2.016 Hydrodynamics

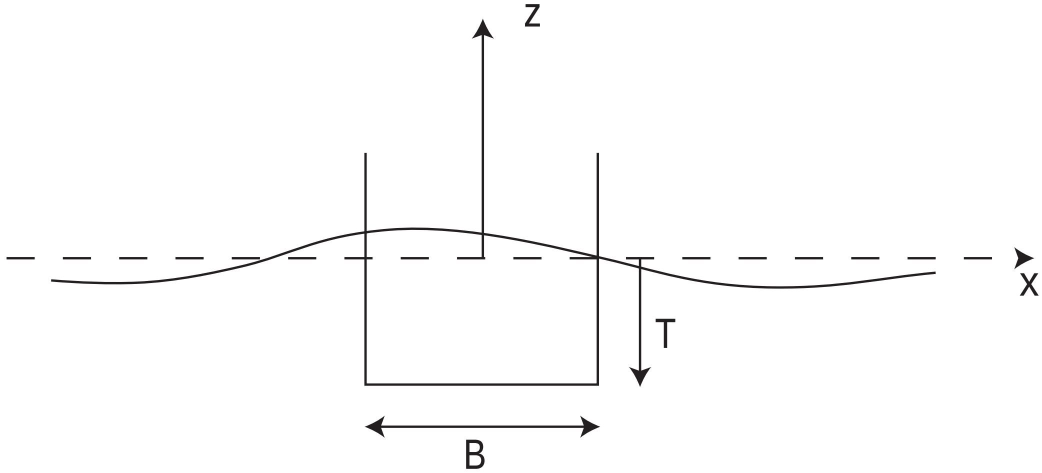
Prof. A.H. Techet

1. Froude Krylov Excitation Force

Ultimately, if we assume the body to be sufficiently small as not to affect the pressure field due to an incident wave, then we can neglect diffraction effects completely. This assumption comes from the *Froude-Krylov hypothesis* and assures a resulting excitation force equivalent to the Froude-krylov force:

$$F^{FK}(t) = -\rho \int\int \frac{\partial \phi_I}{\partial t} n \, dS \quad (8.1)$$

Vertical Froude-Krylov Force on a Single Hull Vessel



Deep water incident wave potential is:

$$\phi_I = \frac{a\omega}{k} e^{kz} \quad Re\left\{ i \, e^{i(\omega t - kx)} \right\} \quad (8.2)$$

The force in the vertical direction is found from the incident potential using eq. (8.1) along the bottom of the vessel. Here the normal in the z-direction, n_z, is negative: $n_z = -1$, so the force per unit length in the z-direction is

$$F_z^{FK} = Re\left\{\int_{-B/2}^{B/2} -\rho\, i\, \omega \frac{ia\omega}{k} e^{-kT}\ e^{i(\omega t-kx)}\ dx\right\} \tag{8.3}$$

$$= Re\left\{\frac{\rho o\omega^2}{k^2} a\, e^{-kT}\ e^{i\omega t}\left(e^{-ikB/2} - e^{ikB/2}\right)\right\} \tag{8.4}$$

$$= Re\left\{2\rho\frac{\omega^2}{k^2} a\, e^{-kT}\ e^{i\omega t}\ \sin(kB/2)\right\} \tag{8.5}$$

Recall that $sin\ z\ =\ \frac{e^{iz}-e^{-iz}}{2i}$.

Using the vertical velocity we can rewrite the force in terms of the velocity.

$$w(t) = Re\left\{a\omega\, e^{kz}\ i\ e^{i(\omega t-kx)}\right\} \tag{8.6}$$

$$\dot{w}(t) = Re\left\{-a\omega^2\ e^{kz}\ e^{i(\omega t-kx)}\right\} \tag{8.7}$$

$$\dot{w}(x=0, z=0, t) = Re\left\{a\ \omega^2\ e^{i\omega t}\right\} \tag{8.8}$$

Now we can write the force in the vertical direction as a function of the vertical (heave) acceleration,

$$F_z = Re\left\{\frac{2\rho}{k^2} e^{-kT}\ sin(kB/2)\ \dot{w}(0,0,t)\right\}. \tag{8.9}$$

