2.161 Signal Processing: Continuous and Discrete
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

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Problem Set 3 Solution: Analog Filter design

Problem 1: Use the nomenclature in the class handout. For both filters:

$$\frac{1}{1+\epsilon^2} = 0.5 \longrightarrow \epsilon = 1, \qquad \frac{1}{1+\lambda^2} = 0.1 \longrightarrow \lambda = 3$$

(a) For Filter A, Butterworth design:

$$N \geq \frac{\log(\lambda/\epsilon)}{\log(\Omega_r/\Omega_c)} = \frac{\log(3)}{\log(1.5)} = 2.71$$

Therefore select $N = 3$.

(b) For Filter A, Chebyshev design:

$$N \geq \frac{\cosh^{-1}(\lambda/\epsilon)}{\cosh^{-1}(\Omega_r/\Omega_c)} = \frac{\cosh^{-1}(3)}{\cosh^{-1}(1.5)} = 1.831$$

Therefore select $N = 2$.

(c) For Filter B, Butterworth design:

$$N \geq \frac{\log(\lambda/\epsilon)}{\log(\Omega_r/\Omega_c)} = \frac{\log(3)}{\log(1.175)} = 6.81$$

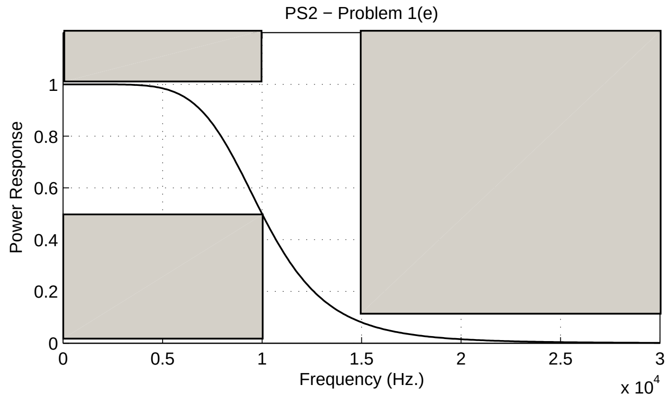
Therefore select $N = 7$.

(d) For Filter B, Chebyshev design:

$$N \geq \frac{\cosh^{-1}(\lambda/\epsilon)}{\cosh^{-1}(\Omega_r/\Omega_c)} = \frac{\cosh^{-1}(3)}{\cosh^{-1}(1.175)} = 3.02$$

Therefore select $N = 4$.

(e)
```
% Design the filter
[A,B]=butter(3,2*pi*10000,'s');
filt=tf(A,B);
% Create a frequency vector
w=[0:2*pi*100:2*pi*30000];
% Compute the freq resp. at the frequencies in the vector
[MAG, PHASE] = bode(filt, w);
% Plot the response
plot(w/(2*pi), squeeze(MAG).^2);
grid;
xlabel('Frequency (Hz.)');
ylabel('Power Response');
title('PS2 - Problem 1(e)');
```



(f) For the standard MATLAB functions we need to design a filter with $\Omega_c = 1$ rad/s. In the following script we have done an implicit conversion by specifying $\Omega_r = 1.175$ rad/s. The following script designs the bandpass filter, plots the power response on a linear scale, and makes the Bode plots as requested:

```
% Problem Set 2, Prob 1(f)
% lp2bp() reqires a prototype lp filter with unity wc.
% Note we specify the rejection band as being 10db down
[b,a]=cheby2(4,10,1.175,'s');
% Convert to a band-pass filter with center frequency
% as the geometric  mean of the band edges
[pb,pa]=lp2bp(b,a,2*pi*sqrt(5000*15000),2*pi*10000);
bpsys=tf(pb,pa);
% Plot the power response
w=[0:2*pi*100:2*pi*30000];
[mag,phase]=bode(bpsys,w);
plot(w/(2*pi),squeeze(mag).^2);
title('PS2 Prob 1(f): Bandpass Filter Design');
xlabel('Frequency (Hz)');
ylabel('Power Response');
%
figure
bode(bpsys)
```

