2.161 Signal Processing: Continuous and Discrete
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.161 Signal Processing - Continuous and Discrete
Fall Term 2008

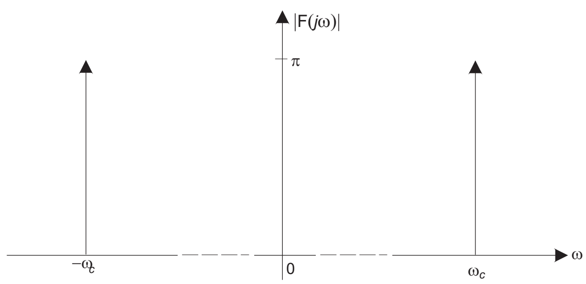
Solution of Problem Set 4

Assigned: Oct. 2, 2008 **Due:** Oct. 9, 2008

Problem 1:
We have shown in class (see the Fourier handout):

$$\begin{aligned}\mathcal{F}\left\{\sin(\omega_c t)\right\} &= -j\pi\left(\delta(\omega-\omega_c)-\delta(\omega+\omega_c)\right)\\ \mathcal{F}\left\{\cos(\omega_c t)\right\} &= \pi\left(\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right)\end{aligned}$$

(a) When $f_{audio} \equiv 0$, $f_{AM}(t) = \sin(\omega_c t)$, and from above



(b) There are several ways of doing this. For example

(i) Expand $f_{AM}(t)$ into sinusoidal components using this trigonometric relationship:

$$\cos\alpha\sin\beta = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\beta-\alpha))$$

$$\begin{aligned}f_{AM}(t) &= (1+af_{audio}(t))\sin(\omega_c t)\\ &= (1+0.5\,(0.5\cos(2\pi\cdot 1000t)+0.25\cos(2\pi\cdot 2000t))\sin(\omega_c i)\\ &= \sin(\omega_c t)\\ &\qquad +0.25\cos(2\pi 1000t)\cdot\sin(\omega_c t)\\ &\qquad +0.125\cos(2\pi 2000t)\cdot\sin(\omega_c t)\\ &= \sin(\omega_c t)\\ &\qquad +0.125\left(\sin((\omega_c+2000\pi)t)+\sin((\omega_c-2000\pi)t)\right)\\ &\qquad +0.0625\left(\sin((\omega_c+4000\pi)t)+\sin((\omega_c-4000\pi)t)\right)\end{aligned}$$

and take the Fourier transform of each of the five components:

$$\begin{aligned} F_{AM}(j\omega) &= -j\pi\{(\delta(\omega-\omega_c)-\delta(\omega+\omega_c)) \\ &\quad + \quad 0.125\,(\delta(\omega-(\omega_c-2000\pi))-\delta(\omega+(\omega_c-2000\pi))) \\ &\quad + \quad 0.125\,(\delta(\omega-(\omega_c+2000\pi))-\delta(\omega+(\omega_c+2000\pi))) \\ &\quad + \quad 0.0625\,(\delta(\omega-(\omega_c-4000\pi))-\delta(\omega+(\omega_c-4000\pi))) \\ &\quad + \quad 0.0625\,(\delta(\omega-(\omega_c+4000\pi))-\delta(\omega+(\omega_c+4000\pi)))\ \} \end{aligned}$$

giving a total of 10 impulse components in the spectrum.