Let's look at the case where $\omega \to 0$ the wavenumber, $k = \omega^2/g \to 0$, also goes to zero and the following simplifications can be made:

$$e^{kt} \simeq 1 - kT \tag{8.10}$$

$$\sin(kB/2) \simeq kB/2 \tag{8.11}$$

to yield a simplified heave force.

$$F_z^{FK} \simeq Re\left\{2\rho\,\frac{\omega^2}{k^2}\,a\,(1-kT)\,(kB/2)\,e^{i\omega t}\right\} \tag{8.12}$$

$$\simeq Re\left\{\rho\;g\;aB\left(1-\frac{\omega^2}{g}T\right)e^{i\omega t}\right\} \tag{8.13}$$

If we look at the case where $\omega \to 0$ and consider the heave restoring coefficient, $C_{33} = \rho\;g\;B$, and the free surface elevation, $\eta(x,t) = Re\left\{a\,e^{i(\omega t - kx)}\right\}$ we can rewrite this force as

$$F_z^{FK} \simeq Re\left\{C_{33}\;\eta(x=0,t)\right\} \tag{8.14}$$

Horizontal Froude-Krylov Force on a Single Hull Vessel

The horizontal force on the vessel above can be found in a similar fashion to the vertical force.

$$F_x = \int\!\!\int_{S_B} \rho\frac{\partial \phi_I}{\partial t}\,n_x\;dS \tag{8.15}$$

$$= Re\left\{\rho\;i\;\omega\,\frac{i\omega a}{k}\int_{-T}^{0} e^{kz}\;dz\left[\,e^{i(\omega t - kB/2)} - e^{i(\omega t + kB/2)}\right]\right\} \tag{8.16}$$

$$= Re\left\{i\rho\frac{a\omega^2}{k}\left(1-e^{-kT}\right)e^{i\omega t}\;2\sin(kB/2)\right\} \tag{8.17}$$

As frequency approaches zero similar simplifications can be made like above for the vertical force:

$$F_x(t) \simeq Re\left\{i\rho\frac{a\omega^2}{k}\,(KT)\,e^{i\omega t}\;2\,kB/2\right\} \tag{8.18}$$

$$u(t) = Re\left\{a\omega\,e^{kz}\;e^{i(\omega t - kx)}\right\} \tag{8.19}$$

$$\dot{u}(t) = Re\left\{i\;a\;\omega^2\;e^{kz}\;e^{i(\omega t - kx)}\right\} \tag{8.20}$$

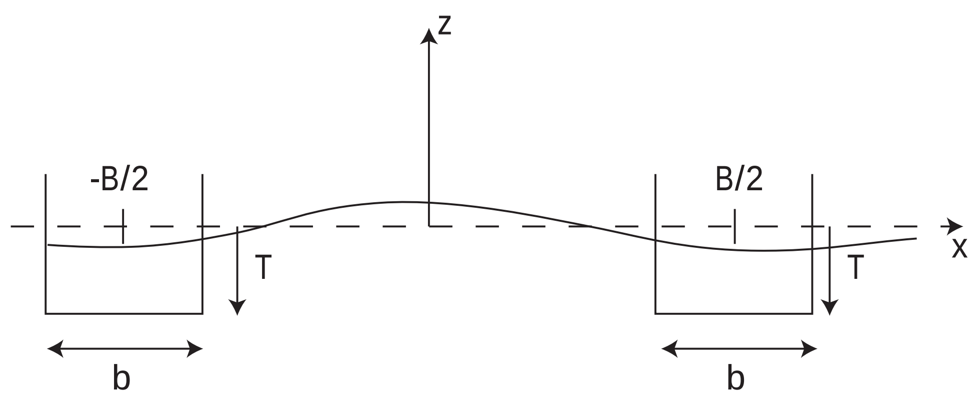
$$F_x(t) \simeq Re\left\{\rho TB\,\dot{u}(x=0,z=0,t)\right\} \tag{8.21}$$

