Lecture 5: Solution Method for Beam Deflections

5.1 Governing Equations

So far we have established three groups of equations fully characterizing the response of beams to different types of loading. In Lecture 2 relations were established to calculate *strains from the displacement field.*

$$\epsilon(x,z) = \epsilon^\circ(x) + z\kappa \tag{5.1}$$

where

$$\epsilon^\circ(x) = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \quad , \quad \kappa = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \tag{5.2}$$

The above geometrical relation are independent on equilibrium and apply to any kind of materials.

The second set of equations, derived in Lecture 3, is the equilibrium requirement

$$\frac{\mathrm{d}V^*}{\mathrm{d}x} + q(x) = 0 \quad - \quad \text{force equilibrium} \tag{5.3}$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0 \quad - \quad \text{moment equilibrium} \tag{5.4}$$

$$\text{where} \quad V* = V + N\frac{\mathrm{d}w}{\mathrm{d}x} \quad \text{is the effective shear.} \tag{5.5}$$

$$\frac{\mathrm{d}N}{\mathrm{d}x} = 0 \tag{5.6}$$

Eliminating V and V^* between the above equations, the beam equilibrium equation was obtained (See Eq. (3.74))

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + N\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + q = 0 \tag{5.7}$$

The derivation of the equilibrium is valid for all types of materials. In the theory of moderately large deflections, the equilibrium is coupled with the kinematics.

The third group of equation define the material behavior and relates the generalized strains to generalized forces

$$N = EA\epsilon^\circ \tag{5.8}$$

$$M = EI\kappa \tag{5.9}$$

Independence of geometry and equilibrium on constitutive equation allows to develop the general framework of a solver in the Finite Element codes. The constitutive equations can then be added as a user Defined Subroutines.

Let's consider first the infinitesimal deformations (small rotations for which the term $\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2$ vanish in Eq. (5.2) and the term $\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0$ in Eq. (5.7). Then from Eqs. (5.2, 5.4 and 5.8) one obtains

$$EA\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0 \tag{5.10}$$

Eliminating the curvature and bending moments between Eqs. (5.2, 5.7 and 5.9), the beam deflection equation is obtained

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q(x) \tag{5.11}$$

The concentrated load P can be treated as a special case of the distributed load $q(x) = P\delta(x - x_0)$, where δ is the Dirac delta function.

Let's consider first Eq. (5.4) for the axial displacement. The boundary conditions in the x-direction are

$$(N - \bar{N})\delta u = 0 \tag{5.12}$$

The general solution for $u(x)$ is

$$\frac{\mathrm{d}u}{\mathrm{d}x} = D_1\,,\ \ u = D_1 x + D_0 \tag{5.13}$$

There are two integration constants, and two boundary conditions are needed. There are only four combinations of boundary conditions:

1. Beam restricted from axial motion, see Fig. (5.1).

$$u(x = 0) = u(x = l) = 0 \tag{5.14}$$

This gives rise to the solution of two algebraic equation

$$0 = D_0 + D_1 \cdot 0 \tag{5.15a}$$

$$0 = D_0 + D_1 l \tag{5.15b}$$

which gives $D_0 = D_1 = 0$ and $u(x) = 0$. This is a trivial case, for which the axial force $N = EA\frac{\mathrm{d}u}{\mathrm{d}x}$ vanishes as well.

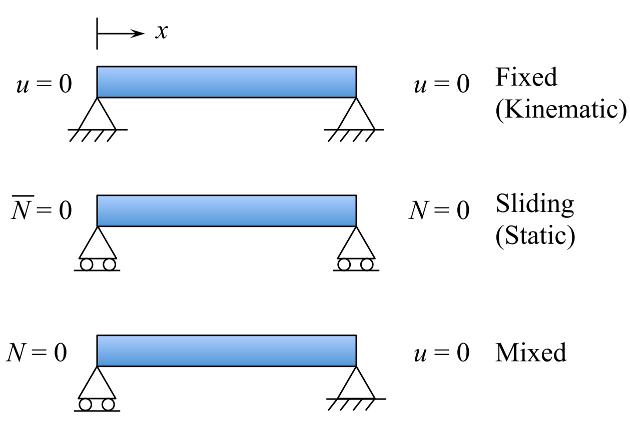

Figure 5.1: Three combinations of in-plane boundary conditions for $u(x)$.

