PROFESSOR: This week is my second pair of lectures. Last week the two lectures were about first order differential equations, and this week second order. Those are the two big topics in differential equations.

Let me start with most basic second order equation. We see the second derivative and the function itself, and we don't see yet the first derivative term. This is the nice case, when I just have y double prime and y. In general, I-- I'm taking constant coefficients today. Because if the coefficients depend on time, the problem gets much, much harder now.

So let's stay with constant coefficients, meaning we have a mass, for example, we have a spring. The stiffness of the spring is k, the mass is m, and the y, the unknown displacement, tells us the movement of the mass. The classical problem. You will have seen it before.

Because you have an exam this afternoon, I wanted to start with things that-- they are about second order equations, but they're still close to the exam idea, particularly the idea of exponentials. With constant coefficients, that's the fundamental message. Exponentials in, exponentials out. But it's not quite so clear when we had first order, y prime equal ay, we knew that the exponent was a. The solution was e to the at.

Now we've got second derivatives coming in, and it won't be so much e to the at type thing. Either the at was growth for a positive, decay for a negative. Now we're going to see oscillation. It's still exponentials, but oscillation. Things going up and down, things going around. Harmonic motion, you call it. Sines and cosines.

And sines and cosines connect to complex exponentials. So that instead of e to the at-- so now oscillations-- they're going to be coming from e to the i omega t. In other words, instead of an a, we're going to have an i omega.

Or, if we like to stay real, we can stay with cos-- cosine-- and sine. And actually, I've written two real guys there, so I better have two complex ones. And it will turn out to be plus or minus. There are two frequencies there. Plus i omega, minus i omega, and they turn into cosine and sine.

So in this case, with no damping term, we can stay entirely real without creating any problems. We can work with cosines and sines.

The first question is, what's omega. What is the frequency of oscillation. And of course, another similar picture would be a pendulum, a linear pendulum, swinging side to side, keeping time, because that frequency will stay constant.

Always I'll start with zero on the right hand side. Just look at these equations. Constants there. I'm looking for solutions. And I'm looking for null solutions, looking for the natural motion of this spring, the natural up and down motion of this spring.

Classical problem. Won't be brand new, but it's the right starting point for the full second order equation. It'll get a little complicated on Wednesday, when damping gets in there. The formula got a little messy, because you've got a mass-- you've got an m-- and a k, still, but you also will have a damping constant. Then complex numbers really come in. Here they're optional.

So this is my equation to solve. Because we don't have a first derivative, a cosine will solve that. So let me look for-- I could look for exponentials. Maybe I should do that first, look for an exponential solution. Yeah, that's a good idea. And let me not jump ahead to know that the exponent has that i omega form. Let me discover it.

So I look for no solutions-- because I have that zero there-- no solutions of the form e to the st, some exponent. Plug it in. That's the message with constant coefficients. Look for exponentials, substitute them in, discover what s will be. So let's just do that.

This is the most basic step. For null solutions will be exponentials, I substitute into the equation. I get ms squared from two derivatives. It will bring s down twice. This is just ke to the st, and I'm looking for null solutions. Zero. No forcing. So this is undamped, unforced. Undamped, unforced. Natural motion.

What do I do now? Plugged in an exponential, got this equation. And the beauty is that the exponentials cancel. An exponential is never zero, so I can safely divide by it. So I cancel those, and I get ms squared plus k equals zero. The key equation-- and it's so simple-- it's just we're doing algebra now. The calculus, the derivative we took when we plugged it in, but now it's an algebra question. And of course, solving that system is easy. There's no s term, no damping term.

So the frequency, s, is-- put k on the other side, divide by n. s is-- s squared, let's say-- is k over m-- is minus k over m. Critical point.

That tells me, with that minus sign there, that s is an imaginary number. A complex number has a real part and an imaginary part. In this case, all imaginary. No real part at all.

It's natural to think-- s is the square root of that, so I'm going to write-- everybody writes-- s, the frequency s, is i omega. So if I plug that in, I have omega squared equal k over m. i squared and the minus 1 deal with each other. So the frequency omega-- here is the great fact-- square root of k over m. That's-- yes?

AUDIENCE: What difference does having imaginary parts to answer affect the oscillation?

PROFESSOR: To-- OK. Oscillation, just pure oscillation-- which is what we would see here with no damping-- is the frequency, e to the-- the solution-- the displacement, I could write-- the displacement up and down, y, will involve e to the i omega t, and e to the minus i omega t.