Where $\rho TB = \rho\forall$, and $\forall$ is the vessel volume such that we are left with the surge force

$$F_x \simeq \rho\forall\, \dot{u} \tag{8.22}$$

$$F_z \simeq C_{33}\eta + \rho\forall\dot{w} \tag{8.23}$$

Multi Hulled Vessel



Again, let's make a few basic assumptions: ($b/\lambda << 1$), ($B/\lambda \sim 1$), ($a < b$), and ($b \sim T$). Let's look at the force in the x-direction:

$$F_x^{FK} \simeq \rho\, bT\, \dot{u}(x = -B/2, z = 0, t) + \rho\, bT\, \dot{u}(x = B/2, z = 0, t)$$

$$\eta(x,t) = a\cos(\omega t - kx) \tag{8.24}$$

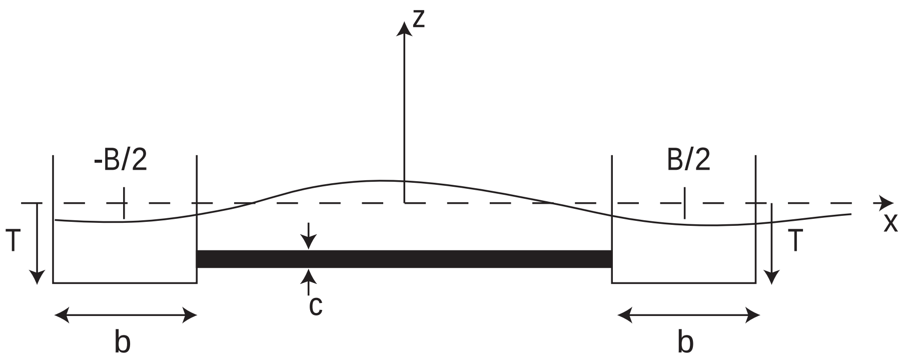
$$\dot{u}(x,z,t) = -a\,\omega^2\, e^{kz}\, \sin(\omega t - kx) \tag{8.25}$$

$$F_x^{FK} \simeq \rho bT\, (-a\,\omega^2)\left\{\sin(\omega t + kB/2) + \sin(\omega t - kB/2)\right\} \tag{8.26}$$

$$\simeq -2\rho\, b\, T\, (a\,\omega^2)\cos(kB/2)\sin(\omega t) \tag{8.27}$$

Note that when $kB/2 = \pi/2$ (or $B = \lambda/2$) then $F_x^{FK}(t) = 0$.

4.1. Multi Hulled Vessel with additional pontoon



Use the same assumptions from above to find the x-force adjusted for the additional pontoon between the two hulls.

$$\begin{aligned} F_x^{FK} \simeq \quad & -2\,\rho\, b\, T\,(a\,\omega^2)\cos(kB/2)\sin(\omega t) \\ & +c\; p(x=-B/2+b/2, z=0, t) \\ & -c\; p(x=B/2-b/2, z=0, t) \end{aligned} \tag{8.28}$$

The last two terms are the adjustment to the force for the addition of the pontoon, $\delta F_x^{FK}(t)$.

Pressure is found from the incident potential: $p(x,z,t)=\rho g\; a\; e^{kz}\cos(\omega t-kx)$.

$$\delta F_x^{FK} = -2\;\rho\; g\; a\sin(\omega t)\sin\left(\frac{k}{2}(B-b)\right) \tag{8.29}$$

For $B >> b$ using $g=\omega^2/k$ we get a force:

$$F_x^{FK}(t) \simeq -2\;\rho\; a\;\omega^2\;\sin(\omega t)\left\{bT\cos(kB/2)+\delta/2\;\;\sin(kB/2)\right\} \tag{8.30}$$

2. Forces on Large Structures

For discussion in this section we will be considering bodies that are quite large compared to the wave amplitude and thus the inertial component of force dominates over the viscous forces. Typically we can neglect the viscous force when it is less that 10% of the total force, except near sharp edges and separation points. We must be careful to consider wave diffraction when the wavelength is less that 5 cylinder diameters.

If we assume that the viscous effects can be neglected and we consider the case of irrotational flow, then we can write the velocity field in terms of the potential function, $\phi(x, y, z, t)$.

$$V(x, y, z, t) = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \tag{8.31}$$

The governing equation of motion is given by the Laplace equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{8.32}$$

Given a body in the presence of the wave field we much consider the relevant boundary conditions on the free surface, the seafloor and the body. Boundary conditions are, on the bottom,

$$\nabla\phi \cdot \hat{n} = \frac{\partial\phi}{\partial n} = 0, \tag{8.33}$$

and, on the body,

$$\nabla\phi \cdot \hat{n} = \frac{\partial\phi}{\partial n} = V_B \cdot \hat{n} = V_{Bn}. \tag{8.34}$$

At the free surface the kinematic and dynamic boundary conditions must be satisfied. The linearized free surface boundary conditions are both taken about $z = 0$ as we are accustomed to, given that $a/\lambda << 1$,

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \tag{8.35}$$

$$\eta = -\frac{1}{g}\frac{\partial\phi}{\partial t}. \tag{8.36}$$

In the case of a free floating body we must also take into consideration the wave field generated by the body motion alone. At some distance far away from the body, the potential function must take into consideration the waves radiating from the body.