2. Beam allowed to slide in the x-direction on both ends.

$$\bar{N} = N = 0 \quad \text{at} \quad x = 0 \text{ and } x = l \tag{5.16}$$

The axial force is proportional to $\frac{\mathrm{d}u}{\mathrm{d}x}$. From Eq. (5.13) we can see that the gradient of u is zero along the entire beam. So, if $\bar{N} = 0$ or $\frac{\mathrm{d}u}{\mathrm{d}x}$ vanishes at one end, say $x = 0$, $D_1 = 0$ and automatically $\bar{N} = 0$ is satisfied at the other end, $x = l$. The integration constant D_0 is undetermined meaning that the rigid body translation of the entire beam is allowed.

3. In order to prevent the rigid body translation, one end of the beam, say $x = 0$, must be fixed against motion in the x-direction. Thus

$$\bar{N} = 0 \text{ or } \frac{\mathrm{d}u}{\mathrm{d}x} = 0 \quad \text{at} \quad x = 0 \tag{5.17a}$$

$$u = 0 \quad \text{at} \quad x = l \tag{5.17b}$$

which are precisely the boundary conditions for the third case. From Eq. (5.13) we get

$$D_1 = 0 \tag{5.18a}$$

$$D_1 l + D_2 = 0 \;\rightarrow\; D_2 = 0 \tag{5.18b}$$

Now, the axial displacement vanishes, $u(x) = 0$ but the rigid body translation is eliminated.

For all the above three cases of kinematic static and mixed boundary conditions, the axial force was zero.

4. If one end of the beam (bar) is loaded by a given force $\bar{N}$ and the other one is fixed, the boundary conditions (BC) are

$$\begin{aligned} &N = -\bar{N}\ ,\ EA\frac{\mathrm{d}u}{\mathrm{d}x} = 0 \qquad && \text{at } x = 0 \\ &u = 0 && \text{at } x = l \end{aligned} \tag{5.19}$$

$$D_1 = -\frac{\bar{N}}{EA}\ ,\ D_2 = \frac{\bar{N}l}{EA} \tag{5.20}$$

and the solution is

$$u(x) = \frac{\bar{N}}{EA}(l - x) \tag{5.21}$$

The case in which the nonlinear term is retained in Eq. (5.2) is much more interesting. This will be dealt with in the section on moderately large deflection of beams.

We now turn our attention to the solution of the beam deflection, Eq. (5.11). This is the fourth-order linear inhomogeneous equation which requires four boundary conditions. There are four types of boundary conditions, defined by

$$(M - \bar{M})\delta w' = 0 \tag{5.22a}$$

$$(V - \bar{V})\delta w = 0 \tag{5.22b}$$

For the sake of illustration, we select a pin-pin BC for a beam loaded by the uniform like load q, Fig. (5.2).

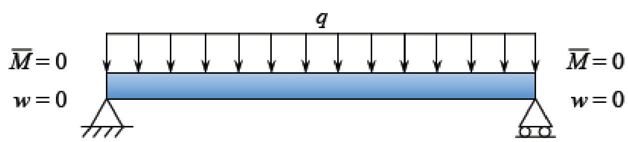

Figure 5.2: Pin support allows for rotation but not for vertical translation.

The bending moment is proportional to the curvature. Eq. (5.11) is then subjected to the following boundary conditions:

$$w(x=0) = w(x=l) = 0 \tag{5.23a}$$

$$\left.\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right|_{x=0} = \left.\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right|_{x=l} = 0 \tag{5.23b}$$

Let's integrate the differential equation four times with respect to x:

$$\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = \frac{qx}{EA} + C_1 \tag{5.24a}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{qx^2}{EA2} + C_1 x + C_2 \tag{5.24b}$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{qx^3}{EA6} + \frac{C_1 x^2}{2} + C_2 x + C_3 \tag{5.24c}$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{qx^4}{EA24} + \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4 \tag{5.24d}$$

Substituting the BC into the general solutions, one gets

$$0 = C_2 \tag{5.25a}$$

$$0 = \frac{ql^3}{2EA} + C_1 l + C_2 \tag{5.25b}$$

$$0 = C_4 \tag{5.25c}$$

$$0 = \frac{ql^4}{24EA} + \frac{C_1 l^3}{6} + \frac{C_2 l^2}{2} + C_3 l + C_4 \tag{5.25d}$$