We've got second order equation. Let me just go back to that key point. When we have second order equations, we look for two-- we expect and we want and we need two-- solutions. There will be two.

I didn't put it here, and I should. s, the frequency, is plus or minus i omega, because in both cases when we square, it comes out right. So we get two frequencies, and here they are.

So let's see how to answer your question. The presence of this i is only telling me that, essentially, I've got sines and cosines. That's really what-- when it's a pure imaginary number-- I would call that a pure imaginary number, there's no real part at all-- then equally cos omega t and sine omega t.

I can now, if I want, go real. I can say, OK, these were the general null solutions. Let me put this down, then. The null solution-- I'm looking only right now at null solutions-- is some combination of e to the i omega t and e to the minus i omega t. That's what we got from plugging in e to the st, discovering that s was an imaginary number, and we got these guys.

But equally-- equally-- yn is a combination of cos omega t and sine omega t. And maybe you'll like those better. I think everybody practically likes those better.

Do you see that these guys are the same as these guys? The c's are a little different because, well, we know that we can switch from one to the other. We remember that basic fact that e to the i omega t is cos omega t plus i times sine omega t. You're used to maybe seeing that omega t as theta, e to the i theta is cos theta plus i sine theta.

And e to the minus i omega t, of course, is cos omega t minus i sine omega t.

I hope you won't think I'm filling the blackboard with formulas, because I'm really just writing down-- well anyway, they're beautiful formulas.

So if I have these guys, then I have these and vice versa. If I have cos-- how would you write cos omega t using the exponentials? I want to just see totally clearly that I can go back and forth between complex imaginary exponentials and cosines and sines.

So how would I, I want to go in the opposite direction and write the cosine and the sine as combinations of these, just to show if I've got combinations of one, I've got combinations of the other. Combinations of these are the same as combinations of those. So what is cos omega t in terms of these guys?

AUDIENCE: [INAUDIBLE] some of them divided by 2?

PROFESSOR: Exactly. If I add those two, this part cancels. I've got two of these, so I have to divide by 2, as you say. It's a half of the first plus a half of the second.

And how about sine omega t? Sine omega t is always slightly more annoying, because it's the one-- it's the imaginary part that brings in an i. What would be the same formula? How could I produce sine omega t out of that?

Yes?

AUDIENCE: The difference divided by 2i.

PROFESSOR: Yes. If I take the difference, that'll cancel the cosines. So I'm going to take e to the i omega t minus e-- minus e to the minus i omega t. Take the difference.

But then I've got 2i multiplying this sine. Up here I had a 2, but now I've got-- when I take the subtract, these i's are in there, so I divide by 2i.

So this just tells me that I can go either way. Next time, we'll see what happens when there is damping and there are complex numbers instead of pure i omegas.

We're golden here. We've found the great quantity with the right units. The right units of omega are 1 over time. Actually the units are radians per second, would be the typical appropriate unit. Radians per second.

I'll use the word frequency for that, but there's another definition of frequency, cycles per second. I just want to think about steady motion around a circle. So this tells me how many radians per second. And if this is 2pi-- if omega happened to be 2pi-- then I would go once around the circle. If omega was 2pi, then when t reached one, I would be around the circle. Let me draw a circle in a minute.

So there's a 2pi here hiding behind the word radians. And in many cases, you'll want also a definition in cycles per second. So f is omega divided by the 2pi, and that's in cycles per second. Full revolutions per second. And that's hertz. I think I misspoke last time in confusing these two, so let's get them straight here. There's no complicated math in here, it's just a factor 2pi, but of course that factor is important.

So a typical frequency in everyday life would be like f, 60 cycles per second, 120pi radians per second. So I'm going around in a circle.

Now I'm ready to have initial conditions. This connects, again, to the afternoon exam. We found the general solution with some constants, like here. Let's keep that real form. And now those constants get determined by the initial conditions. Conditions plural, because we have an initial position, like I stretch it-- maybe I stretch it and let go. Maybe I stretch the spring and then I let go. What happens?

By stretching it, I'm giving it an initial displacement. And I'm giving it zero initial velocity, because I stretched it and just let go. Another possibility would be to strike it. If I hit that mass, that would be a different initial condition. What would be the initial condition if it's sitting there in equilibrium quietly minding its own business and I hit it? Then I've given it an initial velocity, with initial displacement zero. So those would be the two extreme possibilities. Pull it down, let go, or strike it when it's sitting in equilibrium.

