

Supplement to Lecture 10

Dynamics of a DC Motor with Pinion Rack Load and Velocity Feedback

As an extension to Lecture 10, here we will analyze a DC motor connected to a pinion rack with a mass–damper load. The motor is connected in a velocity–feedback configuration with a differential op–amp providing the error signal as shown in Figure 1 below. This is similar to the flywheel that we have been studying in the labs, with the important difference that now the flywheel is assumed to be connected to a rack and pinion converting angular motion to translational motion. Such conversions occur often in mechanical systems.

Plant transfer function

We begin by analyzing the plant, *i.e.*, the motor connected directly to a voltage source input $v_s(t)$ and velocity output $v(t)$. The result that we are about to derive is referred to as the “plant model.” After we know the plant model, we can proceed to model the entire system which includes the velocity feedback loop.

To analyze the plant, we begin by writing the by–now familiar electrical equation of motion that is derived from KVL on the electrical circuit including the motor:

$$v_s(t) - i(t)R - v_e(t) = 0 \Rightarrow v_s(t) = i(t)R + K_v\omega(t), \quad (1)$$

where we used the relationship $v_e = K_v\omega$ for the motor’s back–emf and we have neglected the inductance. In the Laplace domain, this equation becomes

$$V_s(s) = I(s)R + K_v\Omega(s). \quad (2)$$

Next we must write the mechanical equation of motion as torque balance on the motor’s shaft. The motor’s torque is provided by the electrical current through the windings, and is $K_m i(t)$. There are three torques counteracting the motor: *(i)* the pinion inertia J , *(ii)* the rotational viscous damper D , and *(iii)* the torque exercised by the translational elements (mass M and translational viscous damper f_v) via the pinion gear. Since at the moment we do not know how to express torque *(iii)*, let us denote it by T_p . The time–domain equation of motion of the motor shaft then becomes

$$K_m i(t) = J\dot{\omega}(t) + D\omega(t) + T_p(t). \quad (3)$$

In the Laplace domain, the equation of motion is re–written as

$$K_m I(s) = Js\Omega(s) + D\Omega(s) + T_p(s). \quad (4)$$

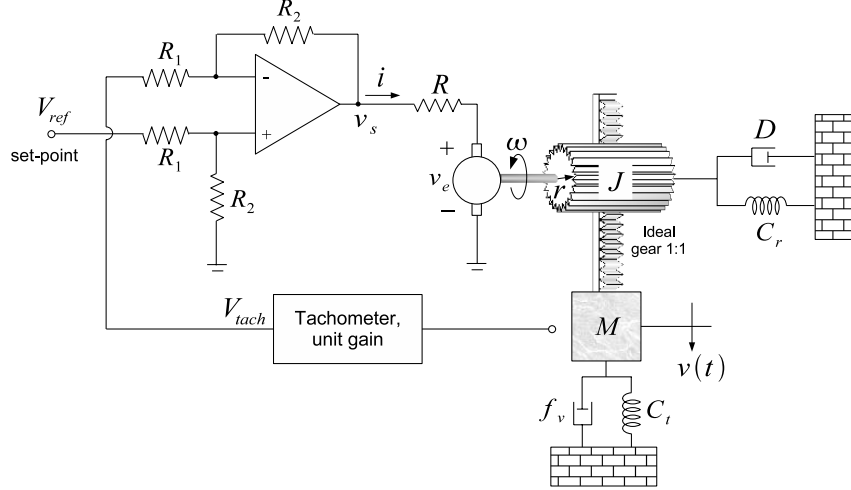


Figure 1: Schematic of the feedback system to control the velocity of a pinion

We still have to find $T_p(s)$. This is done by force–balance on the translational part of the load, *i.e.* the pinion’s rack that carries the mass M . If $F_p(t)$ denotes the force by the pinion on the rack, then in the time domain we have

$$F_p(t) = M\dot{v}(t) + f_v v(t), \quad (5)$$

or, in the Laplace domain,

$$F_p(s) = MsV(s) + f_v V(s). \quad (6)$$

We can now relate the pinion torque T_p to the pinion force F_p and the angular velocity ω to the translational velocity v . Assuming that the pinion is perfectly cylindrical and neglecting backlash or other nonlinearities due to the gears, then in the time–domain we have

$$T_p(t) = rF_p(t), \quad (7)$$

$$v(t) = r\omega(t). \quad (8)$$

In the Laplace domain, we equivalently have

$$T_p(s) = rF_p(s), \quad (9)$$

$$V(s) = r\Omega(s). \quad (10)$$

Finally, we need to collect all the above results towards a “plant transfer function” of the form $V(s)/V_s(s)$ (Laplace transform of the output velocity v divided by the Laplace transform of the input voltage v_s .) To do this, we must eliminate the intermediate dynamical variables $I(s)$, $\Omega(s)$, $T_p(s)$, and $F_p(s)$. This is done by first substituting (6),

