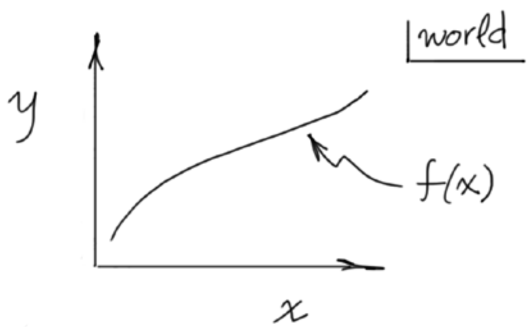
Interpolation

Introduction



$$f : x \longrightarrow f(x)$$

Approximation:

accuracy

replace $x \longrightarrow f(x)$

with $x \longrightarrow (\mathcal{A}f)(x) \approx f(x)$

cost

$$\mathcal{A} : \begin{cases} \mathcal{I}, \mathcal{P}, \ldots \\ \mathbb{P}_p, \text{ ad hoc}, \ldots \end{cases}$$

How is $\mathcal{A}f$ <u>connected</u> to f?

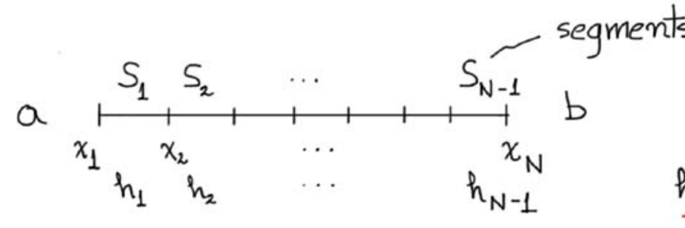
What <u>is</u> $\mathcal{A}f$?

Piecewise-Constant, "what"

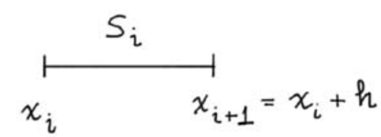
Left-Endpoint "how"

Formulation

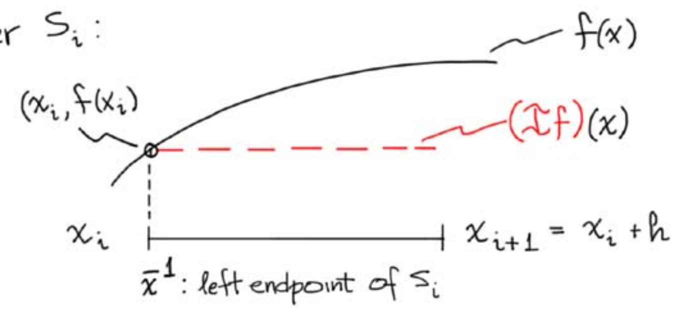
interval $a \le x \le b$



h (equi)



over S_i:



what

$(\mathcal{I}f)(x)$: constant over S_i

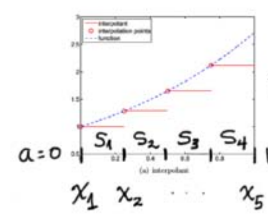
$(\mathcal{I}f)(x_i) = f(x_i)$ how

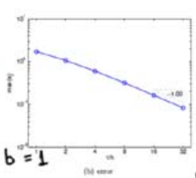
$\left.\right\}$ $(\mathcal{I}f)(x) = f(x_i)$

$\uparrow$

$\bar{x}^1$

over $a \le x \le b$:





$N = 5$

$h = (b-a)/N-1$

$= 1/4$

DEMO

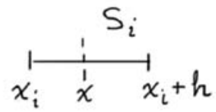
Error Analysis

f' continuous

$$|f(x) - (\mathcal{I}f)(x)| = |f(x) - f(x_i)| \qquad x \text{ in } S_i$$

$$= \left| \int_{x_i}^{x} f'(\xi)\, d\xi \right|$$

$$\leq \int_{x_i}^{x} |f'(\xi)|\, d\xi$$



$$\leq \max_{x \text{ in } S_i} |f'| \int_{x_i}^{x} d\xi$$

$$\leq h \max_{x \text{ in } S_i} |f'| \qquad \text{any } x \text{ in } S_i$$

$$e_i \equiv \max_{x \text{ in } S_i} |f(x) - (\mathcal{I}f)(x)| \leq h \max_{x \text{ in } S_i} |f'|$$