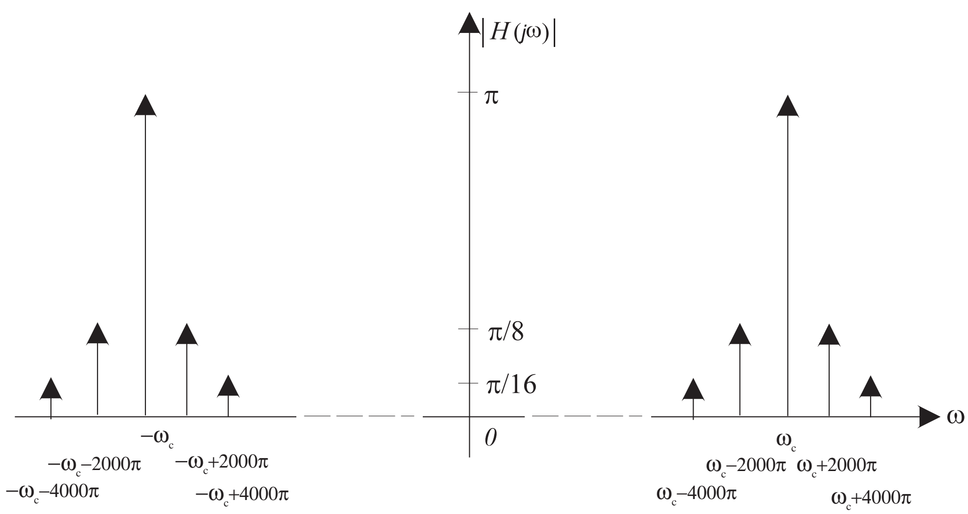
(ii) Alternatively you can recognize that the expansion to

$$f_{AM}(t) = \sin(\omega_c t) + 0.25\cos(2\pi 1000t)\cdot\sin(\omega_c t) + 0.125\cos(2\pi 2000t)\cdot\sin(\omega_c t)$$

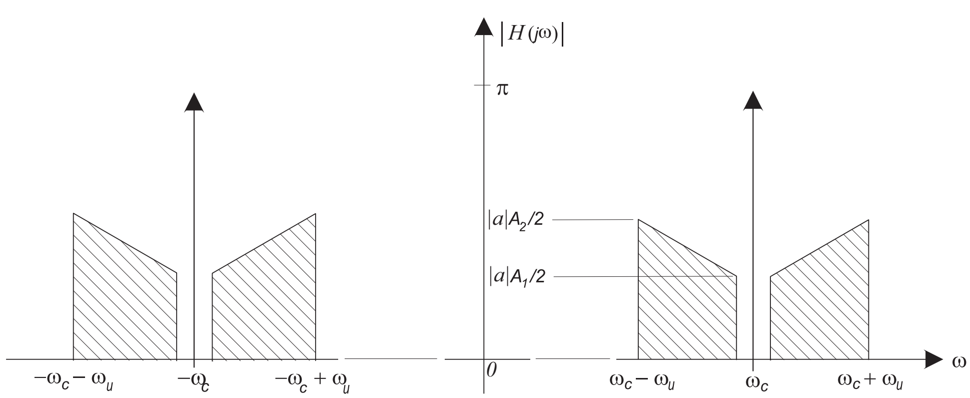
involves time-domain products and these will result in frequency-domain convolutions, so that

$$F_{AM}(j\omega) = F_c(j\omega) + \frac{1}{2\pi}F_c(j\omega)\otimes F_{1000}(j\omega) + \frac{1}{2\pi}F_c(j\omega)\otimes F_{2000}(j\omega)$$

where $F_c(j\omega) = \mathcal{F}\{\sin(\omega_c t)\}$, $F_{1000}(j\omega) = \mathcal{F}\{0.25\cos(2000\pi t)\}$, and $F_{2000}(j\omega) = \mathcal{F}\{0.125\cos(4000\pi t)\}$. The same result as in (i) will follow.



(c) The following generalizes the results of part (b)



(d) From the above figure it can be seen that the required bandwidth is $2\omega_u$ rad/s.

Problem 2:

This problem uses the time-reversal property of the Fourier transform, if $\mathcal{F}\{f(t)\} = F(j\omega)$ then $\mathcal{F}\{f(-t)\} = F(-j\omega)$, and if $f(t)$ is real then $F(-j\omega) = \overline{F}(j\omega)$, so that $\mathcal{F}\{f(-t)\} = \overline{F}(j\omega)$.

Method(1)

1. $G(j\omega) = F(j\omega)H(j\omega)$.
2. $X(j\omega) = \overline{G}(j\omega)H(j\omega)$.
3. $Y(j\omega) = \overline{X}(j\omega) = \overline{\overline{F}(j\omega)H(j\omega)}H(j\omega) = F(j\omega)\overline{H}(j\omega)H(j\omega) = F(j\omega)\left|H(j\omega)\right|^2$.