(9) and (10) to (4) to obtain

$$\begin{aligned} K_m I(s) &= (Js + D) \frac{V(s)}{r} + r(Ms + f_v) V(s) \Rightarrow \\ I(s) &= \left(\frac{J + r^2 M}{r K_m} s + \frac{D + r^2 f_v}{r K_m} \right) V(s). \end{aligned} \quad (11)$$

Now that we have a relationship between current $I(s)$ and translational velocity $V(s)$, we substitute it in (2) along with (10) to obtain

$$V_s(s) = \left(\frac{J + r^2 M}{r K_m} s + \frac{D + r^2 f_v}{r K_m} \right) R V(s) + K_v \frac{V(s)}{r}. \quad (12)$$

We are basically done; all that's left is to rewrite (12) in proper transfer function form, clearly indicating the location of the pole:

$$\frac{V(s)}{V_s(s)} = \frac{\frac{r K_m}{R(J + r^2 M)}}{s + \frac{D + r^2 f_v + K_m K_v / R}{J + r^2 M}}. \quad (13)$$

Substituting the (*revised*) numerical values $R = 1\Omega$, $K_m = 1\text{N} \cdot \text{m}/\text{A}$, $K_v = 1\text{V} \cdot \text{sec}/\text{rad}$, $J = 0.1\text{kg} \cdot \text{m}^2$, $D = 0.5\text{kg} \cdot \text{m}^2/(\text{rad} \cdot \text{sec})$, $M = 9\text{kg}$, $f_v = 5\text{kg}/\text{sec}$ and $r = \sqrt{0.1}\text{m} = 0.3162\text{m}$, we obtain

$$\frac{V(s)}{V_s(s)} = \frac{0.3162}{s + 2} \left[\frac{\text{m}/\text{sec}}{\text{V}} \right]. \quad (14)$$

Feedback transfer function

The velocity output is measured by a tachometer and fed back via a differential amplifier. It is important to note that the tachometer *converts the velocity to a voltage*; that voltage is then used as input in the negative terminal of the op-amp. The positive terminal is connected to a *reference voltage*. We will now take a few steps to appreciate the significance of the feedback and reference voltages.

Looking first at the feedback loop, we model the tachometer as producing a voltage V_{tach} equal in numerical value to the translational velocity v of the mass. This means that

$$V_{\text{tach}}(t) \quad [\text{Volts}] = 1 \left[\frac{\text{Volts}}{\text{m} \cdot \text{sec}} \right] v(t). \quad (15)$$

This relationship expresses an ideal “linear transducer with unit gain.” It is especially important to note that the unit conversion requires the presence of a unity gain with units Volts/(m/sec). In many practical situations the relationship is not unity; we would then have to take transduction into account as a non-unity gain in the feedback path. Moreover, many transducers are approximately linear but *saturate* or exhibit other types of nonlinearities if the measured variables (the velocity, in this case) become too large. In the interest of keeping this discussion as clear as possible, we will

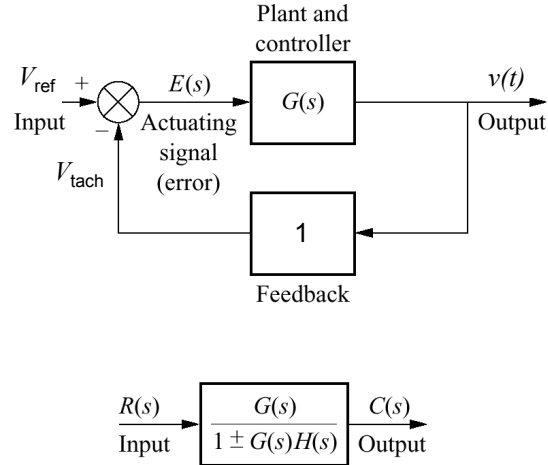


Figure 2: Block diagram representing the physical system of Figure 1.