2.1. The Total Wave Potential

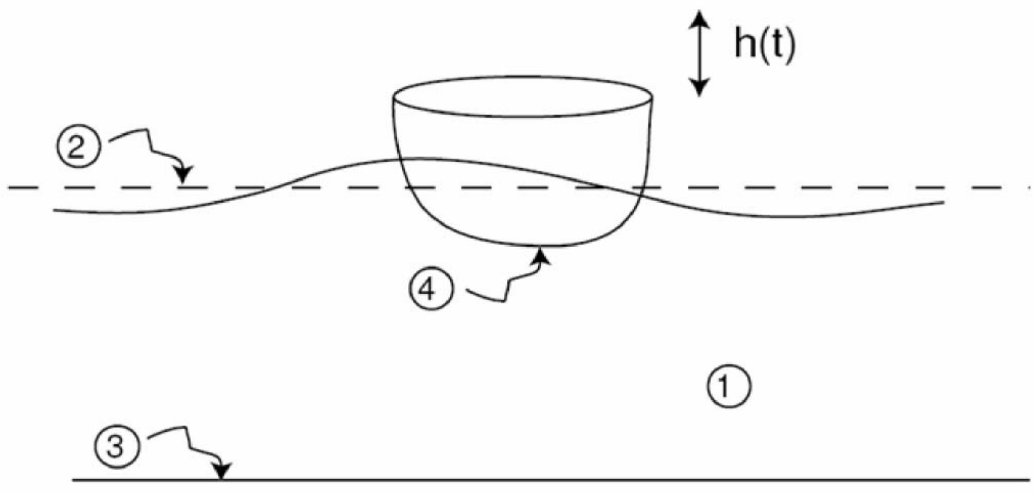


Figure 2.1. Boundary conditions for the total potential must be met at three places: sea floor (3), free surface (2), structure surface (4). The continuity equation must be satisfied within the fluid (1).

Due to the nature of potential flow and linear waves it is possible to sum multiple potential functions to obtain the total potential representative of the complete flow field. Each component of the total potential must also satisfy the appropriate boundary conditions. For linear waves incident on a floating body the total potential is a sum of the undisturbed incident waved potential, $\phi_I(x,y,z,t)$, the diffraction potential, $\phi_D(x,y,z,t)$, due to the presence of the body when it is motionless, and the radiation potential, $\phi_R(x,y,z,t)$, representing the waves generated (radiating outwards) by a moving body. For a permanently fixed body the radiation potential is non-existent (ie. $\phi_R(x,y,z,t)=0$).

$$\phi(x,y,z,t)=\phi_I(x,y,z,t)+\phi_D(x,y,z,t)+\phi_R(x,y,z,t) \tag{8.37}$$

It is good here to note the important conditions on each component of the total potential. The incident potential is formulated from that of a free wave without consideration for the presence of the body. Therefore $\phi_I(x,y,z,t)$ satisfies only the free surface boundary conditions and the bottom boundary condition, in addition to the Laplace equation.

The diffraction potential, $\phi_D(x, y, z, t)$, must also satisfy the Laplace equation, the free surface and the bottom boundary conditions. In order to compensate for the disturbance of the incident wave around the body by the an additional condition at the body boundary such that the normal gradient of the diffraction potential is equal but opposite in sign to the normal gradient of the incident potential:

$$\frac{\partial \phi_D}{\partial n} = -\frac{\partial \phi_I}{\partial n}. \tag{8.38}$$

The radiation potential satisfies the same conditions as the incident potential as well as an additional condition on the moving body boundary in the absence of incoming waves. On the body boundary the normal gradient of the velocity potential must equal the normal velocity of the body.