Anyway, we've got two initial conditions. You see why-- y double prime is showing up because essentially we've got Newton's Law. This is Newton's Law. Mass times acceleration is equal to minus ky-- that's the, with the minus sign, and the all-important minus sign, that's the acceleration. That's a force, sorry.

Mass, acceleration, this thing with a minus sign is the force, and the force is pulling back. If y is stretching, the force is restoring.

Let me just go ahead with what you know. The initial conditions. And I want to solve my double prime plus ky equals 0. So I'm still talking about the unforced with given y of 0 and y prime of 0.

Just think for a moment. Could you do that? This is the most basic second order equation. We know what the solutions look like. Let's do this one in a box, cosines and sines. We know what omega is. Omega had to be square root of k over m. Then the equation was solved.

All I've got left is to get c1 and c2. All I have left is to match-- choose c1 and c2 to match the two initial conditions. So let me just do that.

What are c2 and c2? At time zero, I have to match an initial displacement. So at time zero, this is a 1, cosine of zero is a one, and that's a zero. So at t equals 0, I have y of 0. The displacement matches c1 times cosine of omega t, which is a 1, plus c2 times 0. I'll put it in there plus c2 times 0.

C1 cos 0 plus c2 sine 0.

I've learned c1. And also-- what do I do next? I want to get c2. And where is c2 coming from? Now I would like to know what's the coefficient of the-- the initial conditions are supposed to determine that coefficient. It'll be that initial condition that determines it. y prime of 0. The initial velocity should match the derivative.

OK, so what's the derivative? y prime. So the derivative of the cosine will be a sine. And that will disappear at t equals 0. The derivative of this sine will be a cosine with a factor omega. So I'll have y prime of 0 will be the c2 omega cos of 0. Which is what? That tells me c2.

You could do all this without my pointing the way. I'm solving this equation. I have the solution in general form with two constants. Now I'm determining those constants, and cosine and sine just determine them perfectly because cosine is 1 and sine is 0 at the start. So we've got the answer.

The solution is y of t is c1 is y of 0 cos omega t, and c2 is-- now you'll notice, c2 is y prime at 0 divided by omega. y prime of 0 divided by omega sine omega t. There we go. Finished. Finished. Unforced problem solved.

Everybody in this room could get to that point. Let me make some comments about that. It's a combination of cosine and sine. They're both running at the same frequency, omega.

I'm going to give a special name to that frequency, omega, this famous formula, all-important. Lots of physics in that formula. I call that the natural frequency, because the next step will be to drive the system by a driving frequency, which would be different from omega. So we need to-- we've got 2 omegas. Actually when I first wrote the book I thought, we've got to keep these two separate. Everybody has to keep them separate. My first attempt was to use little omega and big omega for the two. I concluded after looking at it for a while that it was better to be more conventional. People had figured out a good way to do it.

And the good way is to call this the natural frequency and put a subscript, m. So all the omegas that you see on this board should be omega n. I can change them all, but let me just change it here. I'll change omega n. So that's omega n. We've only got that one omega right now, because we don't have a driving term yet.

So natural frequency has the advantage, which kind of made me smile, that the n stands for natural, and everybody calls it the natural frequency. And n also stands for null, and we're talking here about the null solution, because there's no forcing. So I could have subscripts on all the y's. Eventually I'll need subscripts on the y to separate what we've done. This is really yn of t, the null solution.

Good? Now we could take one more step. This is a combination of cosine and sine. And we learned last time that that could be put in a polar form, but I don't plan to do this. Let me just say I could do it. This would be some amplitude, some gain-- maybe g for gain-- no, a for amplitude is good-- times-- what is this second optional form, which I'm just going to write here, say that we could do it, remember a little about it, but not make a big deal-- what is it I'm after here?

I'm looking to write this combination of cosine and sine, which is two oscillations, a cosine curve and a sine curve, but with the same frequency. Then I can combine them into a single cosine, a single cosine of omega nt. But now what else have I got in this form? There's a phase shift, minus phi. Thanks.

So there's an a phi, two constants, or there's y of 0, y prime of 0, two constants. Let me not write again the formula for a or for phi, I don't plan to do anything with it. It just could be done.

In other words, what we've done so far is just to see that the single spring oscillates with the frequency omega n. That's really what we've done. A single spring oscillates with a frequency omega n.

