Chapter 4
Planar Kinematics

Kinematics is *Geometry of Motion*. It is one of the most fundamental disciplines in robotics, providing tools for describing the structure and behavior of robot mechanisms. In this chapter, we will discuss how the motion of a robot mechanism is described, how it responds to actuator movements, and how the individual actuators should be coordinated to obtain desired motion at the robot end-effecter. These are questions central to the design and control of robot mechanisms.

To begin with, we will restrict ourselves to a class of robot mechanisms that work within a plane, i.e. *Planar Kinematics*. Planar kinematics is much more tractable mathematically, compared to general three-dimensional kinematics. Nonetheless, most of the robot mechanisms of practical importance can be treated as planar mechanisms, or can be reduced to planar problems. General three-dimensional kinematics, on the other hand, needs special mathematical tools, which will be discussed in later chapters.

4.1 Planar Kinematics of Serial Link Mechanisms

Example 4.1 Consider the three degree-of-freedom planar robot arm shown in Figure 4.1.1. The arm consists of one fixed link and three movable links that move within the plane. All the links are connected by revolute joints whose joint axes are all perpendicular to the plane of the links. There is no closed-loop kinematic chain; hence, it is a serial link mechanism.

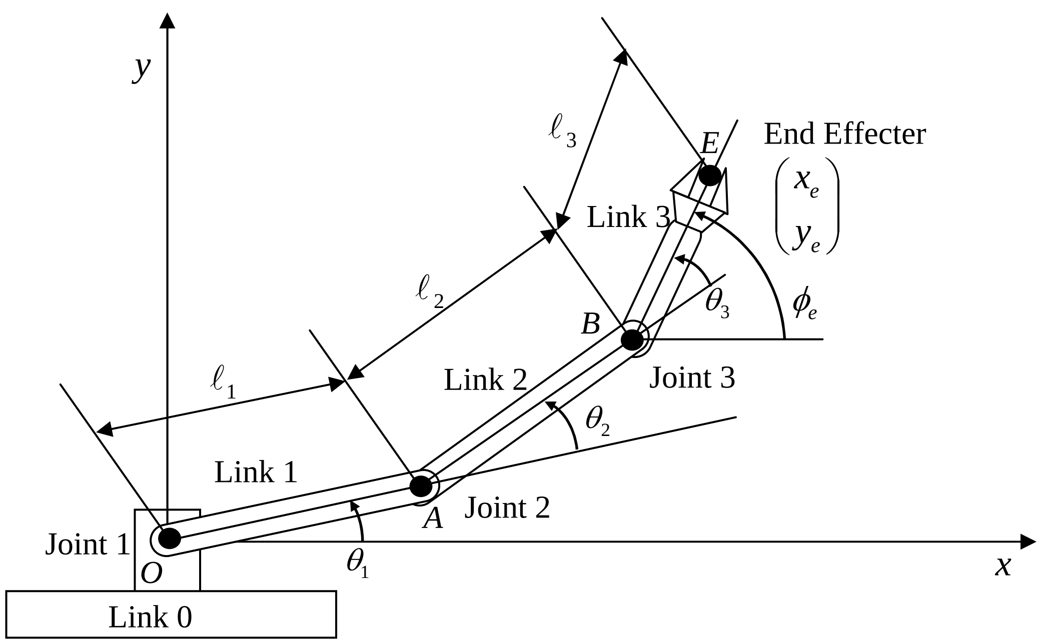

Figure 4.1.1 Three dof planar robot with three revolute joints

To describe this robot arm, a few geometric parameters are needed. First, the length of each link is defined to be the distance between adjacent joint axes. Let points *O, A,* and *B* be the locations of the three joint axes, respectively, and point E be a point fixed to the end-effecter. Then the link lengths are $\ell_1 = \overline{OA}$, $\ell_2 = \overline{AB}$, $\ell_3 = \overline{BE}$. Let us assume that Actuator 1 driving link 1 is fixed to the base link (link 0), generating angle θ_1, while Actuator 2 driving link 2 is fixed to the tip of Link 1, creating angle θ_2 between the two links, and Actuator 3 driving Link 3 is fixed to the tip of Link 2, creating angle θ_3, as shown in the figure. Since this robot arm performs tasks by moving its end-effecter at point E, we are concerned with the location of the end-effecter. To describe its location, we use a coordinate system, *O-xy*, fixed to the base link with the origin at the first joint, and describe the end-effecter position with coordinates x_e and y_e. We can relate the end-effecter coordinates to the joint angles determined by the three actuators by using the link lengths and joint angles defined above:

$$x_e = \ell_1 \cos\theta_1 + \ell_2 \cos(\theta_1 + \theta_2) + \ell_3 \cos(\theta_1 + \theta_2 + \theta_3) \qquad (4.1.1)$$

$$y_e = \ell_1 \sin\theta_1 + \ell_2 \sin(\theta_1 + \theta_2) + \ell_3 \sin(\theta_1 + \theta_2 + \theta_3) \qquad (4.1.2)$$