The solution of the above system is

$$C_1 = -\frac{ql}{2} \tag{5.26a}$$
$$C_2 = 0 \tag{5.26b}$$
$$C_3 = \frac{ql^3}{12} \tag{5.26c}$$
$$C_4 = 0 \tag{5.26d}$$

The load-displacement relation becomes

$$w(x) = \frac{qx}{24EA}(l^3 - 2lx^2 + x^3) \tag{5.27}$$

Differentiating Eq. (5.27) twice, the expression for the bending moment is

$$M(x) = \frac{qx}{2}(l - x) \tag{5.28}$$

and differentiating again, the shear force becomes

$$V(x) = \frac{\mathrm{d}M}{\mathrm{d}x} = \frac{q}{2}(l - 2x) \tag{5.29}$$

Plots of the normalized bending moments and shear forces are shown in Fig. (5.3).

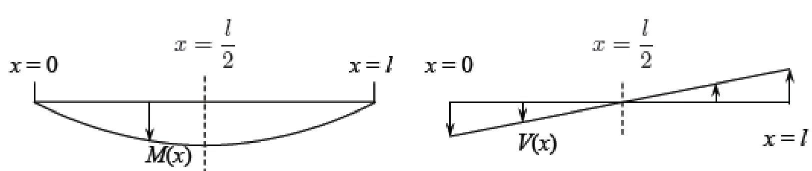

Figure 5.3: Parabolic distribution of the bending moment and linear variation of the shear force.

The shear force $V = EI\dfrac{\mathrm{d}^3 w}{\mathrm{d}x^3}$ is seen to vanish at the *mid-span* of the beam. Also the slope $\dfrac{\mathrm{d}w}{\mathrm{d}x}$ is zero at this location. We have proved that at the symmetry plane

$$V(x = \frac{l}{2}) = 0 \tag{5.30a}$$
$$\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=\frac{l}{2}} = 0 \tag{5.30b}$$

Inversely, if the problem is symmetric, that Eq. (5.30) must hold at the symmetry plane. As an alternative formulation, one can consider a half of the beam with the symmetry BC.

Can you solve the above problem and *compare* it with solution of the pin-pin beam, Eq. (5.27)?

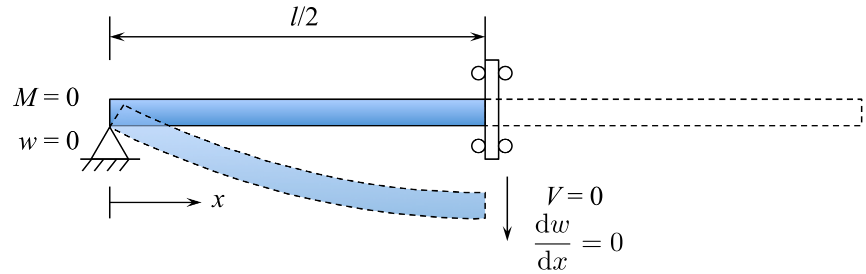

Figure 5.4: Simply-supported plate with symmetry boundary conditions.

It should be mentioned that the pin-pin supported beam is a statically determinate structure. Therefore the distribution of shear forces and bending moments can be determined from the equilibrium equation alone. *Can you do it and get correctly the signs*?

The purpose of Lecture 5 is to present properties of the governing equations and solutions. interested students are referred to end chapter of problem sets where many beams with different loading and BC are considered. Also the recommended reference book and monographs present solution to some common beam problems.

5.2 General Properties of the Beam Governing Equation: General and Particular Solutions

Recall from the Calculus that solution of the inhomogeneous, linear ordinary differential equation is a sum of the general solution of the homogeneous equation w_{g} and the particular solution of the inhomogeneous equation w_{p}. The property of homogeneity means that $f(Ax) = Af(x)$. The homogeneous counterpart of Eq. (5.11) is

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = 0 \quad \text{or} \quad \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = 0 \tag{5.31}$$

and its solution, obtained by four integrations is the third order polynomial

$$w_{\mathrm{g}}(x) = \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4 \tag{5.32}$$

