2.161 Signal Processing: Continuous and Discrete
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.161 Signal Processing – Continuous and Discrete

The Dirac Delta and Unit-Step Functions[1]

1 The Dirac Delta (Impulse) Function

The Dirac delta function is a non-physical, singularity function with the following definition

$$\delta(t) = \begin{cases} 0 & \text{for } t \neq 0 \\ \text{undefined} & \text{at } t = 0 \end{cases} \tag{1}$$

but with the requirement that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1, \tag{2}$$

that is, the function has unit area. Despite its name, the delta function is not truly a function. Rigorous treatment of the Dirac delta requires *measure theory* or the *theory of distributions.*

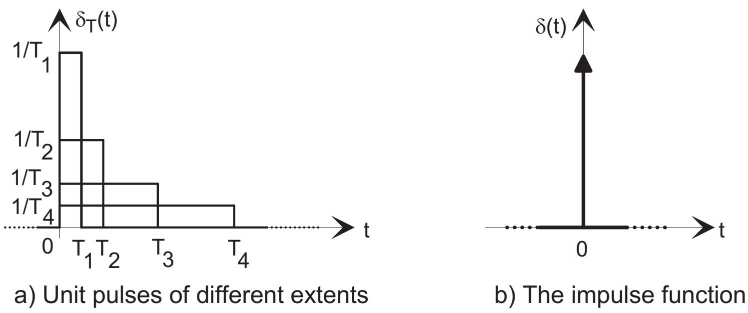

Figure 1: Unit pulses and the Dirac delta function.

Figure 1 shows a *unit pulse* function $\delta_T(t)$, that is a brief rectangular pulse function of extent T, defined to have a constant amplitude $1/T$ over its extent, so that the area $T \times 1/T$ under the pulse is unity:

$$\delta_T(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1/T & 0 < t \leq T \\ 0 & \text{for } t > T. \end{cases} \tag{3}$$

The Dirac delta function (also known as the impulse function) can be defined as the limiting form of the unit pulse $\delta_T(t)$ as the duration T approaches zero. As the extent T of $\delta_T(t)$ decreases, the amplitude of the pulse increases to maintain the requirement of unit area under the function, and

$$\delta(t) = \lim_{T \to 0} \delta_T(t). \tag{4}$$

The impulse is therefore defined to exist only at time $t = 0$, and although its value is strictly undefined at that time, it must tend toward infinity so as to maintain the property of unit area in the limit.

[1]D. Rowell, September 5, 2008

In addition to the one-sided unit pulse described above, the delta function may be considered as the limit of several other functions

$$\delta(t) = \lim_{a \to 0} \delta_a(t)$$

where $\delta_a(t)$ is sometimes called a *nascent* delta function. The following are some commonly used approximations:

$\delta_a(t) = \begin{cases} \dfrac{1}{a} & \text{for } \lvert t \rvert \le \dfrac{a}{2} \\ 0 & \text{for } \lvert t \rvert > \dfrac{a}{2} \end{cases}$	"rect" function
$\delta_a(t) = \begin{cases} \dfrac{2}{a} - \dfrac{4}{a^2}\lvert t \rvert & \text{for } \lvert t \rvert \le \dfrac{a}{2} \\ 0 & \text{for } \lvert t \rvert > \dfrac{a}{2} \end{cases}$	triangular function
$\delta_a(t) = \dfrac{1}{a\sqrt{\pi}} e^{-t^2/a^2}$	Gaussian function
$\delta_a(t) = \dfrac{1}{\pi} \dfrac{a}{a^2 + t^2}$	Cauchy-Lorentz distribution
$\delta_a(t) = \dfrac{1}{a} e^{-\lvert t/a \rvert}$	Cauchy φ function
$\delta_a(t) = \dfrac{1}{a\pi} \left(\dfrac{\sin(t/a)}{t/a} \right)$	sinc function
$\delta_a(t) = \dfrac{1}{a\pi} \left(\dfrac{\sin(t/a)}{t/a} \right)^2$	sinc^2 function

All of these functions have the property that as $a \to 0$, the function becomes impulse-like in the neighborhood of $t = 0$, with unit area under the curve.

