar steps could be followed and the resulting equation would be similar. Pressure gradient is always balanced by gravity, acceleration, viscous forces, and any other external body forces.

Gauge and vacuum pressure:

Pressure is usually referred to in one of two ways:

1) Absolute, or total, pressure
2) Relative to ambient (atmospheric) pressure

Since most pressure instruments are differential measurement devices, meaning they measure the pressure in the fluid relative to atmospheric pressure, absolute pressure is a commonly used quantity.

Pressure is either greater or less than the ambient (atmospheric) pressure:

1) $p > p_a$	Gage Pressure	$p(gage) = p - p_a$
2) $p < p_a$	Vacuum pressure	$p(vacuum) = p_a - p$

In order to get the total (absolute) pressure, the atmospheric pressure must be known.

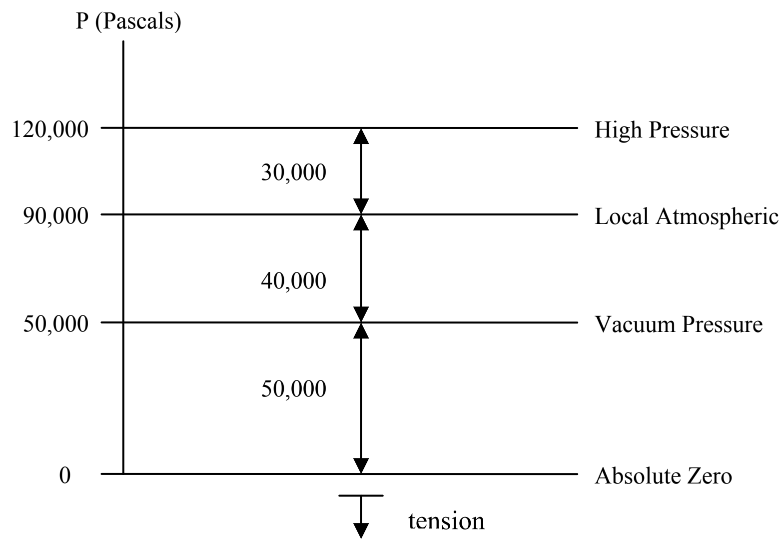

Figure 2.3: Relative pressure chart

Hydrostatic Force on a Wall:

Recall the equation for the pressure gradient in a liquid is

$$\vec{\nabla} p = -\rho \vec{g} \; . \tag{2.19}$$

Which, in the vertical direction this translates to

$$\frac{dp}{dz} = -\rho g \; , \tag{2.20}$$

or

$$dp = -\rho g \; dz \; . \tag{2.21}$$

Integrating in the z-direction we get pressure as a function of depth:

$$\int_{p}^{p_a} dp = -\int_{z} \rho g \; dz \tag{2.22}$$

$$p - p_a = -\rho g (z - H) = \rho g (H - z) \tag{2.23}$$

Note that pressure increases with depth with a constant slope, ρg. Pressure is either considered relative to a *reference* pressure or in *absolute* terms. Most pressure gauges are differential measurement devices that measure pressure relative to ambient (or atmospheric) pressure. Thus it is important to keep in mind the effects due to atmospheric pressure in your laboratories and calculations.

Pressure is isotropic, and therefore pressure is the same on a vertical or horizontal surface at depth *h*. Take for instance the ocean bottom and a vertical seawall that extends to the bottom. At depth H, the horizontal pressure acting on the wall is equivalent to that pressure acting along the entire seafloor at that same depth.

Horizontal Force acting on a vertical surface:

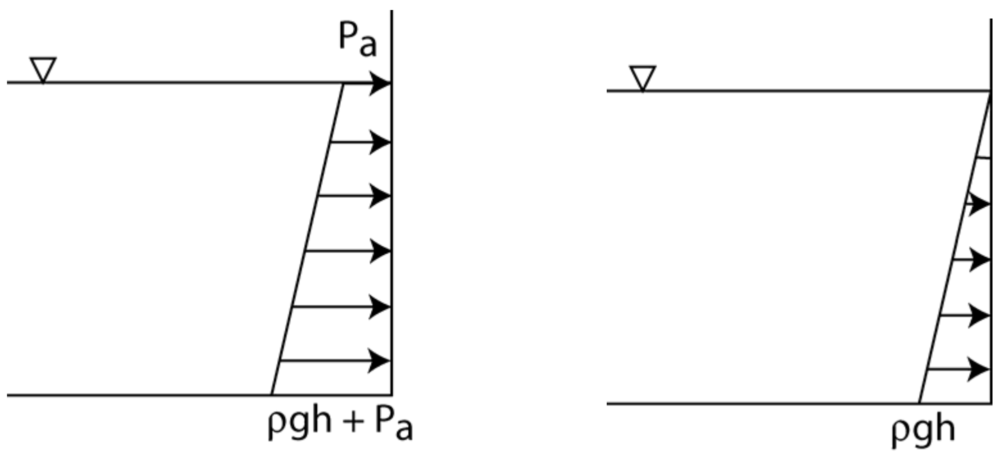

Figure 2.4: Absolute Pressure vs. Gauge pressure

Using absolute pressure formulation we can find the force on the wall from the pressure as a function of depth:

$$p - p_a = -\rho g\left(z - h\right) \tag{2.24}$$

Thus the elemental force acting in the x-direction due to the pressure is

$$dF_x = p\,dA = p\,w\,dz\ , \tag{2.25}$$

where *w* is the width of the wall into the page.