The equivalent transfer function is

$$H_{eq} = \left|H(j\omega)\right|^2$$

which is real, that is with zero phase shift.

Method(2)

1. $G(j\omega) = F(j\omega)H(j\omega)$.
2. $X(j\omega) = \overline{F(j\omega)}H(j\omega)$.
3. $Y(j\omega) = G(j\omega) + \overline{X(j\omega)} = F(j\omega)H(j\omega) + \overline{\overline{F(j\omega)}H(j\omega)} = 2F(j\omega)\Re\{H(j\omega)\}$.

The equivalent transfer function is

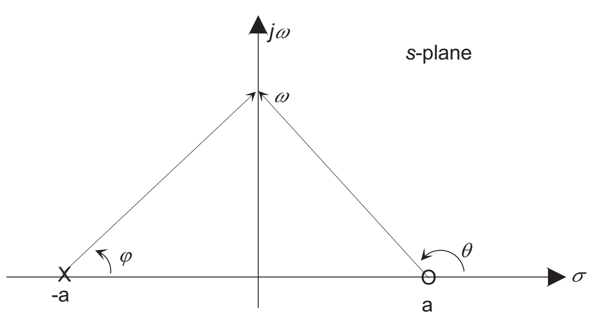
$$H_{eq} = 2\Re\{H(j\omega)\}$$

which is real, that is with zero phase shift.

Note that because it squares the frequency response magnitude, method (1) will have a sharper cut-off characteristic than method (2).

Problem 3:

Note: There was a typo in the problem statement, that stated that the phase-shift at 50 Hz should be $-\frac{\pi}{2}$. The intention was for a phase shift of $\frac{\pi}{2}$ rad. The all-pass transfer function will only generate a phase-lead, therefore an extra inversion is required to create a lag. No penalty is imposed for missing this point.



$$\angle H(j\omega) = \theta - \phi = 2\theta - \pi$$

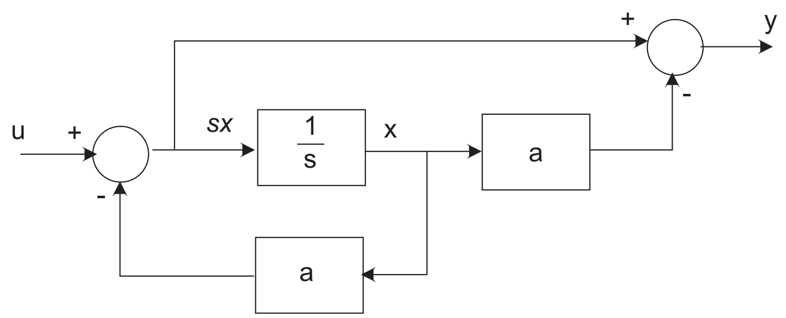
since $\theta + \phi = \pi$, then:

$$\angle H(j\omega) = \pi - 2\tan^{-1}\left(\frac{\omega}{a}\right).$$

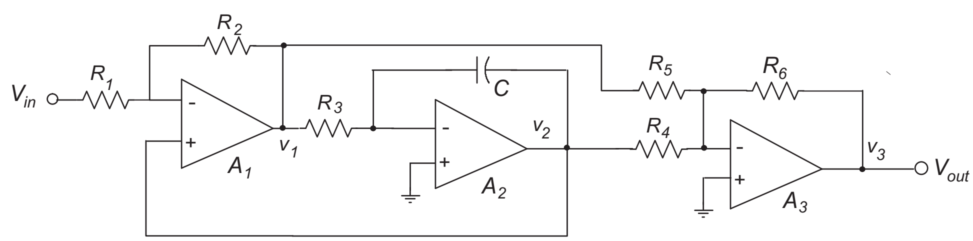
First Solution: this solution requires three op-amps for $\frac{\pi}{2}$ phase shift.
At 50 Hz:

$$\angle H(j\omega) = \frac{\pi}{2} = \pi - 2\tan^{-1}\left(\frac{100\pi}{a}\right).$$

giving $a = 100\pi$. The filter can be achieved with the following block diagram:



Consider the following circuit, which is derived from Fig. 7 in the op-amp filter handout:



For amplifier A_1

$$\frac{V_{in} - v_-}{R_1} = \frac{v_1 - v_-}{R_2} \quad \text{but} \quad v_- = v_2$$

and for amplifier A_2

$$v_2 = -\frac{1}{R_3 C s} v_1 \quad \text{or} \quad v_1 = -R_3 C s v_2$$

Eliminating v_1

$$\frac{v_2}{V_{in}} = \frac{R_2}{R_1 R_3 C s + (R_1 + R_2)}.$$

For amplifier A_3

$$\begin{aligned} V_{out} &= -\frac{R_6}{R_5} v_1 - \frac{R_6}{R_4} v_2 \\ &= \left(\frac{R_6 R_3 C}{R_5} s - \frac{R_6}{R_4}\right) v_2 \end{aligned}$$

and substituting for v_2 gives the transfer function

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{R_2 R_6}{R_1 R_5} . \frac{s - R_5/(R_3 R_4 C)}{s + (R_1 + R_2)/(R_1 R_3 C)}$$

For an all-pass filter with $a = 100\pi$ we require

$$\frac{R_5}{R_3 R_4 C} = 100\pi \quad \text{and} \quad \frac{R_1 + R_2}{R_1 R_3 C} = 100\pi.$$

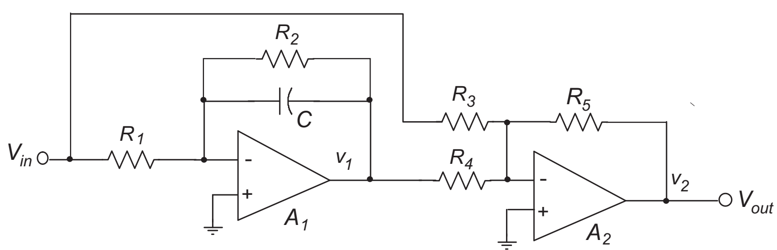
Let $C = 1\ \mu$F, $R_3 = R_4 = 10$ kΩ, then $R_5 = 31.42$ kΩ.
Let $R_1 = 10$ kΩ, then $R_2 = 21.142$ kΩ.
For unity gain we require

$$\frac{R_2 R_6}{R_1 R_5} = 1, \quad \text{or} \quad R_6 = \frac{R_1 R_5}{R_2} = 14.64 \text{ k}\Omega.$$

Alternative Solution: this solution requires only two op-amps for $-\frac{\pi}{2}$ phase shift!
Write the transfer function as:

$$H(s) = \frac{s - a}{s + a} = 1 - \frac{2a}{s + a}$$

and simply implement a first-order block and a summer. Consider the circuit shown below:



For amplifier A$_1$, $v_1/V_{in} = -Z_{in}/Z_f$, where $Z_{in} = R_1$ and $Z_2 = R_2/(R_2 C s + 1)$ are the input and feedback impedances respectively. Then

$$\frac{v_1}{V_{in}} = -\frac{1}{R_1 C} . \frac{1}{(s + 1/(R_2 C))}$$

Amplifier A$_2$ is simply an inverting summer, and

$$\begin{aligned} v_{out} &= -\frac{R_5}{R_3} V_{in} - \frac{R_5}{R_4} v_1 \\ &= -\frac{R_5}{R_3} V_{in} + \frac{R_5}{R_4 R_1 C} . \frac{1}{(s + 1/(R_2 C))} V_{in} \end{aligned}$$

Let $R_3 = R_4 = R_5$, then

$$\frac{V_{out}}{V_{in}} = -\left(1 - \frac{1/(R_1 C)}{(s + 1/(R_2 C))}\right).$$

which actually has the implicit inversion required in the (erroneous) problem statement. Let $R_3 = R_4 = R_5 = 10$ kΩ, and $C = 0.1\mu$F. Then

$$\frac{1}{R_2C} = a = 100\pi \text{ giving } R_2 = 31.83\text{k}\Omega$$

$$\frac{1}{R_1C} = 2a = 200\pi \text{ giving } R_1 = 15.91\text{k}\Omega$$

Problem 4:

(a) Define Ω_c as the either upper or lower -3db (0.707) response frequencies. So from the definition of Ω_c:

$$|H_{bp}(j\Omega_c)| = \frac{1}{\sqrt{2}} = |\frac{\frac{\Omega_p}{Q}j\Omega_c}{(j\Omega_c)^2 + \frac{\Omega_p}{Q}j\Omega_c + \Omega_p^2}|$$

$$\frac{1}{\sqrt{2}} = \frac{\frac{\Omega_p}{Q}\Omega_c}{|\Omega_p^2 - \Omega_c^2 + \frac{\Omega_p}{Q}j\Omega_c|}$$

Now define $\alpha = \frac{\Omega_c}{\Omega_p} > 0$, then:

$$\frac{1}{\sqrt{2}} = \frac{\frac{\alpha}{Q}}{|1 - \alpha^2 + \frac{\alpha}{Q}j|}$$

$$\frac{1}{2} = \frac{(\frac{\alpha}{Q})^2}{(1-\alpha^2)^2 + (\frac{\alpha}{Q})^2}$$

$$1 + \alpha^4 - 2\alpha^2 + \frac{1}{Q^2}\alpha^2 = 2\frac{1}{Q^2}\alpha^2$$

$$\alpha^4 - 2(1 + \frac{1}{2Q^2})\alpha^2 + 1 = 0$$

α^2 values can be found from:

$$\alpha^2 = (1 + \frac{1}{2Q^2}) \pm \sqrt{(1 + \frac{1}{2Q^2})^2 - 1}$$

$$\alpha^2 = (1 + \frac{1}{2Q^2}) \pm \sqrt{\frac{2}{2Q^2} + \frac{1}{(2Q^2)^2}}$$

$$\alpha^2 = (1 + \frac{1}{2Q^2}) \pm \frac{1}{|Q|}\sqrt{1 + (\frac{1}{2Q})^2}$$

Now define equation roots as $\alpha_u > \alpha_l > 0$:

$$\alpha_u^2 = (1 + \frac{1}{2Q^2}) + \frac{1}{|Q|}\sqrt{1 + (\frac{1}{2Q})^2}$$

$$\alpha_l^2 = (1+\frac{1}{2Q^2}) - \frac{1}{|Q|}\sqrt{1+(\frac{1}{2Q})^2}$$

Now note that for a stable filter $Q > 0$ and hence:

$$\begin{aligned}
\Omega_u &= \Omega_p\sqrt{1+\frac{1}{2Q^2}+\frac{1}{Q}\sqrt{1+\left(\frac{1}{2Q}\right)^2}} \\
\Omega_l &= \Omega_p\sqrt{1+\frac{1}{2Q^2}-\frac{1}{Q}\sqrt{1+\left(\frac{1}{2Q}\right)^2}}
\end{aligned}$$

(b)

$$\Delta = (\Omega_u - \Omega_l) \ > \ 0$$

$$\Delta^2 = \Omega_p^2(\alpha_u^2 + \alpha_l^2 - 2\alpha_u\alpha_l)$$

$$\Delta^2 = \Omega_p^2(2(1+\frac{1}{2Q^2}) - 2\sqrt{(1+\frac{1}{2Q^2})^2 - \frac{1}{Q^2}(1+(\frac{1}{2Q})^2)})$$

$$\Delta^2 = \Omega_p^2(2+\frac{2}{2Q^2} - 2\sqrt{(1+\frac{1}{4Q^4}+\frac{2}{2Q^2}-\frac{1}{Q^2}-\frac{1}{4Q^4}})$$

$$\Delta^2 = \Omega_p^2(2+\frac{2}{2Q^2} - 2\sqrt{1})$$

$$\Delta^2 = \Omega_p^2(\frac{1}{Q^2})$$

$$|\Delta| = \frac{\Omega_p}{|Q|}$$

$$\Delta = \frac{\Omega_p}{Q}$$

Alternatively, we can further simplify Ω_u and Ω_l by realizing that we have a complete square form under square root and proceed from there:

$$\begin{aligned}
\Omega_u &= \Omega_p\sqrt{1+\frac{1}{2Q^2}+\frac{1}{Q}\sqrt{1+\left(\frac{1}{2Q}\right)^2}} = \Omega_p\sqrt{(\sqrt{1+\frac{1}{(2Q)^2}}+\frac{1}{2Q})^2} = \Omega_p(\sqrt{1+\frac{1}{(2Q)^2}}+\frac{1}{2Q}) \\
\Omega_l &= \Omega_p\sqrt{1+\frac{1}{2Q^2}-\frac{1}{Q}\sqrt{1+\left(\frac{1}{2Q}\right)^2}} = \Omega_p\sqrt{(\sqrt{1+\frac{1}{(2Q)^2}}-\frac{1}{2Q})^2} = \Omega_p(\sqrt{1+\frac{1}{(2Q)^2}}-\frac{1}{2Q})
\end{aligned}$$

$$\Delta = (\Omega_u - \Omega_l) = \frac{\Omega_p}{Q}$$

(c) Note that $\Omega_p = 100 \cdot (2\pi)\ rad/s$ and $\Delta = 10 \cdot (2\pi)\ rad/s$:

$$H_{bp}(s) = \frac{\frac{\Omega_p}{Q}s}{s^2 + \frac{\Omega_p}{Q}s + \Omega_p^2}$$

$$H_{bp}(s) = \frac{\Delta s}{s^2 + \Delta s + \Omega_p^2}$$

$$H_{bp}(s) = \frac{20\pi s}{s^2 + 20\pi s + (200\pi)^2}$$

<u>Problem 5:</u> Given $y(t) = \sin(\omega_0 t)$ and $y_1(t) = \sin(\omega_1 t)$ with $\omega_0 < \omega_N = \frac{\pi}{\Delta T} < \omega_1$

(a) Assume $\omega_1 = 2k\omega_N - \omega_0$ for k a positive integer, then

$$y_1(t) = \sin((2k\omega_N - \omega_0)t) = \sin((2k\frac{\pi}{\Delta T} - \omega_0)t)$$

and when $t = n\Delta T$, for integer N,

$$y_1(n\Delta T) = \sin(-\omega_0 N\Delta T) = -\sin(\omega_0 N\Delta T) = -y(n\Delta T)$$

(b) Assume $\omega_1 = 2k\omega_N + \omega_0$ for k a positive integer, then

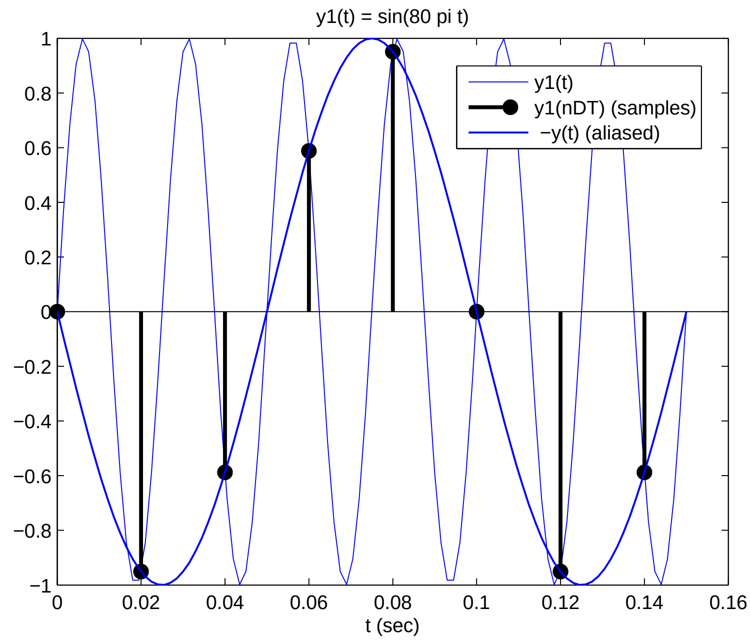
$$y_1(t) = \sin((2k\omega_N + \omega_0)t) = \sin((2k\frac{\pi}{\Delta T} + \omega_0)t)$$