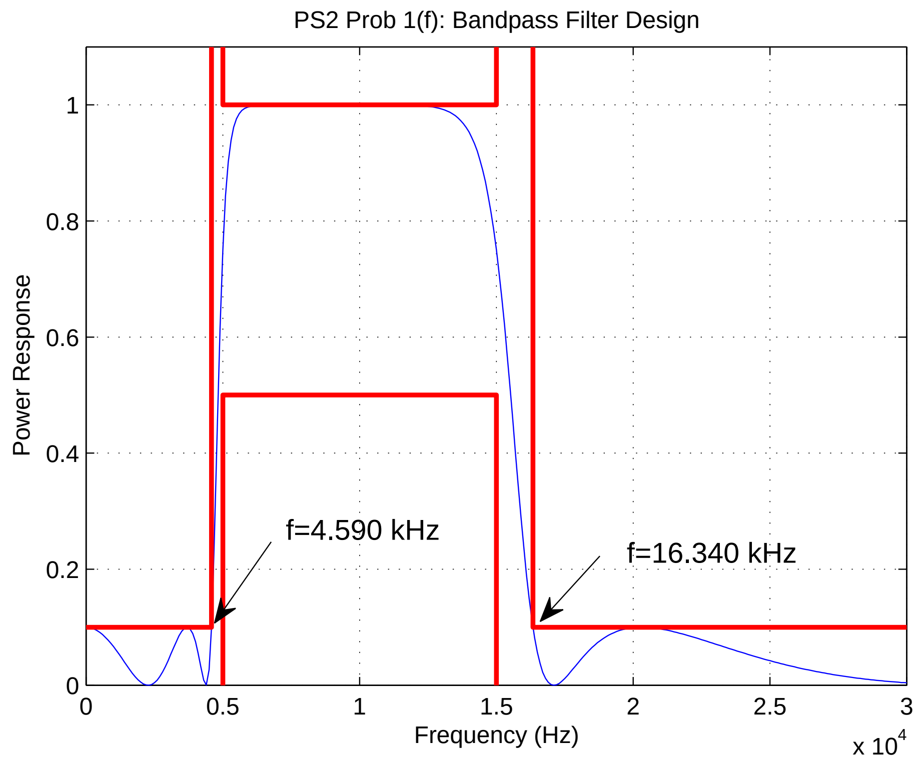
Note that since we were given the specs for the prototype lpf, we have no control over the stop-band edges. We can however compute them using Table 2 in the class handout:

$$g(s) = \frac{s^2 + \Omega_o^2}{\Delta\Omega s}$$

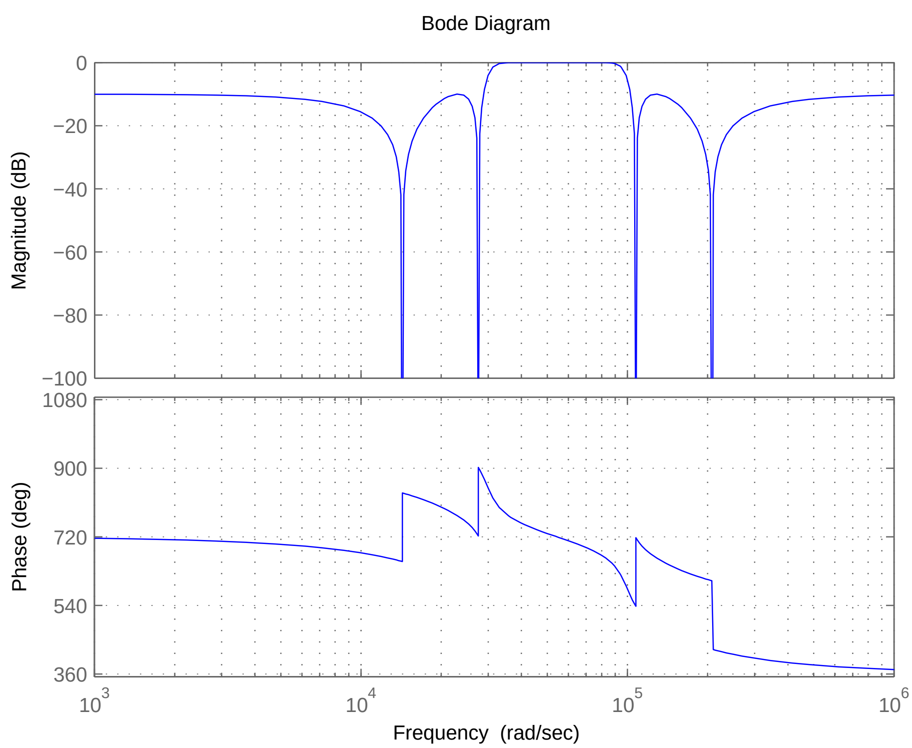
so that the mapping of frequency Ω_r in the prototype to Ω_r in the band-pass filter is given by the absolute value of the roots of

$$\begin{aligned} \Omega^2 - \Delta\Omega\Omega_r\Omega - \Omega_o^2 &= 0 \\ \text{or } \Omega^2 - (2\pi 10000) \times 1.175\Omega - (2\pi 5000) \times (2\pi 15000) &= 0 \end{aligned}$$

which gives $f_{rl} = 4.590$ kHz and $f_{ru} = 16.340$ kHz as indicated on the plot.