To determine the force on the wall we must consider the pressure acting on both sides of the wall. Let's assume that on the left of the wall water of depth, h, is exerting force F_1 on the wall. The right side of the wall is open to the air with atmospheric pressure acting over the height of the wall. Under this setup we must account for the atmospheric pressure on both sides of the wall. The elemental force in the x-direction at some depth z below the free surface is formulated on the left and right sides of the wall to determine the total force acting on the wall:

Left: $$dF_{x1} = (P_a + \rho g(h-z))\, w\, dz$$

Right: $$dF_{x2} = -P_a\, w\, dz$$

Total force: $$dF_x = dF_{x1} + dF_{x2} = \underline{P_a w\, dz} + \rho g(h-z) w\, dz - \underline{P_a\, w\, dz}$$

The two underlined terms cancel and we are left only with the effect due to the presence of the water. Integrating the pressure over the depth

$$F = \int_0^h \rho g(h-z)dz = \tfrac{1}{2}\rho g w h^2 \qquad (2.26)$$

If use gauge pressure to calculate the force on the wall, we already take into account the effect of atmospheric pressure and can directly calculate the force from:

$$p_g = \rho g(h-z) \quad \text{(gauge pressure)} \qquad (2.27)$$

$$dF = p_g dA = \rho g(h-z) w\, dz \qquad (2.28)$$

Resulting in the same force found when we used absolute pressure.

The force acting on the wall is a "distributed load" which acts along the entire depth of the wall. However it is possible to determine how this force "acts" on the wall using a simple moment balance:

$$dMo = z \times dF \qquad (2.29)$$

where *Mo* is the moment on the wall about the origin (for now lets consider the origin the bottom of the wall) and z is the moment arm perpendicular to the force direction.

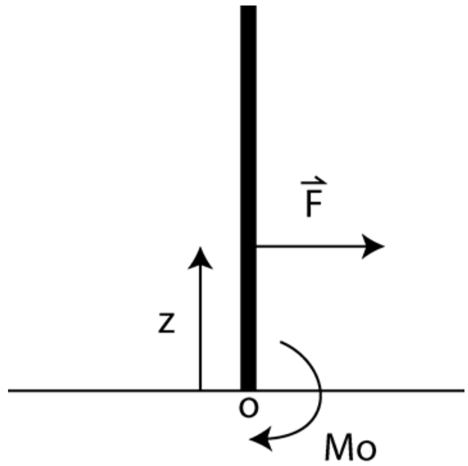

Figure 2.5: Moment balance on the wall about the origin o with z taken positive from the ground up.

The elemental moment about the origin due to the hydrostatic pressure is:

$$dMo = \rho g(h - z)\, w\, z\, dz \tag{2.30}$$

Integrating leads to the total moment about the origin:

$$Mo = \tfrac{1}{6}\rho g w h^3 \tag{2.31}$$

By definition the product of the force F times the moment arm $\bar{z}$ must equal the moment M:

$$M = F\,\bar{z} \tag{2.32}$$

Such that the point at which the force acts on the wall is $\bar{z}$:

$$\bar{z} = \frac{\tfrac{1}{6}\rho g w h^3}{\tfrac{1}{2}\rho g w h^2} = \tfrac{1}{3}h \tag{2.33}$$

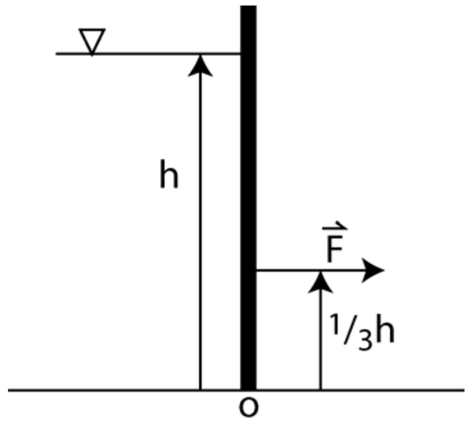

Figure 2.6: The point at which the resultant force acts is known as the *center of pressure*.

We can use a similar approach to the problem of pressure on a sloped wall. This is left for a homework exercise. Since atmospheric pressure acts everywhere then gage pressure is the ideal pressure to use in these exercises. The resultant forces and moments on the sloped wall can be found using simple geometry and then extended to the case of a "V"-shaped ship hull.

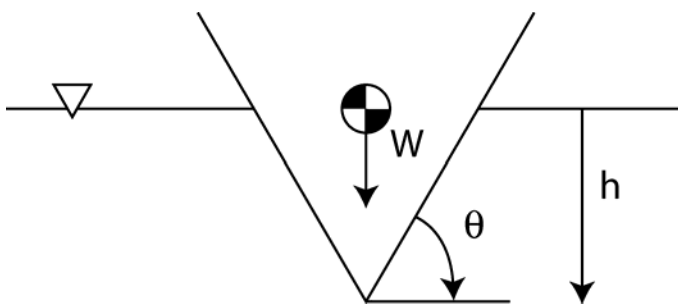

Figure 2.7: V-shaped Hull