Saying that makes me think, let me look ahead to the linear algebra part of the course. So where is linear algebra going to come in? It's going to come in for a system of springs. When I have another spring. Can I draw another spring and another mass and another spring? Say six springs, six masses. Then-- and they could be different k's, different m's, or not.

Then we've got six displacements-- six differential equations-- coupled together, because the whole system is coupled together. So what happens at that point? That's the point where linear algebra, where matrices are coming in. You want to see what's the point of matrices. It's not a separate course by any means. It's a most necessary part, because a single spring happens in reality but also systems today are coupled. Big, actually, there are many, many things. You have an electric circuit with thousands or tens of thousands of elements. You have a coupled system with many gears, many oscillations going on. So we need matrices at that point.

Can I even just add one more word about the language? When we had-- here we have a frequency of motion for one spring. What are we going to have for two springs or six springs? The motion will be a combination of six different frequencies. And so you'll see that it's a much more interesting, much more not so simple motion. A combination of six pure frequencies. And those frequencies are determined from the six eigenvalues of the matrix.

I'm just using that word looking ahead. We will have a 6x6 matrix to describe the coupled system. That matrix will have six eigenvalues. It will tell us six natural frequencies, and our solution will be a combination of all six oscillations. Here, it's 1. Here it's 1. That spring is not there.

So the problem we've solved now is the fundamental, basic problem, and I have to-- next step is forcing. I now want to add a force that drives the motion. In general, it could be any function of time. Calling it f of t. So that's what I'm going to put in now.

But in reality, very, very, very often f of t is also a simple harmonic motion. It's also a cosine. But at a different frequency, at a driving frequency. So I'm going to-- the next equation to solve is to put in cosine-- let's stay real for now-- at another, driving frequency. At a driving frequency. And of course, it could have an amplitude. But let me take that amplitude as 1 to keep things simple.

So now I'm talking about forced motion. Can we solve it? How can we solve this equation? Let me take out the 0 or-- take out the 0-- equals cosine omega t. With a different omega.

If the two omegas were the same, if the driving frequency is the same as the natural frequency, the formulas have to be slightly adjusted. There's still an answer, but it's a case of resonance and you have to look separately. But let's say, no. Let's say omega d is different from omega n. How are you going to solve this?

I have to think myself. How do I solve that. Let's start a fresh board.

my double prime plus ky equals cosine of omega dt-- or often, I won't put the d. I don't have to put the d anymore. Omega will now represent the driving frequency, because I've got omega n, the natural frequency, as the square root of k over m.

What am I looking for now? I found the null solution. I'm looking for a particular solution. I'm trying to keep the whole thing systematic. Null solutions are now dealt with. Took a little more time than just ce to the at for first order equations, because we've now got a two-dimensional collection of null solutions, but we've got them. Now I'm taking a forcing term.

So I'm looking for a particular solution. I'm looking for any solution to this equation.

I'm looking for a particular guy. What do you suggest? Again, it's a neat problem because of that particular forcing term, a cosine, an oscillation.

So I'm going to look for yp is some gain times [INAUDIBLE]. This is the next and, fortunately, a highly, highly important case, in which the particular solution has the same form as the forcing term. It's just a multiple of the forcing term. That's best possible. That's best possible, is to have the forcing term reveal to me-- the forcing term immediately reveals a particular solution.

Once I know what I'm looking for, what do I do? Substitute it in. So I substitute that particular solution in here. And notice everything is going to be a cosine, mgy double prime. So what do I get when I plug this in for that guy? I want to-- you can do it quickly, but let's stay together and do it together, because we can with this case.

What happens when I plug that in and take its second derivative? I get the g. And then what's the second derivative?

AUDIENCE: [INAUDIBLE].

PROFESSOR: We have a negative, because two derivatives of the cosine bring out a minus omega d, will come out twice. And I'll keep writing omega d for a moment, but then I'll give up on the d. Cosine of omega dt. And then k times this, g cosine of omega dt, equals the forcing term, cosine of omega dt.

It worked. This is one of that small family of nice functions where the solution has the same form as the function. Actually that list of what you could call best possible forcing functions, where the form of the forcing function tells you the form of the solution. That's a small family. But it's fortunately a very important one. Cosines, sines are included, and we'll see all the other guys that are included.

Most forcing functions we couldn't just assume that the solution had the same form. It's only these nice ones. But cosines are nice.