The Bode plots are shown below:



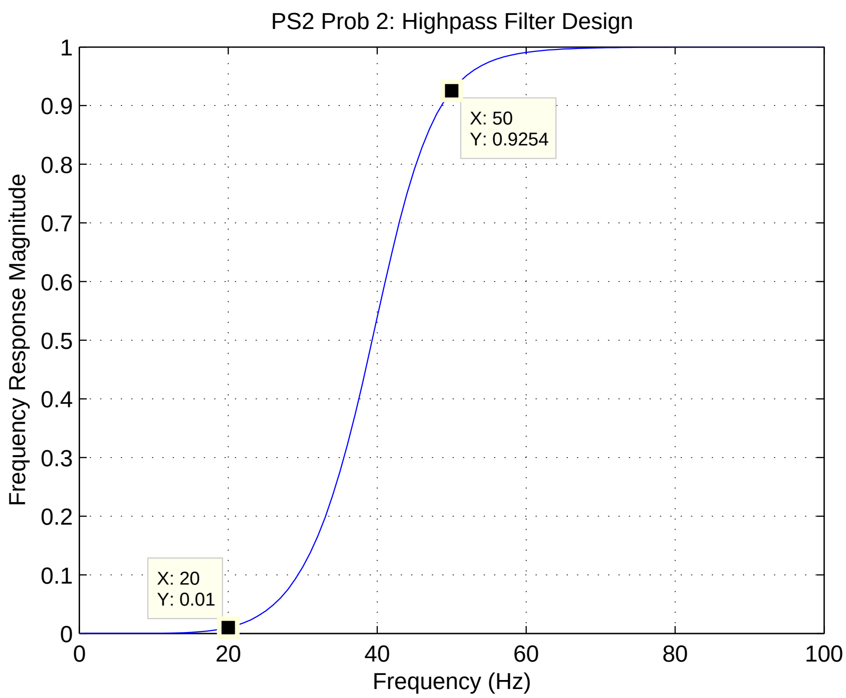
Problem 2: We require a high-pass filter.

$$
\begin{aligned}
\Omega_c &= 2\pi 50 \text{ rad/s} \\
\Omega_r &= 2\pi 20 \text{ rad/s} \\
R_c &= 3 \text{ rad/s} \\
R_s &= 40 \text{ rad/s}
\end{aligned}
$$

Let's choose a Butterworth prototype, and convert to a high-pass filter as described in Section 3.1 of the class handout *Introduction to Continuous Time Filter Design* using the following MATLAB script:

```
wc = 2*pi* 50;
wr = 2*pi*20;
Rc = 3;
Rs = 40;
Wr = wc/wr;
%
[N, Wn] = buttord(1, Wr,Rc,Rs,'s');
[num,den]= butter(N, Wn,'s');
[num_hp, den_hp] = lp2hp(num,den, wc);
hpfilt=tf(num_hp, den_hp)
%
w=[0:2*pi:2*pi*100];
[mag, phase] = bode(hpfilt,w);
plot(w/(2*pi),squeeze(mag));
title('PS2 Prob 2: Highpass Filter Design');
xlabel('Frequency (Hz)');
ylabel('Frequency Response Magnitude');
grid;
```

which produces the frequency response magnitude plot:



which meets the specifications.

<u>Problem 3:</u>

Let the lpf be a unity gain all-pole filter with transfer function

$$H_{lp}(s) = \frac{a_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

then the hiph-pass filter formed as

$$\begin{aligned} H_{hp}(s) &= 1 - H_{lp}(s) \\ &= 1 - \frac{a_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \\ &= \frac{s(s^{n-1} + a_{n-1}s^{n-2} + \ldots + a_1)}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \end{aligned}$$

We note that this is a high-pass filter because

- It has the same number of poles as zeros, indicating unity gain at high frequencies,and
- It has one or more zeroes at the origin, indicating that the gain goes to zero as the frequency approaches zero.

(a) In this case there is a single zero at the origin, while with the transformation method described in the handout, there will be n zeros at the origin.

(b) The attenuation rate as $\Omega \to 0$ will be much higher in the hpf designed by frequency transformation ($20n$ dB/decade as opposed to 20 db/decade.).

(c) From the graphical s-plane interpretation of the frequency response, each zero at the origin contributes $\pi/2$ radians of phase shift at very low frequencies. Thus the hpf designed by frequency transformation will have $+n\pi/2$ rad of phase lead at low frequencies, the filter designed as proposed in this problem will have a phase lead of $\pi/2$ rad.