The particular solution w_{p} of the beam deflection equation, Eq. (5.11) depends on the loading, but not the boundary conditions. For the uniformly loaded beam the particular solution is the first term in Eq. (5.23d). As an illustration, consider the same pin-pin supported beam loaded by the triangular line load

$$q(x) = q_0 \frac{2x}{l}\,, \quad 0 < x < \frac{l}{2} \tag{5.33}$$

where q_0 is the load intensity at mid-span $x = l/2$. The particular solution of this problem, satisfying the governing equation is

$$w_{\mathrm{p}} = \frac{q_0 x^5}{60EIl} \tag{5.34}$$

Then, the full solution is $w(x) = w_{\mathrm{g}} + w_{\mathrm{p}}$.

Beam loaded by concentrated forces (or moments) requires special consideration.

Continuity requirements

A sudden change in the beam cross-section or loading may produce a discontinuous solution. What quantities may suffer a jump and what must be continuous?

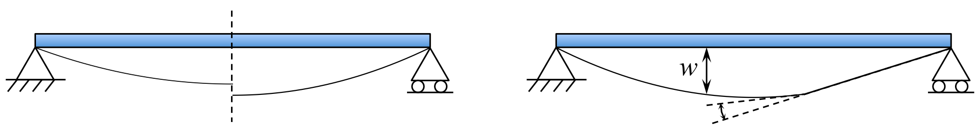

Figure 5.5: The displacement and slope discontinuities are not allowed in beams.

In mechanics the discontinuity of a given function is denoted by a square bracket

$$[f(\xi)] = f(\xi^+) - f(\xi^-) \tag{5.35}$$

where ξ^+ and ξ^- denote the values of the argument on the right and left hand of a discontinuity. In the quasi-static theory of beam

$$[w] = 0 \tag{5.36a}$$

$$\left[\frac{\mathrm{d}w}{\mathrm{d}x}\right] = 0 \tag{5.36b}$$

The discontinuity in the vertical displacement means separation so of course it may not occur. Why then slopes must be continuous for elastic beams? This is simple. A change of slope is called a curvature. A jump in the slope gives an infinite curvature, and thus an infinite bending moments. Such a situation is impossible, because the beam cross-section will go into plastic range, and the beam will no longer stay elastic. Quantities that can be discontinuous are

$$\text{Bending moments} \quad [M] = \bar{M} \tag{5.37a}$$

$$\text{Shear force} \quad [V] = \bar{V} \tag{5.37b}$$

As an illustration, consider a pin-pin supported beam loaded at mid-span by a point force P.

As mentioned earlier, the point load can be considered as a limiting case of a continuous line load with the help of the Dirac delta function

$$q(x) = P\delta(x - \frac{l}{2}), \quad \text{where} \int \delta(x - \frac{l}{2})\mathrm{d}x = 1 \tag{5.38}$$

Even though techniques have been developed to deal with singularity functions for beams, they require to use the apparatus of the mathematical theory of distribution. This

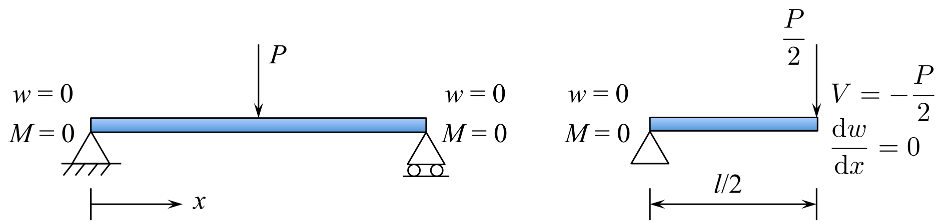

Figure 5.6: Symmetric loading of a beam by a concentrated force.

is not the avenue that we will take. instead, a symmetry boundary condition will be imposed. Now, the concentrated load is not applied inside the beam $0 < x < l$, governed by the inhomogeneous differential equation, but at the boundary. Each half of the beam is carrying half of the load. Therefore, the boundary conditions are

$$\text{at } x = 0 \quad w = 0 \tag{5.39a}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \tag{5.39b}$$

$$\text{at } x = \frac{l}{2} \quad V = -\frac{P}{2} \tag{5.39c}$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = 0 \tag{5.39d}$$