neglect these practical difficulties and assume an ideal unity-gain tachometer; as we proceed with the solution of this ideal case, however, it is important to keep in mind its limitations in actual systems. Also, to avoid notational clutter we will hereafter cease the explicit mention of units in each equation.¹

As you will learn in 2.678 (or other classes on electronics), the voltage at the output of the differential amplifier is given by

$$\frac{R_2}{R_1} (V_{\text{ref}} - V_{\text{tach}}). \quad (16)$$

The motor circuit does not load the op-amp (the reason we use the op-amp in the first place is that it produces a voltage independent of the load—again, under ideal conditions.) Therefore, the source voltage applied to the DC motor in the feedback configuration is

$$V_s(s) = \frac{R_2}{R_1} (V_{\text{ref}} - V_{\text{tach}}) \equiv K (V_{\text{ref}} - V_{\text{tach}}), \quad (17)$$

where we have denoted $K \equiv R_2/R_1$. We will refer to K as the “feedback gain,” as we saw in Lecture 10.

Collecting the above results from the plant and the feedback loop analysis, we can see that the feedback system is adequately represented as shown in Figure 2. The box labeled as “plant & controller” represents the cascade of the op-amp with the plant.² The box labeled “feedback” represents the tachometer.

¹In real life we should be prepared to reproduce the units associated with each equation as needed, and occasionally use them as sanity check!

²The plant transfer function is particularly simple yet it includes the electrical and mechanical dynamics of the DC motor including the rotational part (shaft with pinion) and the translational part (rack geared to the pinion and carrying the mass load.)

Using the closed-loop transfer function that we derived in class and is shown again in Figure 2, we obtain

$$\frac{V(s)}{V_{\text{ref}}(s)} = \frac{0.3162K}{s + 2 + 0.3162K}. \quad (18)$$

Interpretation of the feedback transfer function

To appreciate the effect of feedback in system behavior, we start by comparing the closed-loop transfer function (18) to the plant (open-loop) transfer function (14). We observe the following:

1. The plant's pole is at -2rad/sec ; whereas the closed-loop pole depends on the feedback gain K and is located at $-2 - 0.3162K$. Therefore, the plant's time constant is $\tau = 0.5\text{sec}$ whereas the closed-loop system's time constant is $\tau = 1/(2 + 0.3162K)\text{sec}$. *In this system, as we increase the gain K the closed-loop system pole moves to the left on the s -plane; therefore, the closed-loop system response becomes faster.*
2. The plant's steady state value is $v_{\infty} = 0.1581\text{m/sec}$; whereas the closed-loop system's steady-state value also depends on the feedback gain K and is $v_{\infty} = 0.3162K/(2 + 0.3162K)$. *In this system, as we increase the gain K the closed-loop system's steady-state value approaches 1; therefore, for large K the output velocity is numerically approximately equal to the reference voltage V_{ref} .*

The first observation means that we can specify (“control”) the feedback system's time constant (“speed”) by changing the gain K . The second observation means that, for large K , the feedback system acts to approximately match its output (velocity) to the reference input (voltage.) In other words, if we apply a step function voltage $V_{\text{ref}} = V_0u(t)$ and wait for a few time constants, the feedback system's output will match (numerically) our reference voltage amplitude V_0 . With some thought, you can convince yourselves that, in fact, any time we change V_{ref} externally, the system will “race” to approximately match the velocity v to our new reference; and it will do so at very high speed because high gain K also implies very small closed-loop time constant.

It is important to note that, even though for large feedback gain K the closed-loop system's steady state is *approximately* equal to unity, it is *not exactly* so; the difference is called the “steady state error” and in this case it is

$$e_{\infty} = 1 - \frac{0.3162K}{2 + 0.3162K} = \frac{2}{2 + 0.3162K}. \quad (19)$$

Table 1 illustrates the trends described above. The feedback gain K is specified by changing the resistance R_2 . In practice, one might use a variable resistor (“trimmer”) to make feedback gain adjustments more conveniently.