This three dof robot arm can locate its end-effecter at a desired orientation as well as at a desired position. The orientation of the end-effecter can be described as the angle the centerline of the end-effecter measured from the positive *x* coordinate axis. This end-effecter orientation ϕ_e is related to the actuator displacements as

$$\phi_e = \theta_1 + \theta_2 + \theta_3 \qquad (4.1.3)$$

□

The above three equations describe the position and orientation of the robot end-effecter viewed from the fixed coordinate system in relation to the actuator displacements. In general, a set of algebraic equations relating the position and orientation of a robot end-effecter, or any significant part of the robot, to actuator or active joint displacements, is called ***Kinematic Equations,*** or more specifically, ***Forward Kinematic Equations*** in the robotics literature.

Exercise 4.1

Shown below in Figure 4.1.2 is a planar robot arm with two revolute joints and one prismatic joint. Using the geometric parameters and joint displacements, obtain the kinematic equations relating the end-effecter position and orientation to the joint displacements.

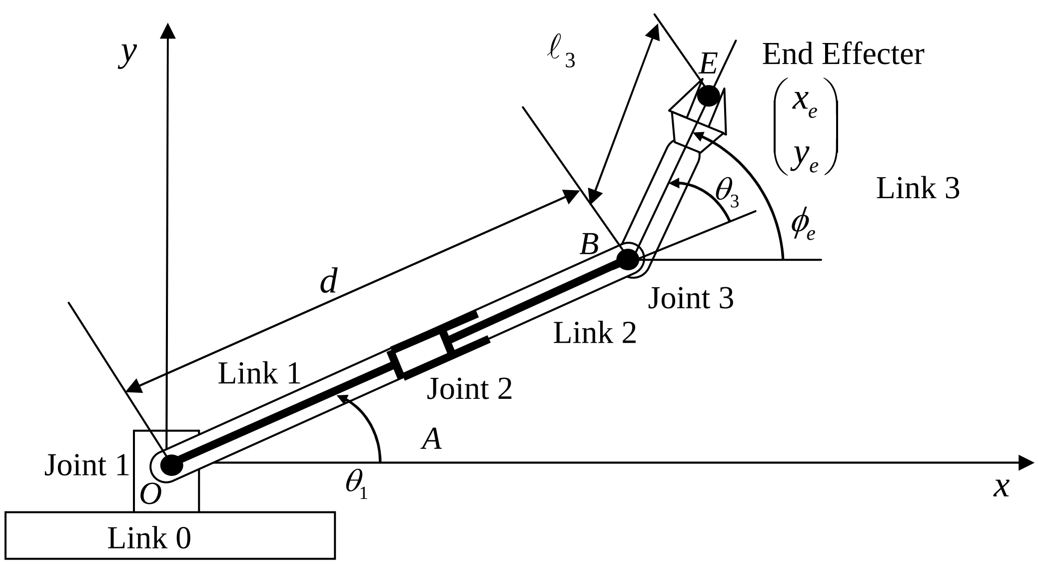

Figure 4.1.2 Three dof robot with two revolute joints and one prismatic joint

Now that the above Example and Exercise problems have illustrated kinematic equations, let us obtain a formal expression for kinematic equations. As mentioned in the previous chapter, two types of joints, prismatic and revolute joints, constitute robot mechanisms in most cases. The displacement of the *i-th* joint is described by distance d_i if it is a prismatic joint, and by angle θ_i for a revolute joint. For formal expression, let us use a generic notation: q_i. Namely, joint displacement q_i represents either distance d_i or angle θ_i depending on the type of joint.

$$q_i = \begin{cases} d_i & \text{Prismatic joint} \\ \theta_i & \text{Revolute joint} \end{cases} \tag{4.1.4}$$

