25   Geometry Optimization

Give the formulas for the radius and length of a closed, half-cylindrical tank (i.e., having a
"D"-like cross-section and flat ends) that will hold a given volume V. Your objective is to
make the tank with minimum surface area. What are the specific dimensions if the volume
is $100m^3$?

The volume and surface area of the tank are given respectively by:

$$V = \frac{\pi}{2}r^2l$$
$$A = \pi rl + \pi r^2 + 2rl$$

*With the volume fixed, we seek the condition where the variation in A is zero, that is,
$\partial A/\partial r = \partial A/\partial l = 0$, subject to the volume constraint. Following the example of Mid-
dendorf and Engelmann (1998), we work this out using Lagrange multipliers, even though
substitution as in the first problem could also be used.*

The area itself is the cost function, and we augment it as

$$A' = A + \lambda(V - \pi r^2 l/2)$$
$$A' = \pi rl + \pi r^2 + 2rl + \lambda(V - \pi r^2 l/2)$$

Now we have to set the derivatives of A' with respect to r and l to zero:

$$\frac{\partial A'}{\partial r} = 0 = \pi l + 2\pi r + 2l - \lambda \pi rl$$
$$\frac{\partial A'}{\partial l} = 0 = \pi r + 2r - \lambda \pi r^2/2$$

*These relations along with the constraint $V - \pi r^2 l/2 = 0$, form three equations in three
unknowns. Use the second equation to write λ in terms of r, and then use the third to write
l in terms of r, and then substitute both of these into the first equation, to end up with our
explicit solution for radius:*

$$r = \left[\frac{V(2+\pi)}{\pi^2}\right]^{1/3}$$

*This is exactly what you get if you use substitution to solve the problem, and in that case,
we would put this r back into the constraint equation to solve for l (a little messy). But
the Lagrange technique gives us more insight than this: if we use the first and second of
the three equations above, we can eliminate λ to find that $l = 2\pi r/(2+\pi)$. This is a new
property of the optimum solution, and we have simply*

$$l = \frac{2\pi}{2+\pi}\left[\frac{V(2+\pi)}{\pi^2}\right]^{1/3}$$

If $V = 100m^3$, we find $r = 3.735m$ and $l = 4.564m$.

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