Because the loading is applied on the boundary, the differential equation becomes homogeneous. The solution of Eq. (5.31) is given by the third order polynomial, substituting the above BC into the solution given by Eq. (5.32), a system of four linear algebraic equations is obtained, where the solution is

$$C_1 = -\frac{D}{2EI},\ C_2 = 0,\ C_3 = \frac{Pl^2}{16EI},\ C_4 = 0 \tag{5.40}$$

The deflection line is given by

$$w(x) = \frac{Px}{48EI}(3l^2 - 4x^2) \tag{5.41}$$

and the central deflection (something to remember) is

$$w_0 = w(x = \frac{l}{2}) = \frac{pl^3}{48EI} \tag{5.42}$$

The plot of the distribution of bending moment and shear forces along the length of the beam determined from the calculated deflection line is shown in Fig. (5.7).

Note that the jump in the internal shear force is equal to the applied force

$$[V] = V_{\text{right}}(x = \frac{l}{2}) - V_{\text{left}}(x = \frac{l}{2}) = P \tag{5.43}$$

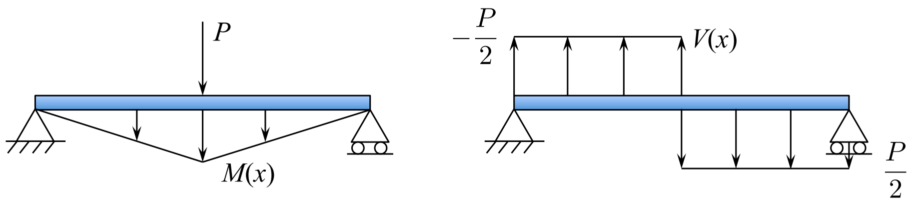

Figure 5.7: Bending moment is continuous at the mid-span, but the shear force is not.

If the point load is not applied at the mid-span but at an arbitrary distance $x = a$, the beam must be divided into two parts $0 < x < a$, $a < x < l$, and each part must be solved independently.

$$\text{First segment } 0 < x < a \quad w^{\mathrm{I}}(x) = \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4 \tag{5.44a}$$

$$\text{Second segment } a < x < l \quad w^{\mathrm{II}}(x) = \frac{C_5 x^3}{6} + \frac{C_6 x^2}{2} + C_7 x + C_8 \tag{5.44b}$$

This gives rise to eight integration constants, four for each side. Would there be enough conditions to determine these constants? The answer is YES.There are two boundary conditions at $x = 0$, four continuity conditions at $x = a$, given by Eqs. (5.36-5.37) and, again, two boundary conditions at $x = l$. In summary

BC, $x = 0$	Continuity, $x = a$	BC, $x = l$
$w = 0$	$[w] = 0$	$w = 0$
$M = 0$	$\left[\frac{\mathrm{d}w}{\mathrm{d}x}\right] = 0$	$M = 0$
	$[M] = 0$	
	$[V] = P$	

(5.45)

Note that there is no concentrated bending moment applied $\bar{M} = 0$ so that the bending moment field is continueous across $x = a$. The concentrated force produces a jump in the distribution of the shear forces, so $\bar{V} = P$.

We leave it to the reader to apply the above condition and solve the problem. More on this problem can be found in two sections of this notes: Problem Sets and Recitations.

The method of superposition says that the deflections and slopes of the beam subjected to a system of loads are equal to the sum of those quantities due to individual loads. In other words the individual results may be superimposed to determine a combined response, hence the term *method of superposition.*

This is a very powerful and convenient method since solutions for many support and loading conditions are readily available in various engineering handbooks. Using the principle of superposition, we may combine these solutions to obtain a solution for more complicated loading conditions.

As an example, consider a clamped-clamped beam loaded by a uniform line load q and concentrated force at the center P. The deflection formulas for the two individual loading are

$$w|_{\text{uniform}} = \frac{qx^2}{24EI}(l-x)^2 \tag{5.46a}$$

$$w|_{\text{point}} = \frac{Px^2}{48EI}(3l-4x) \tag{5.46b}$$

The solution for both loads acting together is

$$w_{\text{total}} = w|_{\text{uniform}} + w|_{\text{point}} \tag{5.47}$$

5.3 Statically Determined Beams

Beam for which the distribution of bending moments and shear forces can be determined from the equilibrium alone are called *statically determinate beams*. For such beams $M(x)$ and $V(x)$ are known and determination of beam deflection will be a much easier task. Combining Eq. (5.9) with Eq. (5.2) one ends up with the following second order linear differential equation

$$-EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = M(x) \tag{5.48}$$

The bending moment, which by itself should satisfy the second order differential equation, Eq. (5.7) should now obey two stress boundary conditions at the beam ends. The static boundary conditions are indicated in Fig. (5.8) for a pin-pin supported and cantilever beam.