So what do I do now? Everything is multiplying cosines, so I just look at-- I have minus m omega squared g-- g is going to factor out-- minus m omega squared and a k times g. Let me remove that off for the moment. I'm canceling cosine omega, so my right hand side is 1. That's it.

We looked for a solution with that simple format, and we found it. Now we know g, the gain. So the solution is-- this is g is 1 over k minus m omega squared times cosine of omega t. And omega is omega d. Omega is omega d now.

Does that look good to you? This is the periodic solution going at the driving. This is what the-- this g is the gain, the driving force. The driving force is 1 times cosine omega d, then that 1 gets multiplied by this number. This is, you could say, the amplifying factor. I guess frequency response would be the right word. Can I bring in that word, response, again?

Response is a word for a solution. It's what comes out. When the input is this, a pure frequency, the output, the response, is a pure frequency-- same frequency, of course-- multiplied by that. That is the frequency response factor.

Notice we could write that a cool way, by remembering that omega squared-- that's wrong as it stands. What have I forgotten in writing k minus m omega squared in that denominator? I forgot a subscript, which is n. Which is n. This is n. This is-- is that right? No. Is it? Or is it d? Maybe I didn't make a mistake. Is it d? You're seeing a kind of critical moment. Which is it?

AUDIENCE: [INAUDIBLE].

PROFESSOR: It's d, isn't it? Yeah. It's d. Sorry. It's d. But when I see this and remember what omega n squared is-- omega n squared is k over m-- I can see that I can get an omega-- I can use this in here to make it even more interesting. So it'll be equals-- let me get this box ready-- cosine of omega dt divided by-- now I just want to rewrite that.

I want to take out an m. I'm going to write this as m times k over m. m times k over m. Safe to do that. Now I have a factor, m, that I can bring out. And what is m multiplying? That's the neat thing. What is m multiplying? k over m is-- omega n squared. And this is minus m omega d squared. Minus omega d squared. That's pretty terrific.

The gain is this multiplier, 1 over m, times that. And we see that the gain is bigger and bigger when the frequency is near the natural frequency. And of course everybody has seen the pictures of that bridge-- wherever the heck was that bridge? Somewhere in the Northwest, I think. You know the bridge I'm talking about?

AUDIENCE: [INAUDIBLE] Tacoma, Washington.

PROFESSOR: Yeah, I think Tacoma, that's right. The Tacoma Narrows Bridge. Right. Tacoma, Washington. Where the natural-- when you build a bridge, you've built in a natural frequency. And then when traffic comes, it's doing a driving frequency. And if you haven't got those two well-separated, you're in trouble, as this shows. Or similarly, when an architect designs a skyscraper, there's going to be a frequency of oscillation, a natural frequency, at which that skyscraper swings. And then there's wind.

Actually I talked yesterday to the-- by chance, the math department is not a very party-going department, but once a year we let it out. And so we had our party at Endicott house out in the suburbs, and all the usual people-- that's all the professors I know-- came, of course. But also, there was a really cool person. He's the key architect for Building 2.

You've noticed that Building 2 is under wraps and we're moved out. And we move back in January 2016. So we've been out a year and a quarter and we have another year and a quarter to go. It's going to be cool. And you may say, well, who cares. But the key point is Building 1 is next, and Building 1 is going to have the same cool addition of a fourth floor. We're putting in a fourth floor, which all the-- Buildings 3, 4, 5, 6 go up to four, but Buildings 2 and 1 stopped at the third floor. But there's a lot of space up there under the roof. And they've discovered they could put a fourth floor up there.

Here was one interesting thing, though. These buildings that we're sitting in are sinking. You know that MIT was built on marshy land, just the way the Back Bay-- which is like the greatest idea in the history of Boston, the Back Bay and the dam that makes the Charles River beautiful-- was built by bringing in trainloads of earth from Needham. So whole mountains and hills in Needham have come into Boston and come here.

So anyway, we're sinking. You may say something like 3/16 of an inch a year is not something to worry about, but now it's been more than 100 years that these buildings have been here. Anyway, not good to sink faster. So the weight had to be controlled. So by putting in a fourth floor, that put in a lot a new weight, and faster sinking, probably by some formula here. Probably there.

So the weight had to get subtracted out. It turns out that the ceiling, the roof to Building 1-- Building 2 and no doubt to Building 1-- was more than a foot thick of concrete. Really heavy. And some more asbestos probably, which we don't want to think about. That's much reduced. A whole lot of weight came out of the roof. I think they probably did the calculation right, so we won't get rain coming through, but it won't weigh as much and the fourth floor is acceptable. All this was a big decision by MIT to pay for that, or to raise money and pay for the new fourth floor. But it's going to be fantastic. And it'll be fantastic in Building 1 also.