$$\frac{\partial \phi_R}{\partial n} = V_B \cdot \hat{n}. \tag{8.39}$$

2.2. Complex form of the Wave Potential

It is often easiest, for the purpose of systems analysis, to write the wave potential in its complex form:

$$\phi(x, y, z, t) = Re\left\{\left(a\hat{\phi}_I + a\hat{\phi}_D\right)e^{i\omega t} + \phi_R\right\}, \tag{8.40}$$

where a is the wave amplitude and ω is the incoming wave frequency. The complex incident potential is $\phi_I = a\,\widehat{\phi_I} e^{i\omega t}$. The amplitude of the incident potential is the wave amplitude times $\widehat{\phi_I}$, which is a function of depth and position in space. The diffraction potential takes the same form in order to satisfy the body boundary condition.

The radiation potential is not *necessarily* directly related to the wave amplitude. Since ϕ_R results from the motion of a floating body in the absence of waves, we must consider the body motion in all six degrees-of-freedom. The vessel motions are prescribed by, x_j, where $j = 1, 2, 3, 4, 5, 6$ (surge, sway, heave, roll, pitch, and yaw). It is customary to write the complex radiation potential in the following form:

$$\phi_R = \sum_{j=1}^{6} \dot{x}_j \, \phi_j \tag{8.41}$$

where $\dot{x}_j$ is the velocity of the body in the j th direction and ϕ_j is the velocity potential due to a unit motion in the j th direction.

2.2.1. Incident Potential Boundary Conditions

The incident potential, $\phi_I(x, y, z, t)$, is considered without knowledge of the presence of any structure. The boundary conditions must be sufficient to arrive at the correct potential far from the structure.

1. $\nabla^2 \phi_I = 0$; Continuity equation.
2. $\frac{\partial^2 \phi_I}{\partial t^2} + g \frac{\partial \phi_I}{\partial z} = 0$; Combined free surface condition.
3. $\frac{\partial \phi_I}{\partial z} = 0$; Bottom boundary condition.
4. Incident potential does not have any knowledge of body in the flow.

Rewriting the incident potential in its complex form, $\phi_I = Re\left\{ a \, \hat{\phi}_I \, e^{i\omega t} \right\}$, we can simplify the boundary conditions even further:

1. $\nabla^2 \hat{\phi}_I = 0$;
2. $-\omega^2 \, \hat{\phi}_I + g \frac{\partial \hat{\phi}_I}{\partial z} = 0$;
3. $\frac{\partial \hat{\phi}_I}{\partial z} = 0$;
4. Incident potential does not have any knowledge of body in the flow.

2.2.2. Diffraction Potential Boundary Condtions

The diffraction potential, $\phi_D(x, y, z, t)$, results from the presence of the structure in the flow field. Without the structure, there would be no wave diffraction. This potential accounts for the alteration of the incident wave train by the structure and we must now include a boundary condition on the body (condition 4).

1. $\nabla^2 \phi_D = 0$; Continuity equation.
2. $\frac{\partial^2 \phi_D}{\partial t^2} + g\,\frac{\partial \phi_D}{\partial z} = 0$; Combined free surface condition.
3. $\frac{\partial \phi_D}{\partial z} = 0$; Bottom boundary condition.
4. $\frac{\partial \phi_D}{\partial n} = -\frac{\partial \phi_I}{\partial n}$; Body boundary condition.

Rewriting the diffraction potential in its complex form, $\phi_D = Re\left\{a\,\hat{\phi}_D\, e^{i\omega t}\right\}$, we can simplify the boundary conditions like we did for the incident potential:

1. $\nabla^2 \hat{\phi}_D = 0$;
2. $-\omega^2\,\hat{\phi}_D + g\,\frac{\partial \hat{\phi}_D}{\partial z} = 0$;
3. $\frac{\partial \hat{\phi}_D}{\partial z} = 0$;
4. $\frac{\partial \hat{\phi}_D}{\partial n} = -\frac{\partial \hat{\phi}_I}{\partial n}$; Body boundary condition.

2.2.3. Radiation Potential Boundary Conditions

The radiation potential, $\phi_R(x, y, z, t)$, is a result of a freely moving structure floating in a quiescent (still) fluid and radiates outwards from the body. This potential does not come into play for a fixed structure, only for a floating body, though the body can be anchored or tethered.