We collectively represent all the joint displacements involved in a robot mechanism with a column vector: $q = [q_1 \quad q_2 \quad \cdots \quad q_n]^T$, where n is the number of joints. Kinematic equations relate these joint displacements to the position and orientation of the end-effecter. Let us collectively denote the end-effecter position and orientation by vector ***p.*** For planar mechanisms, the end-effecter location is described by three variables:

$$p = \begin{bmatrix} x_e \\ y_e \\ \phi_e \end{bmatrix} \tag{4.1.5}$$

Using these notations, we represent kinematic equations as a vector function relating ***p*** to ***q***:

$$p = f(q), \quad p \in \Re^{3x1}, q \in \Re^{nx1} \tag{4.1.6}$$

For a serial link mechanism, all the joints are usually active joints driven by individual actuators. Except for some special cases, these actuators uniquely determine the end-effecter position and orientation as well as the configuration of the entire robot mechanism. If there is a link whose location is not fully determined by the actuator displacements, such a robot mechanism is said to be ***under-actuated***. Unless a robot mechanism is under-actuated, the collection of the joint displacements, i.e. the vector ***q,*** uniquely determines the entire robot configuration. For a serial link mechanism, these joints are independent, having no geometric constraint other than their stroke limits. Therefore, these joint displacements are ***generalized coordinates*** that locate the robot mechanism uniquely and completely. Formally, the number of generalized coordinates is called ***degrees of freedom.*** Vector ***q*** is called joint coordinates, when they form a complete and independent set of generalized coordinates.

4.2 Inverse Kinematics of Planar Mechanisms

The vector kinematic equation derived in the previous section provides the functional relationship between the joint displacements and the resultant end-effecter position and orientation. By substituting values of joint displacements into the right-hand side of the kinematic equation, one can immediately find the corresponding end-effecter position and orientation. The problem of finding the end-effecter position and orientation for a given set of joint displacements is referred to as the *direct kinematics problem.* This is simply to evaluate the right-hand side of the kinematic equation for known joint displacements. In this section, we discuss the problem of moving the end-effecter of a manipulator arm to a specified position and orientation. We need to find the joint displacements that lead the end-effecter to the specified position and orientation. This is the inverse of the previous problem, and is thus referred to as the *inverse kinematics problem.* The kinematic equation must be solved for joint displacements, given the end-effecter position and orientation. Once the kinematic equation is solved, the desired end-effecter motion can be achieved by moving each joint to the determined value.

In the direct kinematics problem, the end-effecter location is determined uniquely for any given set of joint displacements. On the other hand, the inverse kinematics is more complex in the sense that multiple solutions may exist for the same end-effecter location. Also, solutions may not always exist for a particular range of end-effecter locations and arm structures. Furthermore, since the kinematic equation is comprised of nonlinear simultaneous equations with many trigonometric functions, it is not always possible to derive a closed-form solution, which is the explicit inverse function of the kinematic equation. When the kinematic equation cannot be solved analytically, numerical methods are used in order to derive the desired joint displacements.

Example 4.2 Consider the three dof planar arm shown in Figure 4.1.1 again. To solve its inverse kinematics problem, the kinematic structure is redrawn in Figure 4.2.1. The problem is to find three joint angles $\theta_1, \theta_2, \theta_3$ that lead the end effecter to a desired position and orientation, x_e, y_e, ϕ_e. We take a two-step approach. First, we find the position of the wrist, point B, from x_e, y_e, ϕ_e. Then we find θ_1, θ_2 from the wrist position. Angle θ_3 can be determined immediately from the wrist position.