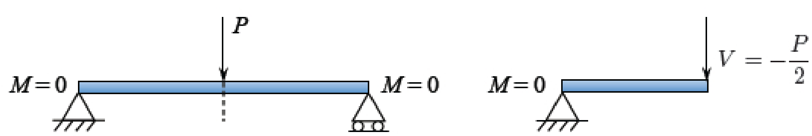

Figure 5.8: The static boundary conditions for a full and half of a beam.

Determination of bending moment and shear force diagrams is the subject of elementary courses in statics, and the general procedure is not explained here. In the case of the simply supported beam with a point load at the mid-span, the bending moments

$$M(x) = \begin{cases} \dfrac{Px}{2}, & 0 < x < \dfrac{l}{2} \\ \dfrac{P(l-x)}{2}, & \dfrac{l}{2} < x < l \end{cases} \tag{5.49}$$

The bending moment vanishes at $x = 0$ and $x = l$.

The corresponding shear force $V = \frac{\mathrm{d}M}{\mathrm{d}x}$ is

$$V(x) = \begin{cases} \dfrac{P}{2}, & 0 < x < \dfrac{l}{2} \\ -\dfrac{P}{2}, & \dfrac{l}{2} < x < l \end{cases} \tag{5.50}$$

At the beam center

$$[V] = \frac{P}{2} - \left[-\frac{P}{2}\right] = P \tag{5.51}$$

Because of shear force discontinuity at the beam center, the solution will be sought for a half of the beam. Each half of the beam is carrying half of the load. We have shown that the bending moment distribution satisfy two satin boundary condition. Therefore the differential equation (5.49) is subjected only to two kinematic boundary conditions

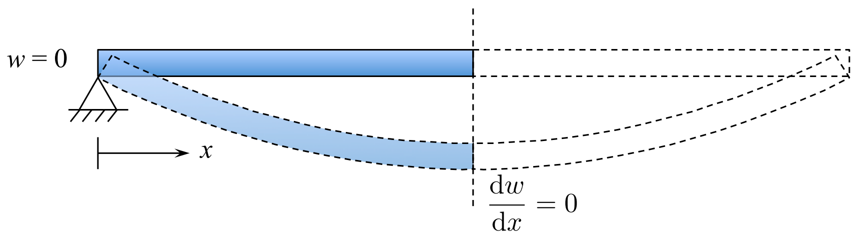

Figure 5.9: Symmetry boundary condition.

Integrating Eq. (5.48) twice one gets

$$-EIw = \frac{Px^3}{12} + C_1 x + C_2 \tag{5.52}$$

The two integration constants, determined from the boundary conditions $w(0) = 0$, $\left.\frac{dw}{dx}\right|_{x=\frac{l}{2}} = 0$, are

$$C_1 = -\frac{Pl^2}{16}, \quad C_2 = 0 \tag{5.53}$$

and the deflection line of the beam is given by

$$w(x) = \frac{Px}{48EI}(3l^2 - 4x^2) \quad 0 < x < \frac{l}{2} \tag{5.54}$$

The second half of the beam is the mirror reflection, by symmetry. In particular, the central deflection $w_o = w(x = \frac{l}{2})$ is expressed by all input parameters of the beam as

$$w_o = \frac{Pl^3}{48EI} \tag{5.55}$$

It will be helpful to remember the above formula for the rest of your professional life.

In summary, determination of deflections of statically determinate beams is much easier than its statically indeterminate counterparts. The governing equation is of the second order, and for symmetric problems there are only two integration constants.

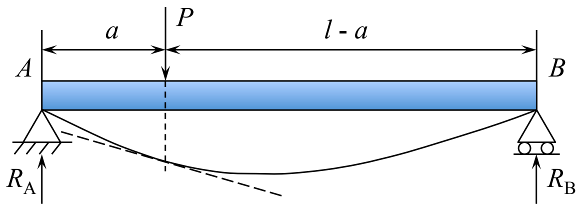

Figure 5.10: Beam under off-center point load.