So all that is discussion of that formula. That's the frequency response, this factor to frequency, omega d, or omega, is this factor. I guess I should say something about resonance. What happens when that formula breaks down? When the driving force equals the natural frequency, then we're dividing by 0, and something is different. The formula isn't right anymore.

What enters in the formula-- let me just tell you what enters, and then we'll see it in a simple example. When I have this repeated thing, two things are equal, what tends to happen is a factor, an extra factor, t, appears. So an extra factor, t, will appear in the case omega n equal omega d. The solution, y, will be some factor, I'll still call it g-- no, I don't want to call it g, let me call it a. There'll be a factor, t, times cosine of omega t.

So in this case, there's really only one frequency. We're driving it. So the oscillation grows. As you know, when you push a child on a swing, the whole point of pushing that child is to push at the natural frequency. You wait for this swing to swing back naturally and you drive it again with that-- at that-- maintain that frequency. And of course you see the amplitude-- the child swing higher and higher.

Presumably you stop pushing before disaster for that child. But that's a case of resonance. And it's what happened in the Tacoma Narrows Bridge, and there was nothing to-- nobody stopped, traffic just kept coming. The movie is amazing, because there's one car that shows up after it's already swinging wildly, some crazy person still driving across.

And you might think, OK, that's ancient history. But you know the bridge in London, the pedestrian bridge, the Millennium Bridge-- it's just a walking bridge across the Thames-- a big feature of modern London, and it had the same problem. It was swaying. People could not walk across. They couldn't keep their balance. So they had change it. So it's not trivial to anticipate.

So now we've solved it-- we've solved the the null equation with no force, and we've solved the driving force equal to a cosine. And of course, we could do a sine. What other driving force should we do?

I think we should do a delta function. I think we have to understand the fundamental solution is the case when, if we can solve it-- there's always this general rule, if we can solve with a delta function, that will give us a formula for every driving force, because every function is some combination of delta functions. So if we could do it with a delta-- really the great right hand sides are-- well, cosines and sines I'll include as great right hand sides. Those are the exponentials in disguise.

So the great right hand sides are really exponentials at different frequencies and delta functions. Delta of impulses. So now I want to find the impulse response. That's the next-- that's really a job.

At this point, in these last 20 minutes when I solve my double prime plus ky equal a delta function-- well, what I was going to say was I'm now taking you to something that you won't see on the exam this afternoon. But maybe you will. Delta function, right hand side. I haven't seen it yet. Or I haven't looked recently. You won't see second derivatives, I guess.

So what is it? So now this is of the form with an f of t, a very special f of t. And that very special f of t makes that extremely easy to solve. That's really my point here, is it it's going to be a cinch to solve that, and we practically have done it already to solve that with a delta function.

And the reason is sort of physical. We have here our spring. And what am I doing with that force? I'm hitting the mass. I'm striking the mass. Let me say, and I'll write it on the board, the point I want to make about this.

That point is that this equation with a delta function force starting from 0-- say, y of 0 equal y prime of 0 equals 0, let's give it starting from rest-- it starts from rest by hitting it. And that hit, that impulse, is in no time at all. It's not stretched out. It's hit over one second. So this has the same solution. This is the beauty. This is why we can solve it so easily. Same solution as-- let me write it and see what you think-- as my double prime plus ky equal 0. We know how to solve those. With-- it's still, when I hit it-- when I hit it, what happens in that split second?

In that split second, it doesn't have time to move. It doesn't move. It still has y of 0 equals 0. But in that split second, we've given it a velocity. We've given it a velocity. And that velocity will be y prime. The initial velocity is 1-- because here I had a 1-- over an m. We have to have the units right.

So here's a point, and we will stay with it. We'll come back to this point next time. Maybe the first thing for you to take in is the fact that it's such a nice thing. We have this equation with this mysterious delta function, and I'm saying that the solution is the same as this equation with no force, but starting from a mass.

I'm tempted to take an example to make this point. Let me take an example where the whole thing is a lot simpler. y double prime equal delta of t. I've taken the spring away, so the k is gone, the mass is 1.