$x_i \leq x \leq x_{i+1}$

$$\Downarrow$$

$$e_{max} \equiv \max_{\text{all } S_i} |f(x) - (\mathcal{I}f)(x)| \leq h \max_{\text{all } S_i} |f'|$$

$a \leq x \leq b$ $\qquad$ $1/N-1$

or

$$e_{max} \leq C h \qquad C = \max_{\text{all } S_i} |f'|$$

independent of h

$$e_{max} \leq C h \quad \text{for any } h$$

$$\Downarrow$$

$$e_{max} \leq C h \quad \text{as } h \to 0 \qquad \text{"big Oh"}$$

$$e_{max} = O(h)$$

also (here) $\qquad f''$

$$e_{max} \sim C h \iff \frac{e_{max}}{C h} \to 1 \quad \text{as } h \to 0$$

asymptotic

sooner or later

$$e_{max} \leq C h^p : \qquad (\text{as } h \to 0)$$

convergence: $e_{max} \to 0$ as $h \to 0$

convergence rate : order p

how fast

$p = 1$: first order

(e.g. piecewise-constant, left-endpoint)

$p = 2$: second order

⋮

$$e_{max} \sim C h^p \qquad h \to 0$$

$$\Downarrow$$

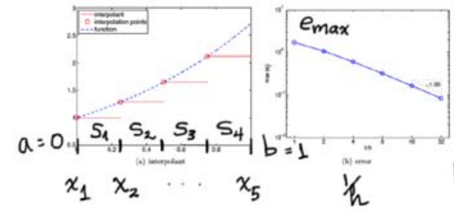
$$\log(e_{max}) \sim \log C + p \log(h)$$

$$= \log C - p \log(1/h)$$

$$= \log C - p \log((N-1)/(b-a))$$

error

intercept ("$h = 1$") slope number of segments



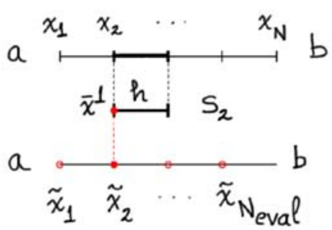
$N = 5$

$h = (b-a)/(N-1)$

$= 1/4$

DEMO $(b - a = 1;\ 1/h = (N-1))$

Operation Count (storage)



$N-1$ segments

$N_{eval} = N-1$ evaluation points

$h = (b-a)/N_{eval}$

Offline:

evaluate (and store) $\tilde{x}_i \to f(\tilde{x}_i),\ 1 \le i \le N_{eval}$

expensive

Online: given x

(i) find x_{i^*}: $x_{i^*} \le x \le x_{i^*+1}$ segment which contains x

h: $O(1)$ FLOPs h_i: $O(\log N_{eval})$, $O(N_{eval})$ FLOPs

$i^* = \text{floor}(x/h) + 1$ binary chop comparison

(ii) "look up" $f(\tilde{x}_{i^*})$ $O(1)$ FLOPs

Nomenclature:

FLOPs : Floating Point Operations

$z = 2 + 3*4$ 2 FLOPs

$O(g(K))$: K: "size" of problem N_{eval}

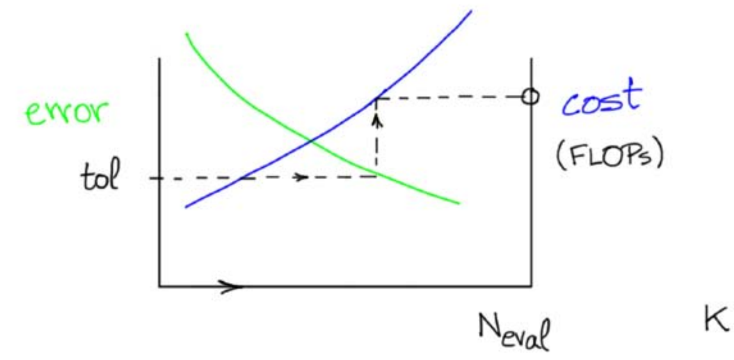
operation count $= O(g(K))$ FLOPs

$\Updownarrow$

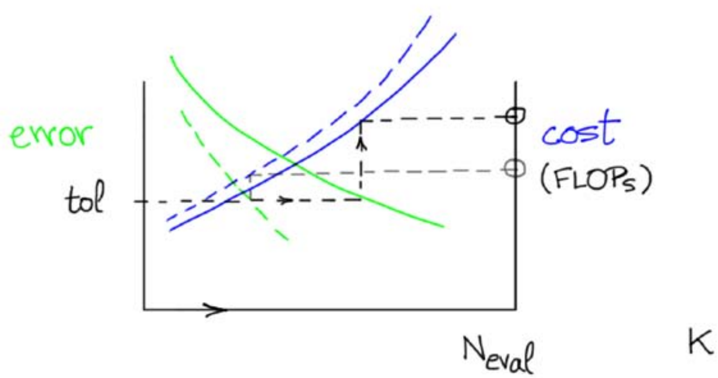
operation count $\leq c\,g(K)$ FLOPs as $K \to \infty$

[e.g.: $O(K^2 + K) = O(K^2)$]

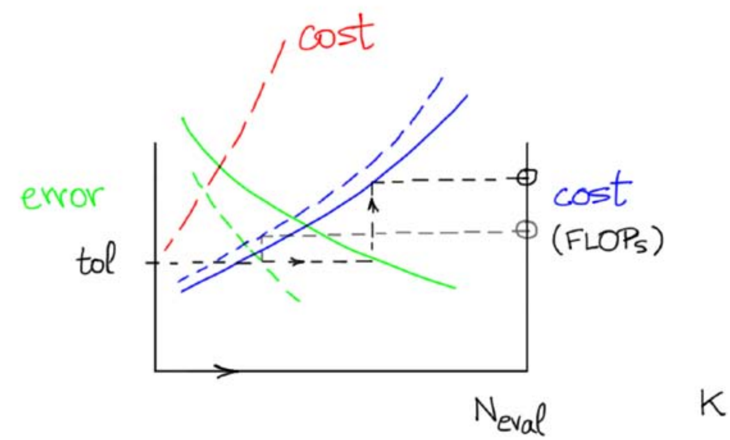
The Game:



Method I



Method II

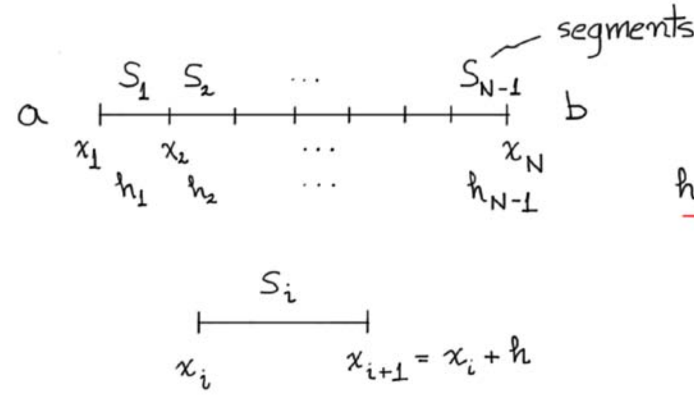


Method II
poorly implemented

Piecewise-Linear

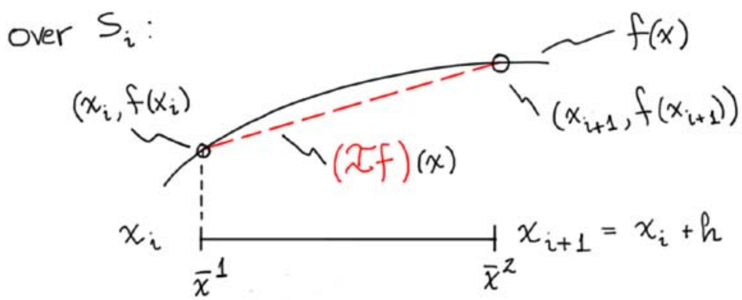
Formulation

interval $a \le x \le b$



h (equi)