The impulse function is used extensively in the study of linear systems, both spatial and temporal. Although true impulse functions are not found in nature, they are approximated by short duration, high amplitude phenomena such as a hammer impact on a structure, or a lightning strike on a radio antenna. As we will see below, the response of a causal linear system to an impulse defines its response to all inputs.

1.1 Properties

1.1.1 Shift

An impulse occurring at $t = a$ is $\delta(t - a)$.

1.1.2 The strength of an impulse

Because the amplitude of an impulse is infinite, it does not make sense to describe a scaled impulse by its amplitude. Instead, the *strength* of a scaled impulse $K\delta(t)$ is defined by its area K.

1.1.3 The "Sifting" Property of the Impulse

When an impulse appears in a product within an integrand, it has the property of "sifting" out the value of the integrand at the point of its occurrence:

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a) \tag{5}$$

This is easily seen by noting that $\delta(t-a)$ is zero except at $t = a$, and for its infinitesimal duration $f(t)$ may be considered a constant and taken outside the integral, so that

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)\int_{-\infty}^{\infty} \delta(t-a)dt = f(a) \tag{6}$$

from the unit area property.

1.1.4 Scaling

A helpful identity is the scaling property:

$$\int_{-\infty}^{\infty} \delta(\alpha t)dt = \int_{-\infty}^{\infty} \delta(u)\frac{du}{|\alpha|} = \frac{1}{|\alpha|}$$

and so

$$\delta\left(\alpha t\right) = \frac{1}{|\alpha|}\delta(t).$$

1.1.5 Laplace Transform

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = 1$$

by the sifting property.

1.1.6 Fourier Transform

$$\mathcal{F}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t)e^{-j\Omega t}dt = 1$$

by the sifting property.

2 Practical Applications of the Dirac Delta Function

- The most important application of δt in linear system theory is directly related to its Laplace transform property, $\mathcal{L}\left\{\delta(t)\right\} = 1$. Consider a SISO LTI system with transfer function $H(s)$, with input $u(t)$ and output $y(t)$, so that in the Laplace domain

  $$Y(s) = H(s)U(s).$$

  If the input is $u(t) = \delta(t)$, so that $U(s) = 1$, then $Y(s) = H(s).1$, and through the inverse Laplace transform

  $$y(t) = h(t) = \mathcal{L}^{-1}\left\{H(s)\right\}.$$

  where $h(t)$ is defined as the system's *impulse response*. The impulse response completely characterizes the system, in the sense that it allows computation of the transfer function (and hence the differential equation).

- The impulse response $h(t)$ is used in the convolution integral.
- In signal processing the delta function is used to create a Dirac comb (also known as an impulse train, or Shah function):

$$\Delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

is used in sampling theory. A continuous waveform $f(t)$ is *sampled* by multiplication by the Dirac comb

$$f^*(t) = f(t)\Delta_T(t) = \sum_{n=-\infty}^{\infty} f(t - nT)\delta(t - nT),$$

where $f^*(t)$ is the sampled waveform, producing a train of weighted impulses.

3 The Heaviside (Unit-Step) Function

The unit-step function $u_s(t)$, also known as the Heaviside function, is a discontinuous function with a value 0 for negative arguments, and unity for positive arguments, as shown in Fig. 2.

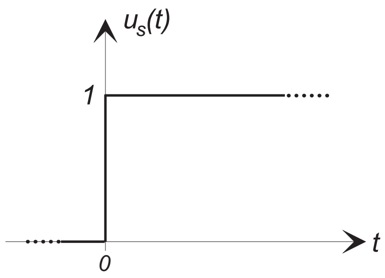

Figure 2: The Heaviside, or unit-step function.