What's the solution to y double prime equal delta of t? If we concentrate on this example, we're good for today. So my point is the same solution as-- now, what's the other problem? I'm just repeating here, but making it simple by taking k equals 0 and m equal 1. So the same solution as y double prime equals 0, with y of 0 equal what, and y prime of 0 equal what. I just wanted to repeat here what I've said there, and then we'll solve it and we'll see that it's all true.

If I look for a solution to y double prime equal delta starting from 0-- this was starting from 0-- if I say that's the same as this, what should y of 0 be here?

AUDIENCE: Zero.

PROFESSOR: Zero, right. It hasn't had time to move. It hasn't had time to move. But in that instant, what happened to y prime? It jumped to 1. That's right. That's right. Exactly.

Now just solve that equation for me. Solve this example for me. Suppose y double prime-- yeah. Here we go. What's the solution if y double prime is 0? What are the solutions to y double prime equals 0?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Constant and linear. a plus bt, right, have second derivative 0. Now what's the solution that starts from 0 that kills the a and has slope 1? What's the answer to that question?

AUDIENCE: [INAUDIBLE].

PROFESSOR: t. t. The solution to this equation is a ramp. It's zero everything in this course, is zero up until time 0. At time 0, in this example, all the action happens. Everything happens. And what happens is it gets a velocity of 1, and the solution is y equal to t. y is 0 here, of course. At that point, that's the key point, t equals 0, right there-- it gets a slope.

We don't have a step function. There's no jump in y. The jump is in y prime, the y prime the velocity jumped from 0 to 1. That's exactly-- I think when I introduced delta functions and drew a picture. What is the derivative, the first derivative, y prime, for that guy? Let's just review, because this is what we've seen already.

The first derivative is--

AUDIENCE: [INAUDIBLE] step.

PROFESSOR: A step. And the second derivative is delta. The second derivative of this is the first derivative of a step. The derivative of a step is 0 everywhere except at the step, at the jump when it jumps to 1. So that's the solution in this example.

And now to end the lecture, let's solve it in this example. Again, let me just say-- why do I like this forcing term? Mathematically, I like it because, if I can solve that guy-- as we're doing, we are solving it-- if I can solve that one, I can solve all forces. Over here, I could solve when I had a very happy f of t, a perfect f of t, where I could guess the answer and push through.

Now with a delta, I can build everything out of delta functions. That's why I like it mathematically. Why do I like it physically? Because it's a very physical thing to have an impulse. That happens in real time, in real things. And by the way, let's just, before I write down any more formula, what would-- I would like to be able to solve it for a step function. [? Heavy thud. ?] I would like to be able to do that one.

I'm going to have to erase something, or I'll write it right above just for the moment. I would also like to solve my double prime plus ky equal a step function. So I would call the solution, y, the step response. And what would be a step function start? A step function start would be like turning a switch. Suddenly things happen.

That's forcing by a step, so I'm looking for the step response. And how do you think these two are related? I look at the relation at the right hand sides. What's the relation of this step to the delta? Yeah?

AUDIENCE: One's a derivative.

PROFESSOR: One's a derivative of the other. And we've got linear equations. So the right hand sides. The step response, y step, and the delta response, y delta-- I'll use a different letter for this because it's so important. One is the derivative of the other. The great thing about linear equations is we have linear equations, differentiation, integration. Those are linear operations. The step function is just like a steady-- anyway. I was going to-- I won't-- is the integral of the delta. Step function is the integral of the delta, so the step response is the integral of the delta response.

I guess to finish the lecture, why don't we solve this problem, which looks tricky because it's got a delta. Instead, we'll solve this problem, which doesn't look tricky at all. It's exactly what we started the lecture with. Zero forcing and some initial conditions. So let me just finally make space for the big deal from today's lecture, which would be the fundamental solution with a force by a delta.

I'm just going to write down the answer when you tell me what it is.

What's the answer to that? What's the solution to this second order constant coefficient unforced equation with those initial conditions? We probably had it here. I may just have erased it. But now let's get it.

So y is y delta. This is the impulse response. y of t-- and I'll give it later another name. So here's a perfect review question. What's the solution to this problem? Everybody remembers-- what are the solutions, what's the general form for the solution to the equation? I'm reviewing today's lecture. The solution to that equation looks like what?

AUDIENCE: [INAUDIBLE].

PROFESSOR: It's a cosine and a sine, right. And then how much of a cosine do we have and how much of a sine do we have? The initial condition will tell me how much of a cosine we have. And what's the answer? None? No cosine. This condition, this initial velocity, will tell me how much of a sine we have, because the sines are the things that have initial velocities.