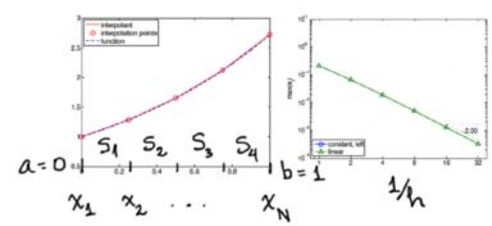
over S_i:



$$\left.\begin{array}{l}(\mathcal{I}f)(x) : \text{linear over } S_i \\ (\mathcal{I}f)(x_i) = f(x_i);\ (\mathcal{I}f)(x_{i+1}) = f(x_{i+1})\end{array}\right\}$$

$$\Rightarrow (\mathcal{I}f)(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \cdot (x - x_i)$$

over $a \le x \le b$



$N = 5$

$h = (b-a)/N-1$

$= 1/4$

DEMO

Error Analysis

f'' continuous

$$e_{max} \equiv \max_{a \le x \le b} |f(x) - (\mathcal{I}f)(x)|$$

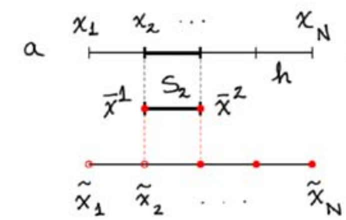
$$\le C h^2 \quad \text{~ p: order} \qquad \Rightarrow O(h^2)$$

for $C = \frac{1}{8} \max_{a \le x \le b} |f''(x)|$

$\Rightarrow$ piecewise-linear is second order

DEMO $(b-a=1)$

Operation Count



N segments

$N_{eval} = N$ evaluation points

$h = (b-a)/(N_{eval} - 1)$

Offline:

evaluate (and store) $\tilde{x}_i \rightarrow f(\tilde{x}_i)$, $1 \le i \le N_{eval}$

expensive

Online: given x

segment which contains x

(i) find x_{i^*}: $x_{i^*} \le x \le x_{i^*+1}$

h: $O(1)$ FLOPs h_i: $O(\log N_{eval})$, $O(N_{eval})$ FLOPs

(ii) "look up" $f(\tilde{x}_{i^*}) \rightarrow r$, $f(\tilde{x}_{i^*+1}) \rightarrow s$

$$(\mathcal{I}f)(x) = r + \frac{s-r}{x_{i^*+1} - x_{i^*}} \cdot (x - x_{i^*}) \qquad 4 \text{ FLOPs}$$

Perspectives

What if

- $f(x)$ is not smooth? $\quad f, f', f'' \ldots$
- $f(x)$ undergoes rapid variation? $\quad f', f'', \ldots$
- we wish to consider higher-order interpolants,
  $(\mathcal{I}f)(x)$: piecewise-quadratic, -cubic, $\ldots$?
- we wish to estimate the error

$$x \to \begin{cases} (\mathcal{I}f)(x), \text{ and} \\ \Delta(x) \text{ such that } |f(x) - (\mathcal{I}f)(x)| \approx \Delta(x) \end{cases} ?$$

What if

- we wish to incorporate derivative conditions,

$$(\mathcal{I}f)'(x_i) = f'(x_i), \ldots \quad ?$$

- evaluation of $f(x)$ is not exact?

$$x \to f(x) + \begin{matrix} \text{error} \sim \text{FP arithmetic}, \ldots \\ \text{or} \\ \text{noise} \sim \text{measurement}, \ldots \end{matrix}$$

- we wish to exploit "ad hoc" information

$$f(x) = \beta_0 + \beta_1/x \qquad \beta_0, \beta_1 \text{ unknown?}$$