The unit-step is an antiderivative of the Dirac delta in the sense that

$$u_s(t) = \int_{-\infty}^{t} \delta(u)du \qquad \text{for } t \neq 0$$

where this relation may not hold (or even make sense) for $t = 0$. The value of $u_s(0)$ is subject to some debate, and is often given as $u_s(0) = 0$, $u_s(0) = 1$, or $u_s(0) = 0.5$. In practice the value rarely matters because $u_s(t)$ usually appears within an integral. However, the value $u_s(0) = 0.5$ has some conceptual appeal, especially when using an analytic approximation to the unit-step.

A useful representation of $u_s(t)$ is

$$u_s(t) = \frac{1 + \mathrm{sgn}(t)}{2}$$

where sgn() denotes the signum (sign) function

$$\mathrm{sgn}(t) = \begin{cases} -1 & \text{for } t < 0 \\ 0 & \text{for } t = 0 \\ 1 & \text{for } t > 0 \end{cases}.$$

Clearly this definition of $u_s(t)$ supports the convention $u_s(0) = 0.5$.

3.1 Analytic Approximations

The limiting form of many sigmoid type functions centered on $t = 0$ may be used as an approximation to the Heaviside function, for example:

$$\begin{aligned} u_s(t) &= \lim_{k\to\infty} \frac{1}{2}\left(1 + \tanh(kt)\right) = \frac{1}{1 + e^{-2kt}} \\ u_s(t) &= \lim_{k\to\infty} \left(\frac{1}{2} + \frac{1}{\pi}\arctan(kt)\right) \\ u_s(t) &= \lim_{k\to\infty} \frac{1}{2}\left(1 + \operatorname{erf}(kt)\right) \end{aligned}$$

3.2 Integral representation

An integral representation is often useful

$$u_s(t) = \lim_{\epsilon\to 0^+} -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\tau + j\epsilon} e^{-jt\tau} d\tau$$

3.3 Properties

3.3.1 Shift

A unit-step occurring at $t = a$ is simply $u_s(t - a)$.

3.3.2 Derivative and Antiderivative

While not strictly differentiable, the derivative of the unit-step is taken to be the Dirac delta

$$\frac{d}{dt}u_s(t) = \delta(t).$$

The integral of $u_s(t)$ is

$$\int_{-\infty}^{t} u_s(\mu)d\mu = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases}$$

which is the unit ramp function $r(t)$.

3.3.3 Laplace Transform

$$\mathcal{L}\left\{u_s(t)\right\} = \frac{1}{s}$$

3.3.4 Fourier Transform

Although the Fourier integral does not converge directly for $u_s(t)$, it is easy to show that

$$\mathcal{F}\left\{u_s(t)\right\} = \frac{1}{2}\delta\left(j\Omega\right) + \frac{1}{j\Omega}$$

See the class handout on Fourier transforms for details.

3.4 Practical Applications of the Unit-Step Function

- Perhaps the most common use of $u_s(t)$ is as a multiplicative weighting function to create causal (one-sided) functions. For example, a causal exponential time function may be expressed as
$$f(t) = u_s(t)e^{-\sigma t}.$$
- As a test input signal for characterizing the response of linear systems throughout linear system theory and feedback control theory.
- As the basis for synthesizing zero-order (*staircase* or *boxcar*) functions. For example a rectangular pulse
$$f(t) = \begin{cases} 0 & \text{for } t < 1.5 \\ 3 & \text{for } 1.5 \le t < 5 \\ 0 & \text{for } t \ge 5 \end{cases}$$
may be written
$$f(t) = 3\left(u_s\left(t - 1.5\right) - u_s\left(t - 5\right)\right).$$
Similarly, a zero-order approximation $r_T(t)$ to a continuous unit ramp $r(t) = t$ for $t > 0$, and updated at intervals T, may be written as a sum of shifted step functions:
$$r_T(t) = \sum_{n=0}^{\infty} u_s(t - nT)$$