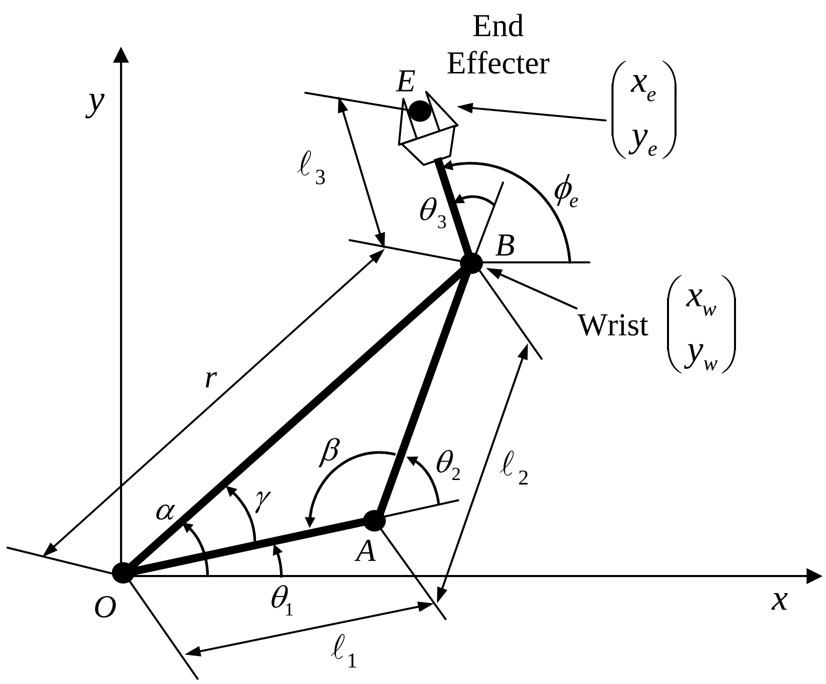

Figure 4.2.1 Skeleton structure of the robot arm of Example 4.1

Let x_w and y_w be the coordinates of the wrist. As shown in Figure 4.2.1, point B is at distance ℓ_3 from the given end-effecter position E. Moving in the opposite direction to the end effecter orientation ϕ_e, the wrist coordinates are given by

$$\begin{aligned} x_w &= x_e - \ell_3 \cos\phi_e \\ y_w &= y_e - \ell_3 \sin\phi_e \end{aligned} \tag{4.2.1}$$

Note that the right hand sides of the above equations are functions of x_e, y_e, ϕ_e alone. From these wrist coordinates, we can determine the angle α shown in the figure.[1]

$$\alpha = \tan^{-1}\frac{y_w}{x_w} \tag{4.2.2}$$

Next, let us consider the triangle *OAB* and define angles β, γ, as shown in the figure. This triangle is formed by the wrist *B*, the elbow *A*, and the shoulder *O*. Applying the law of cosines to the elbow angle β yields

$$\ell_1^2 + \ell_2^2 - 2\ell_1\ell_2\cos\beta = r^2 \tag{4.2.3}$$

where $r^2 = x_w^2 + y_w^2$, the squared distance between O and B. Solving this for angle β yields

$$\theta_2 = \pi - \beta = \pi - \cos^{-1}\frac{\ell_1^2 + \ell_2^2 - x_w^2 - y_w^2}{2\ell_1\ell_2} \tag{4.2.4}$$

Similarly,

$$r^2 + \ell_1^2 - 2r\ell_1\cos\gamma = \ell_2^2 \tag{4.2.5}$$

Solving this for γ yields

$$\theta_1 = \alpha - \gamma = \tan^{-1}\frac{y_w}{x_w} - \cos^{-1}\frac{x_w^2 + y_w^2 + \ell_1^2 - \ell_2^2}{2\ell_1\sqrt{x_w^2 + y_w^2}} \tag{4.2.6}$$

From the above θ_1, θ_2 we can obtain

$$\theta_3 = \phi_e - \theta_1 - \theta_2 \tag{4.2.7}$$

Eqs. (4), (6), and (7) provide a set of joint angles that locates the end-effecter at the desired position and orientation. It is interesting to note that there is another way of reaching the same end-effecter position and orientation, i.e. another solution to the inverse kinematics problem. Figure 4.2.2 shows two configurations of the arm leading to the same end-effecter location: the elbow down configuration and the elbow up configuration. The former corresponds to the solution obtained above. The latter, having the elbow position at point *A'*, is symmetric to the former configuration with respect to line *OB*, as shown in the figure. Therefore, the two solutions are related as

$$\begin{aligned} \theta_1' &= \theta_1 + 2\gamma \\ \theta_2' &= -\theta_2 \\ \theta_3' &= \phi_e - \theta_1' - \theta_2' = \theta_3 + 2\theta_2 - 2\gamma \end{aligned} \tag{4.2.8}$$

Inverse kinematics problems often possess multiple solutions, like the above example, since they are nonlinear. Specifying end-effecter position and orientation does not uniquely determine the whole configuration of the system. This implies that vector ***p*,** the collective position and orientation of the end-effecter, cannot be used as generalized coordinates.