and when $t = n\Delta T$, for integer N,

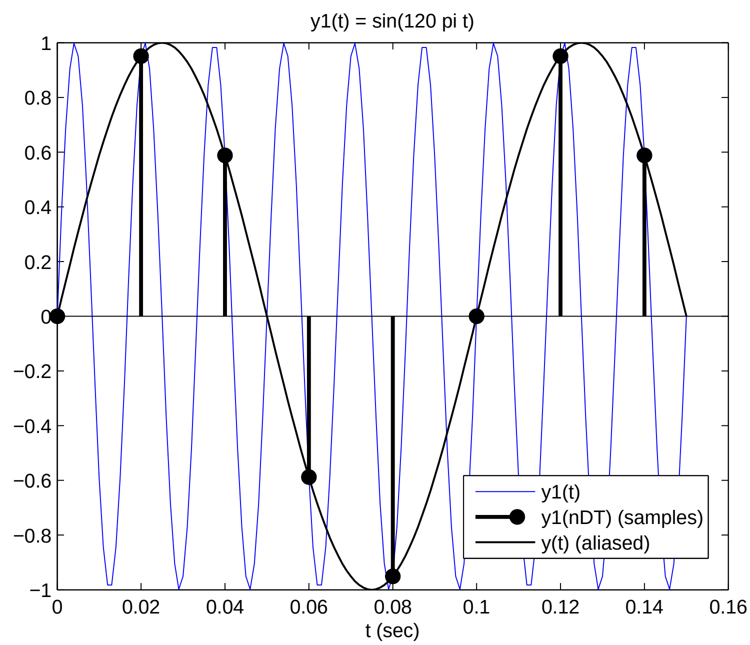
$$y_1(n\Delta T) = \sin(\omega_0 N\Delta T) = y(n\Delta T)$$

(c) To graphically demonstrate the concept of frequency folding and sign changes

(i) Assume $\Delta T = .02$ s. so that $\omega_N = 50\pi$ rad/s (25Hz). Let $\omega_1 = 80\pi$ rad/s (40Hz) so that $k = 1$ and $\omega_0 = 20\pi$ rad/s (10 Hz). The following plot shows $y_1(t) = \sin(80\pi t)$, the samples taken at 0.02 sec intervals, and $-y_0(t)$, showing the out-of-phase aliased component at a frequency of 20π rad/s.



(ii) Assume $\Delta T = .02$ s. so that $\omega_N = 50\pi$ rad/s (25 Hz). Let $\omega_1 = 120\pi$ rad/s (60 Hz) so that $k = 1$ and $\omega_0 = 20\pi$ rad/s (10Hz). The following plot shows $y_1(t) = \sin(120\pi t)$, the samples taken at 0.02 sec intervals, and $y_0(t)$, showing the in-phase aliased component at a frequency of 20π rad/s.



The above examples show that frequencies of 60 Hz and 40 Hz will both be aliased to an apparent component of 10 Hz.

(d) From parts (a) and (b) we see that an under-sampled sinusoid, that is a sinusoids is above the Nyquist frequency ω_N will "fold" into either an in-phase or out-of phase sinusoid, The frequencies of the "folded" sinusoids are given by:

$$\omega_0 = \begin{cases} \omega - 2k\omega_N, & \omega > 2\omega_N \quad \text{(in-phase)} \\ 2k\omega_N - \omega, & \omega < 2\omega_N \quad \text{(out-of-phase)} \end{cases}$$

Given $y(t) = 5\sin(2\pi(25)t) + 2\sin(2\pi(75)t) + 3\sin(2\pi(125)t)$ sampled at 100 samples/sec. Half the sampling frequency is 50 Hz ($\omega_N = 100\pi$ rad/s). Using the equation above we find that the 75Hz sinusoid "folds" into an out-of-phase sinusoid at 25 Hz, and the 125 Hz sinusoid "folds" into a 25 Hz in-phase sinusoid. Therefore the aliased waveform is

$$y(t) = 5\sin(2\pi(25)t) - 2\sin(2\pi(25)t) + 3\sin(2\pi(25)t) = 6\sin(2\pi(25)t)$$

The following plot demonstrates this.