1. $\nabla^2 \phi_R = 0$; Continuity equation.
2. $\frac{\partial^2 \phi_R}{\partial t^2} + g\,\frac{\partial \phi_R}{\partial z} = 0$; Combined free surface condition.
3. $\frac{\partial \phi_R}{\partial z} = 0$; Bottom boundary condition.
4. $\frac{\partial \phi_R}{\partial n} = \nabla \phi_R \cdot \hat{n} = \vec{V}_B \cdot \hat{n}$; Body boundary condition.

In heave motion alone, for example, the body velocity is simply

$$\vec{V}_B = \dot{x}_3\,\hat{k} = \frac{\partial x_3}{\partial t} \tag{8.42}$$

and the body boundary condition (4) becomes

$$\vec{V}_B \cdot \hat{n} = \hat{n} \cdot \dot{x}_3 \, \hat{k} = n_z \, \frac{\partial x_3}{\partial t}. \tag{8.43}$$

Rewriting the radiation potential in its complex form, $\phi_R = Re\left\{a \, \hat{\phi}_R \, e^{i\omega t}\right\}$, we can simplify the boundary conditions:

1. $\nabla^2 \hat{\phi}_R = 0$;
2. $-\omega^2 \, \hat{\phi}_R + g \, \frac{\partial \hat{\phi}_R}{\partial z} = 0$;
3. $\frac{\partial \hat{\phi}_R}{\partial z} = 0$;
4. $\frac{\partial \hat{\phi}_R}{\partial n} = i \, \omega \, n_z$; "Unit motion" condition in heave.

2.2.4. Total Potential Formulation

The total wave potential boundary conditions for unidirectional waves traveling in the positive x-direction are a combination of the above cases. For a two-dimensional potential and a body in heave-only motion:

$$\begin{aligned} \phi(x,z,t) &= \phi_I(x,z,t) + \phi_D(x,z,t) + \phi_R(x,z,t) \\ &= Re\left\{a\left(\hat{\phi}_I(x,z) + \hat{\phi}_D(x,z)\right) e^{i\omega t} + \hat{x}_3 \, \hat{\phi}_R(x,z) \, e^{i\omega t}\right\}. \end{aligned}$$

For any depth,

$$\hat{\phi}_I(x,z) = \frac{i\omega}{k} \, \frac{cosh(k[z+H])}{sinh(kH)} e^{-ikx}, \tag{8.44}$$

and for deep water

$$\hat{\phi}_I(x,z) = \frac{i\omega}{k} e^{kz} \, e^{-ikx}. \tag{8.45}$$

The basic boundary conditions hold for the total potential:

1. $\nabla^2\phi = 0$; Continuity equation.
2. $\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0$; Combined free surface condition.
3. $\frac{\partial\phi}{\partial z} = 0$; Bottom boundary condition.

The body boundary condition for the total potential (in heave motion) is

$$\frac{\partial\phi}{\partial n} = \nabla\phi\cdot\hat{n} = n_z\frac{dx_3}{dt} \tag{8.46}$$

which can be rewritten in its complex form

$$Re\left\{ae^{i\omega t}\left(\nabla\hat{\phi}_I + \nabla\hat{\phi}_D\right)\cdot\hat{n} + \hat{x}_3\, e^{i\omega t}\nabla\hat{\phi}_R\cdot\hat{n}\right\} = Re\left\{i\omega n_z\ \hat{x}_3\, e^{i\omega t}\right\}. \tag{8.47}$$

In order for equation (8.47) to hold true it the following must hold true:

$$\hat{n}\cdot\left(\nabla\hat{\phi}_I + \nabla\hat{\phi}_D\right) = 0 \tag{8.48}$$

and

$$\nabla\hat{\phi}_R\cdot\hat{n} = n_z\ i\omega \tag{8.49}$$

where $\hat{\phi}_I$, $\hat{\phi}_D$, and $\hat{\phi}_R$ are independent of wave amplitude, a, and heave amplitude, $\hat{x}_3$.