The existence of multiple solutions, however, provides the robot with an extra degree of flexibility. Consider a robot working in a crowded environment. If multiple configurations exist for the same end-effecter location, the robot can take a configuration having no interference with

[1] Unless noted specifically we assume that the arc tangent function takes an angle in a proper quadrant consistent with the signs of the two operands.

the environment. Due to physical limitations, however, the solutions to the inverse kinematics problem do not necessarily provide feasible configurations. We must check whether each solution satisfies the constraint of movable range, i.e. stroke limit of each joint.

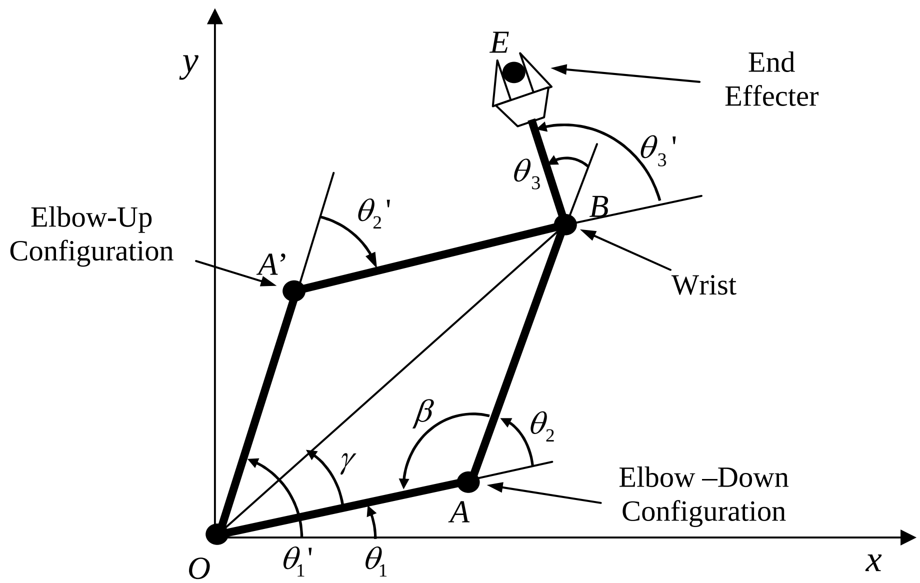

Figure 4.2.2 Multiple solutions to the inverse kinematics problem of Example 4.2

4.3 Kinematics of Parallel Link Mechanisms

Example 4.3 Consider the five-bar-link planar robot arm shown in Figure 4.3.1.

$$x_e = \ell_1 \cos\theta_1 + \ell_2 \cos\theta_2$$
$$y_e = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 \tag{4.3.1}$$

Note that Joint 2 is a passive joint. Hence, angle θ_2 is a dependent variable. Using θ_2, however, we can obtain the coordinates of point A:

$$x_A = \ell_1 \cos\theta_1 + \ell_5 \cos\theta_2$$
$$y_A = \ell_1 \sin\theta_1 + \ell_5 \sin\theta_2 \tag{4.3.2}$$

Point A must be reached via the branch comprising Links 3 and 4. Therefore,

$$x_A = \ell_3 \cos\theta_3 + \ell_4 \cos\theta_4$$
$$y_A = \ell_3 \sin\theta_3 + \ell_4 \sin\theta_4 \tag{4.3.3}$$

Equating these two sets of equations yields two constraint equations:

$$\ell_1 \cos\theta_1 + \ell_5 \cos\theta_2 = \ell_3 \cos\theta_3 + \ell_4 \cos\theta_4$$
$$\ell_1 \sin\theta_1 + \ell_5 \sin\theta_2 = \ell_3 \sin\theta_3 + \ell_4 \sin\theta_4 \quad (4.3.4)$$

Note that there are four variables and two constraint equations. Therefore, two of the variables, such as θ_1, θ_3, are independent. It should also be noted that multiple solutions exist for these constraint equations.