It's worthwhile to question the physical origin of the steady-state error (19) in the velocity feedback system. Recall that the output of the differential amplifier, as function of the voltages applied to the input terminals is given by (16). This can be rewritten as

$$V_s(s) = \frac{R_2}{R_1}E(s), \quad (20)$$

R_1 [k Ω]	R_2 [k Ω]	K	τ [sec]	v_∞ [m/ (sec \cdot V)]	e_∞ (%)
1	1	1	0.4317	0.1365	86.35
1	2	2	0.3799	0.2402	75.98
1	5	5	0.2793	0.4415	55.85
1	10	10	0.1937	0.6126	38.74
1	100	100	0.0297	0.9405	5.95

Table 1: Time constant, steady–state velocity and steady–state error for the velocity control system.

where $E(s)$ represents the Laplace transform of the “error signal” (see Figure 2.) In other words, the motor is driven by a voltage proportional to the difference between the reference voltage and the (transduced) velocity. In response to v_s , the DC motor draws as much current as necessary from the op–amp to drive the rotational/translational loads.

Now consider a motor that is initially at rest and the sequence of events after V_{ref} is turned on: Since $v(t = 0) = 0$, the error signal at the differential amplifier’s input is initially equal to V_{ref} . This produces a voltage v_s and, in turn, provides current to the motor to overcome the load’s inertia and friction and start rotating the shaft. The shaft rotation causes the rack to translate (via the pinion gear coupling) and, since $v(t)$ is now increasing, the feedback signal V_{tach} at the op–amp’s “–” terminal begins to increase as well. This causes the error signal to decrease and, in turn, v_s decreases as well. This sequence continues until the load has reached steady–state, which in this case we can also call *terminal velocity*. At steady state, there is no acceleration, *i.e.* the terms $\dot{\omega}$, \dot{v} in (3), (5) vanish. However, from these two equations we can see that some residual torque is still required to overcome the rotational and translational friction in the system. *The difference between the voltages at the input terminals of the differential amplifier must, therefore, remain equal to e_∞ to provide the requisite voltage (and current) to the DC motor so that it can keep the shaft rotating at constant velocity despite the friction.* The unfortunate outcome is that the output velocity never reaches the numerical value of V_{ref} ; this is, of course, undesirable, but it is an unavoidable property of the feedback topology that we chose here. We’ll learn later that other feedback topologies, e.g. the “PID controller” circuit that we will learn about later, can correct this problem.

The physical description of the previous paragraph explains the origin of the terms “reference voltage” and “error signal.” Moreover, we can also understand why the steady–state error diminishes as the feedback gain K grows larger: the voltage v_s applied to the motor is K times the error signal; therefore, if K increases, then a smaller error signal can drive the DC motor with a sufficiently large voltage v_s to overcome friction at steady state. The terminal velocity would then inch up to a numerical value closer to V_{ref} .

To conclude, we have plotted the unit step response of the system in open–loop configuration and feedback (closed–loop) configuration side–by–side for comparison in