2.3. Body in Heave Motion: Forcing and Equation of Motion

The force on the body in an incident unidirectional wave field can be found from the linearized pressure in the fluid. The pressure is found using the unsteady form of bernoulli's equation:

$$p(t) + \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho\left|\nabla\phi\right|^2 + \rho gz = C(t) \tag{8.50}$$

in order to use this with our linearized wave equations we must linearize the pressure. Ignoring the hydrostatic term and the second order terms the unsteady pressure effects due to the waves is the dynamic pressure. Since C(t) is arbitrary, we set it to zero. The force can now be determined as follows

$$F_z(t) = \iint_{\bar{S}} p(t)\cdot\hat{n}\, ds = \iint_{\bar{S}} \rho\frac{\partial\phi}{\partial t}n_z\ ds \tag{8.51}$$

Expanded in terms of the incident, diffraction, and radiation potentials we have an expression for the total force in the vertical (heave) direction:

$$F_z(t) = Re\left\{\iint_{\bar{S}} \rho i\omega e^{i\omega t} n_z \left(a\left[\hat{\phi}_I + \hat{\phi}_D\right] + \hat{x}_3\hat{\phi}_R\right) ds\right\}, \tag{8.52}$$

where $x_3(t) = \hat{x}_3 e^{i\omega t}$ is the heave motion. Each component of the total potential causes a force on the body. Given a general form of force $F(t) = Re\{\widehat{F}e^{i\omega t}\}$, where $\widehat{F}$ is the amplitude of the forcing function. The total force, $F(t)$, is written as

$$F(t) = Re\left\{a\left(\widehat{F}_I + \widehat{F}_D\right)e^{i\omega t} + \widehat{x}_3\widehat{F}_R e^{i\omega t}\right\} + F_{hydrostatic}, \tag{8.53}$$

where the hydrostatic force in heave is $F_h = -C_{33}\, x_3(t)$. The force amplitudes for the incident, diffraction, and radiation forces are

$$\widehat{F}_I = i\omega\rho \iint_{\bar{s}} n_z \hat{\phi}_I \, dS, \tag{8.54}$$

$$\widehat{F}_D = i\omega\rho \iint_{\bar{s}} n_z \hat{\phi}_D \, dS, \tag{8.55}$$

$$\widehat{F}_R = i\omega\rho \iint_{\bar{s}} n_z \hat{\phi}_R \, dS. \tag{8.56}$$

The amplitude of the radiation force comes from the case of the body heaving in a still fluid (no waves) using from Newton's second law. The radiation force amplitude is simply the sum of the added mass and damping components in heave

$$\widehat{F}_R = \omega^2 A_{33} - i\omega B_{33} \tag{8.57}$$

where A_{33} and B_{33} are the added mass and damping coefficients in the heave direction due to heave motion.

For the complete problem of a body heaving in an incident wave field, we must consider the total force on the body. By Newton's law we can write an equation of motion by equating the inertial force, the body mass times the heave acceleration, to the total force due to the total wave potential: $m\ddot{x}_3 = F(t)$.

$$-\omega^2 m\, \hat{x}_3 e^{i\omega t} = Re\left\{a\left(\widehat{F}_I + \widehat{F}_D\right)e^{i\omega t} + \hat{x}_3\widehat{F}_R e^{i\omega t} + F_h\right\} \tag{8.58}$$

simplifying this expression we get

$$\left\{-\omega^2(m+A_{33})+i\omega B_{33}+C_{33}\right\}\hat{x}_3 = a\left(\widehat{F}_I+\widehat{F}_D\right) \tag{8.59}$$

Using the above equation of motion we can now find the amplitude of heave motion in terms of wave amplitude, inertial and diffraction forces, system natural frequency, system mass, added mass, damping and hydrostatic coefficients.

$$\frac{\hat{x}_3}{a} = \frac{\widehat{F}_I+\widehat{F}_D}{-\omega^2(m+A_{33})+i\omega B_{33}+C_{33}} \tag{8.60}$$

where A_{33}, B_{33}, and C_{33}, are the added mass, damping and hydrostatic coefficients in heave found using strip theory, $a(\widehat{F}_I+\widehat{F}_D)$ is the excitation force amplitude, $a\widehat{F}_I$ is the Froude-Krylov force amplitude, and $a\widehat{F}_D$ is the diffraction force amplitude. All of these depend on the frequency, ω, however the Froude-Krylov force amplitude, diffraction force amplitude, added mass, damping, and hydrostatic coefficients are independent of heave and wave amplitude.