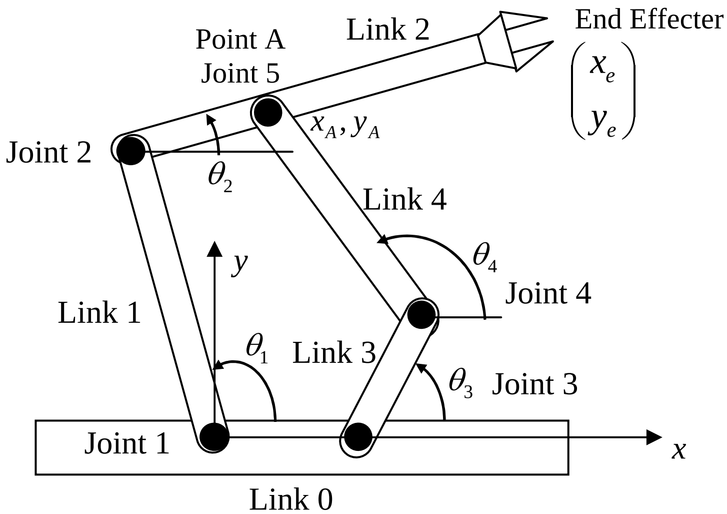

Figure 4.3.1 Five-bar-link mechanism

Although the forward kinematic equations are difficult to write out explicitly, the inverse kinematic equations can be obtained for this parallel link mechanism. The problem is to find θ_1, θ_3 that lead the endpoint to a desired position: x_e, y_e. We will take the following procedure:

Step 1 Given x_e, y_e, find θ_1, θ_2 by solving the two-link inverse kinematics problem.

Step 2 Given θ_1, θ_2, obtain x_A, y_A. This is a forward kinematics problem.

Step 3 Given x_A, y_A, find θ_3, θ_4 by solving another two-link inverse kinematics problem.

Example 4.4 Obtain the joint angles of the dog's legs, given the body position and orientation.

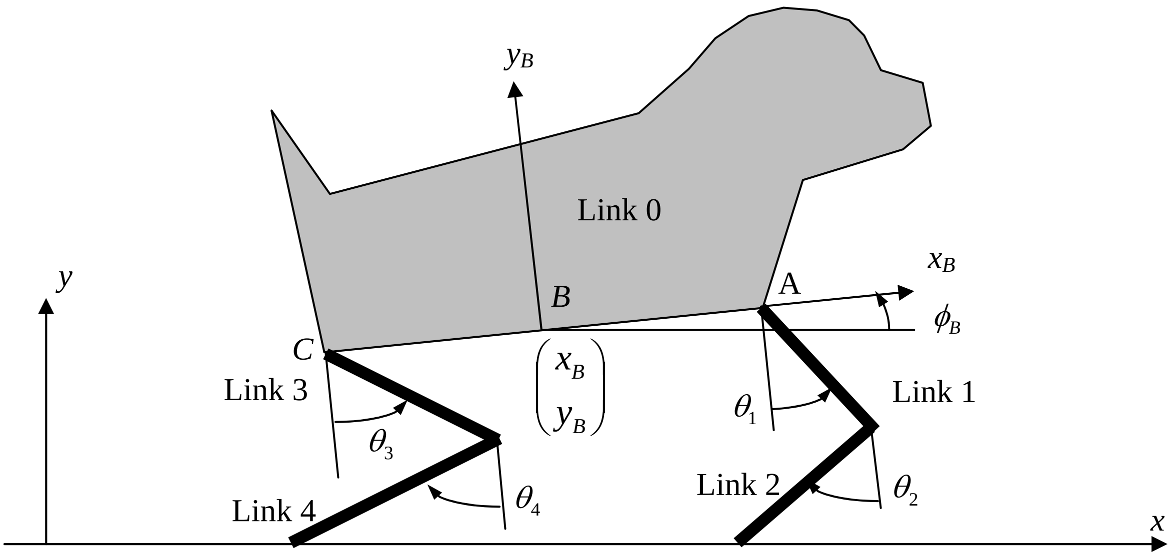

Figure 4.3.2 A doggy robot with two legs on the ground

The inverse kinematics problem:

Step 1 Given x_B, y_B, ϕ_B, find x_A, y_A and x_C, y_C

Step 2 Given x_A, y_A, find θ_1, θ_2

Step 3 Given x_C, y_C, find θ_3, θ_4

4.4 Redundant mechanisms

A manipulator arm must have at least six degrees of freedom in order to locate its end-effecter at an arbitrary point with an arbitrary orientation in space. Manipulator arms with less than 6 degrees of freedom are not able to perform such arbitrary positioning. On the other hand, if a manipulator arm has more than 6 degrees of freedom, there exist an infinite number of solutions to the kinematic equation. Consider for example the human arm, which has seven degrees of freedom, excluding the joints at the fingers. Even if the hand is fixed on a table, one can change the elbow position continuously without changing the hand location. This implies that there exist an infinite set of joint displacements that lead the hand to the same location. Manipulator arms with more than six degrees of freedom are referred to as *redundant manipulators.* We will discuss redundant manipulators in detail in the following chapter.