(d) As indicated above, any system with the same number of poles as zeros will have unity gain at very high frequencies.

Problem 4: There are, of course, many solutions to this problem! Here's one possibility: Design a band-pass filter, "centered" at 60 Hz and with at least 60 dB attenuation at 30 and 60 Hz. Choose the specs

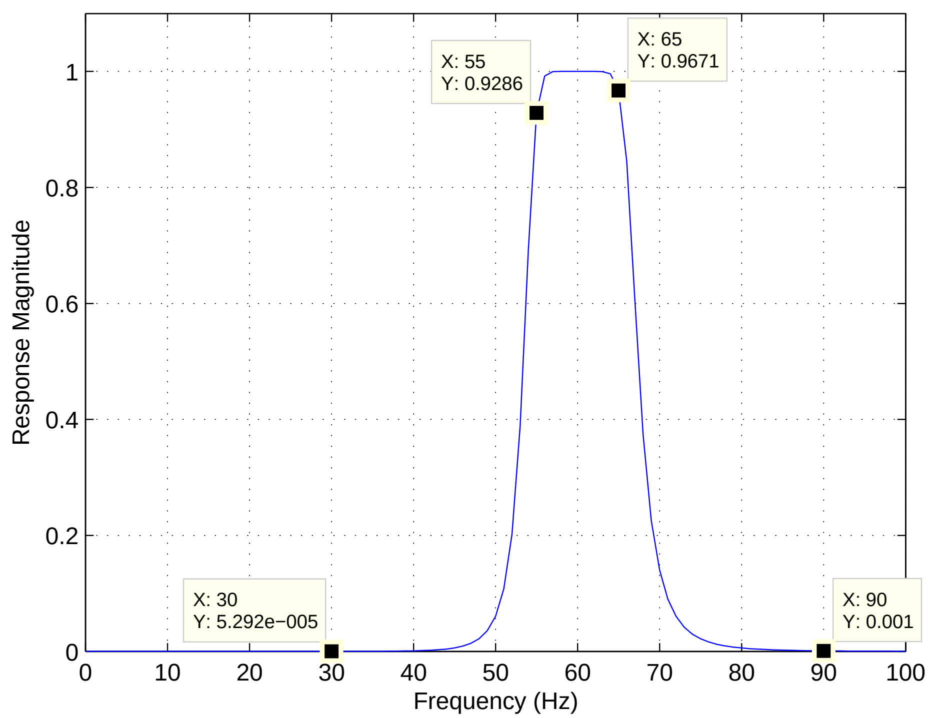
$$\begin{aligned}\Omega_0 &= 2\pi 60 \text{ rad/s}\\ \Omega_{cu} &= 2\pi 65 \text{ rad/s}\\ \Omega_{ru} &= 2\pi 90 \text{ rad/s}\\ \Omega_{rl} &= 2\pi 30 \text{ rad/s}\\ R_c &= 1 \text{ dB}\\ R_s &= 60 \text{ dB}\end{aligned}$$

Let's choose a Butterworth prototype, and convert to a band-pass filter as described in Section 3.1 of the class handout *Introduction to Continuous Time Filter Design* using the following MATLAB script:

```
% Design a band-stop filter
% First define some ctitical frequencies
%  Passband edges
wo  = 2*pi*60;
wcu = 2*pi*65;
wcl = wo^2/wcu;
BW  = (wcu-wcl);
% Stop band edges
wsu = 2*pi*90;
wsl = 2*pi*30;
%  Pass-band and stop-band attenuations:
Rc = 1;
Rs = 60;
% Determine the stop-band edge in the lp prototype
%
W1 = (wo^2-wsu^2)/(BW*wsu);
W2 = (wo^2-wsl^2)/(BW*wsl);
Wr = min(abs(W1),abs(W2));
%
% design the prototype low-pass filter
%
[N,Wn]     = buttord(1,Wr, Rc, Rs, 's');
```

```
[num,den]  = butter(N, Wn, 's');
% Convert to a band-stop filter
[num_bpass,den_bpass] = lp2bp(num,den,wo,BW);
filt = tf(num_bpass,den_bpass);
%
% Plot the frequency response magnitude
%
f=[0:1:100];
[mag,phase]=bode(filt,2*pi*f);
plot(f,(squeeze(mag)));
grid
xlabel('Frequency (Hz)');
ylabel('Response Magnitude');
```

which produces the frequency response magnitude plot:



which meets the specifications since 1 dB attenuation is a gain of $10^{-1/20} = 0.8913$ and 60 db attenuation is a gain of $10^{-60/20} = 0.001$.