So it would be a sine of-- the sine of what? Square root of k over m, right? omega nt, right? And what's the number? What's the number so this has the right-- let me write again what I want. I want y prime at 0 to be 1 over m.

What's the number that I put in there? I've got something, its derivative, at zero. This is some number-- I'll call it little a for the moment, but I want to find out what it is.

Are we right? Yeah? I think we're right. Yeah.

The derivative is at zero, so I just plug that into here, take the derivative at zero-- of course that makes it a cosine, which will be 1-- but it also brings out that factor. So a times-- well, that factor will be 1 over m, and that tells me what a has to be.

Well, this is omega. So a is-- this is omega a equal m. This is 1 over m omega. And that's omega.

Sorry I'm erasing stuff which I-- this is the formula I'm after. Sine omega t over omega. I think we're good. Are we? Yeah? Yeah. I'll come back to this in-- Wednesday is my day to move to damping terms. I've intentionally stayed with undamped equations here, because you're thinking about that level of equation. Damping is going to bring in new stuff, and that should wait till Wednesday.

Shall I recap today? I'll just recap today, and then we're done.

Today started with the unforced equation. We solved it by assuming-- by not thinking ahead, just assume I have an exponential, because the beauty of exponentials is, when I plug it in, the exponential cancels. And that told me that s was pure imaginary. It told me that it had this form, e to the i omega t. And there were two s's. Two possible s's, plus and minus.

I get to make a little comment about this example here. What was omega? What's the natural frequency in this problem? What's the natural frequency here? I guess this is a case where-- what's the natural frequency? I guess this is a case where m is 1 and k is 0, is that right? This does fit into that pattern, but it's a little special. This is a case where m is 1 and k is 0. So what's the natural frequency in this?

AUDIENCE: Zero.

PROFESSOR: Zero. Zero. This is a crazy case of resonance. It's a case in which the natural frequency and the driving frequency, say in this-- I'll have to do it here-- this simplest of all equations is, in a way, special. It's a case when the natural frequency is zero and the driving frequency is zero and they're equal. And what happens with resonance? What's the new formula, the new term that comes in with resonance? It's t.

You saw it happen for this example, and we didn't have to use the word resonance. We knew that we had a ramp. We just used the word ramp, not resonance. But this is a case of resonance. When omega n is zero and omega d is zero. And the factor t up here. Anyway. Just that small comment there.

And now, just going back to the recap. The recap was, we tried exponentials. We learned that they were pure oscillations. We realized that we could do cosines and sines instead, and we did. And we took off. We got the formula. Then of course the-- so this is section 2.1 of the book. And it goes through all those steps carefully.

Section 2.2 of the book tells us about complex numbers, and section 2.3 brings damping in. So that's what's coming next time. So the recap again. We found the null solution, we found a particular solution-- oh there's just one comment I want to make, and then I'm done. Where was our particular solution?

Yeah. This was our particular solution. Here's my comment. Here's my comment. Suppose I want to solve this basic equation starting from a given y of 0 and a y prime of 0. I'm going to do it in two parts, I think. I've got the null solution, and I've got this particular solution.

Now here's my point. If I want to get y of 0-- how shall I say this. You can't just put together-- it's an easy mistake to make-- solve the null equation with the initial conditions and then add in the particular solution. You'd think, I just followed all the rules.

But this particular solution that you added in has a-- at t equals 0, it's not zero. So you have to change. So the correct thing, the correct yp plus yn-- let me make that point. Just a warning.

So in words the warning is, remember that the particular solution has some initial condition-- in that case, g-- and then that is going to affect the right null solution. So again, y is y null plus y particular-- plus y of particular-- so it's some c1 cos omega nt plus some c2 sine omega nt plus this particular guy, g, cosine of omega dt.

All correct. All correct. But now, put in the initial conditions. y of 0 is given. And what do I get on the right hand side when I put in t equals 0? I get c1 here.

What do I get when I put t equals 0 in there? Nothing. What do I get when I put t equals 0 in here? g.

So it's not c1 equal y of 0 anymore. That's the easy mistake that I'm correcting. When you put in this particular solution, it has an initial value. That initial value is going to come in here. So c1, then, the correct c1 is y of 0 minus g. End of story. Just don't be too quick to just add the two pieces and think you can do them completely separately, because you're putting them together. And then you have to put them together in the initial condition.