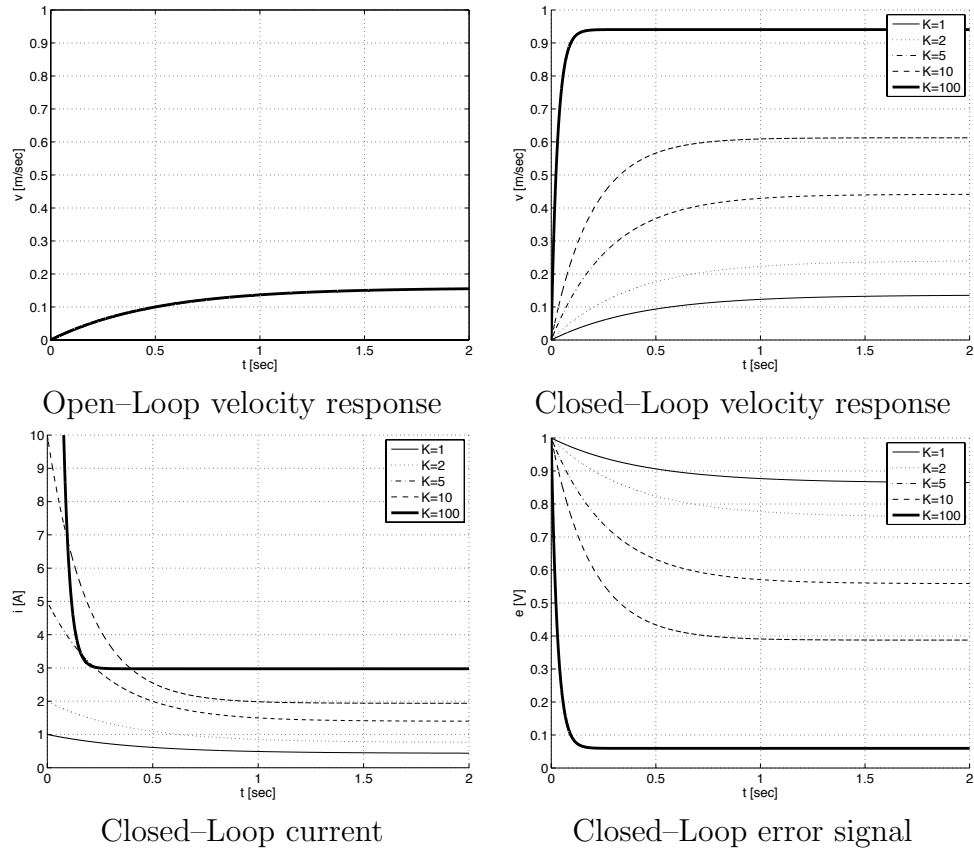


Figure 3: Unit step response of the velocity control system.

Figure 3. The figure also includes the evolution of the current through the DC motor and the error signal applied between the op-amp’s terminals. We can verify the trends described above, especially the time constant and steady state error of the closed-loop step response as we vary the feedback gain K .

We might wish, ideally, to have $K \rightarrow \infty$ because in this limit the steady-state error is eliminated and, moreover, the system’s response is speediest. In practice, this is not possible or even desirable because of several reasons:

- a) We can see from (11) that the current through the DC motor includes a term proportional to the derivative of the translational velocity. If the velocity changes very fast, its derivative becomes large, which means that the motor must draw a large current from the op-amp. Even though in our ideal model the op-amp can produce as much current as we’d like, *real life op-amps can produce a limited maximum current*. If we select the feedback gain K to be large enough to require the DC motor to draw more than the op-amp’s available maximum current, the op-amp will saturate. This is a *nonlinear* effect that we’ve neglected in our analysis.
- b) We recall from Lecture 6 that the DC motor has a finite inductance L which we

can neglect if the system's response is slow enough. If we select the feedback gain K to be large enough to require a response of time constant comparable to the time constant due to L , then L will limit the response speed. This is a *linear* effect that we've neglected in our analysis. In Problem Set 3, we ask you to modify the model to include L , which will allow you to explore feedback dynamics in a second-order system.

- c) There are other phenomena that we've neglected: e.g., the internal gain and input resistance of the op-amp were both assumed to be infinite in the differential amplifier model; the DC motor itself has nonlinearities (e.g. dead zone), etc. Finally, not to mince words, we must always be mindful that if we drive too much current through an op-amp or DC motor, we might burn either or both of them!!

The above caveats do not reduce the value of our simplified model; they just serve as a warning that simple models, powerful as they can be with the intuitive understanding of the physical world that they provide us with, they are limited by their inherent assumptions. A good Engineer must know which is the minimum complexity required to deal effectively with each given situation, and use just that amount of complexity. (This rule is known as "Occam's razor.")

The last question (and perhaps the most relevant one) is: why use feedback at all? From our analysis above, it is clear that by using feedback we can get the system to behave in ways that its natural (open-loop) physics do not allow, e.g. we can get a faster response than the open-loop system. It is also convenient that we can set the desired output (velocity) as an external reference voltage V_{ref} and then have the system race to match the output to our reference input. In many cases, the most important reason for using feedback is *disturbance cancelation*. We have not yet developed the necessary tools to analyze this effect; but it is important to emphasize even now that if the system is subject to a disturbance beyond our control, e.g. electrical noise, mechanical vibrations, etc. which might cause the output to deviate from the desired value V_{ref} , then the feedback loop will act to cancel that disturbance.

A car's cruise control option is a good example of feedback action canceling disturbances; if you have ever used cruise control, you probably know already its utility and its limitations (e.g., if the car needs to climb a steep hill while on cruise control, the engine might not be able to provide enough torque, and the speed then drops or the engine might even stall.) We will examine in detail disturbance cancelation in the near future.

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