5.4 Continuity Conditions, an Example

In Section 5.4 the continuity requirements were formulated, but the system of eight algebraic equations was not solved. Here a complete solution will be presented for a beam loaded by a point force acting at an arbitrary location $x = a$.

The reaction forces are calculated from moment equilibrium:

$$R_A = P\frac{l-a}{l} \tag{5.56a}$$

$$R_B = P\frac{a}{l} \tag{5.56b}$$

The sum of the reaction forces is equal to P. The corresponding bending moments and shear forces are

$$M(x) = \begin{cases} R_A x = \dfrac{P(l-a)x}{l}, \\ R_B(l-x) = \dfrac{Pa(l-x)}{l}, \end{cases} \qquad V(x) = \begin{cases} \dfrac{P(l-a)}{l}, & 0 < x < a \\ -\dfrac{Pa}{l}, & a < x < l \end{cases} \tag{5.57}$$

The jump in the shear force across the discontinuity point $x = a$ is

$$[V] = V^+ - V^- = \frac{P(l-a)}{l} - (-\frac{Pa}{l}) = P \tag{5.58}$$

The bending moments are continuous on both sides, $[M] = 0$. Therefore the static continuity conditions are automatically satisfied at $x = a$. The kinematic continuity conditions, formulated in Eq. (5.36) require displacements and slopes to be continuous. Integrating the governing equations (5.48) with (5.57) in two regions gives

$$-EIw^{\mathrm{I}} = \frac{P(l-a)x^3}{6l} + C_1 x + C_2 \qquad 0 < x < a \tag{5.59a}$$

$$-EIw^{\mathrm{II}} = \frac{Pa}{l}(\frac{lx^2}{2} - \frac{x^3}{6}) + C_3 x + C_4 \qquad a < x < l \tag{5.59b}$$

The four integration constants are found from two boundary condition and two continuity condition

$$w(0) = w(l) = 0,\ w^{\mathrm{I}}(a) = w^{\mathrm{II}}(a), \quad \left.\frac{\mathrm{d}w^{\mathrm{I}}}{\mathrm{d}x}\right|_{x=a} = \left.\frac{\mathrm{d}w^{\mathrm{II}}}{\mathrm{d}x}\right|_{x=a} \tag{5.60}$$

This gives rise to the system of four linear inhomogeneous algebraic equations for C_1, C_2, C_3, and C_4

$$\begin{cases} C_2 = 0 \\ \dfrac{Pal^2}{3} + C_3 l + C_4 = 0 \\ \dfrac{Pba^3}{6l} + C_1 a = \dfrac{Pa}{l}\left(\dfrac{la^2}{2} - \dfrac{a^3}{6}\right) + C_3 a + C_4 \\ \dfrac{Pba^2}{2l} + C_1 = \dfrac{Pa}{l}\left(la - \dfrac{1}{2}a^2\right) + C_3 \end{cases} \tag{5.61}$$

A simple problem has led to a quite complex algebra. Now, you understand why the previous example with eight unknown coefficients was only formulated but not solved. The solution to the system (5.61) is

$$C_1 = -\frac{Pa(a^2 - 3al + 2l^2)}{6l} \tag{5.62a}$$

$$C_2 = 0 \tag{5.62b}$$

$$C_3 = -\frac{Pa(a^2 + 2l^2)}{6l} \tag{5.62c}$$

$$C_4 = \frac{Pa^3}{6} \tag{5.62d}$$

and the final solution of unsymmetrically loaded beam is

$$w^{\mathrm{I}}(x) = \frac{Px\left[a^3 - 3a^2 l - lx^2 + a(2l^2 + x^2)\right]}{6EIl} \qquad 0 < x < a \tag{5.63a}$$

$$w^{\mathrm{II}}(x) = -\frac{Pa(l-x)\left[a^2 + x(-2l + x)\right]}{6EIl} \qquad a < x < l \tag{5.63b}$$

One can easily check that the continuity conditions are met at $x = a$. The above example teaches us that symmetry in nature and engineering not only means beauty, but also brings simplicity.

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