MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Kundu & Cohen 2.6, 2.7, and 2.20

These problems are from chapter 2 in "Fluid Mechanics" by P. K. Kundu, I. M. Cohen, and D.R. Dowling

- (2.6) Show that the condition for the vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$ to be coplanar is:

  $\varepsilon_{ijk} a_i b_j c_k$

- (2.7) Prove the following relationships:

  $\delta_{ij}\delta_{ij} = 3 \quad \varepsilon_{pqi}\varepsilon_{pqj} = 2\delta_{ij}$

- (2.20) Use Stokes theorem to prove that $\nabla \times (\nabla\phi) = 0$ for any single-valued twice-differentiable scalar (ϕ) regardless of the coordinate system.

Solution:

2.6

Condition for three vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$ to be coplanar is

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = 0 \tag{1}$$

Also we know that

$$\underline{b} \times \underline{c} = (\varepsilon_{ijk} b_j c_k \, \underline{e}_i) \tag{2}$$

From (1) and (2):

$$a_i \, \underline{e}_i \cdot (\varepsilon_{ijk} b_j c_k \, \underline{e}_i) = 0 \tag{3}$$

which leads to:

$$\varepsilon_{ijk} a_i b_j c_k \tag{4}$$

2.7

From the definition for Kronecker delta:

$$\delta_{ij} = 0 \; if \; i \neq j \; and \; \delta_{ij} = 1 \, if \; i = j \tag{5}$$

Thus

$$\delta_{ij}\delta_{ij} = \delta_{ii}\delta_{ii} = 1 + 1 + 1 = 3 \tag{6}$$

For proving the other statement ($\varepsilon_{pqi}\varepsilon_{pqj} = 2\delta_{ij}$) we should refer to the definition of the alternating tensors: if there is any repeating index, i.e. $i = j$, or $j = k$, or $i = k$ then $\varepsilon_{ijk} = 0$. This means that in $\varepsilon_{pqi}\varepsilon_{pqj}$ the only non-zero terms are the ones in which p,q,i, and j have four different index values. Since we only have three values for any possible index (1,2, and 3) the mentioned condition for having non-zero terms is only true when $i = j$ (one can easily pick two different values of i and j (e.g. $i = 1$ and $i = 3$)and see that all the terms turn to be zero in $\varepsilon_{pqi}\varepsilon_{pqj}$. Thus we will have the following:

$$\varepsilon_{pqi}\varepsilon_{pqj} = 0 \;\; if \;\; i \neq j \tag{7}$$

$$if \;\; i = j : \;\; \varepsilon_{pqi}\varepsilon_{pqj} = 0 + \varepsilon_{pqi}\varepsilon_{pqi} = 0 + \varepsilon_{pqi}^2 \tag{8}$$

Knowing that for any value of i there are only two remaining non-zero terms in the right hand side of (8) which are either -1 or 1, we will have:

$$if \;\; i = j : \;\; \varepsilon_{pqi}\varepsilon_{pqj} = 0 + (-1)^2 + (1)^2 = 2 \tag{9}$$

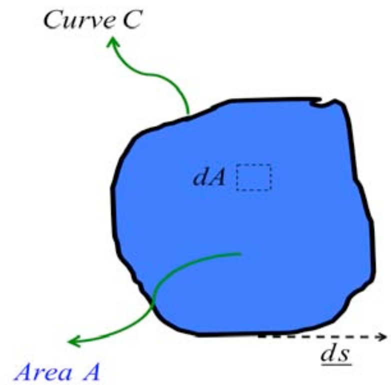
From (7) and (9):

$$\varepsilon_{pqi}\varepsilon_{pqj} = 2\delta_{ij} \tag{10}$$

2.20

From Stokes' theorem:

$$\int_A (\underline{\nabla} \times \underline{u}) \cdot \underline{dA} = \oint_C \underline{u}.\underline{ds} \tag{11}$$

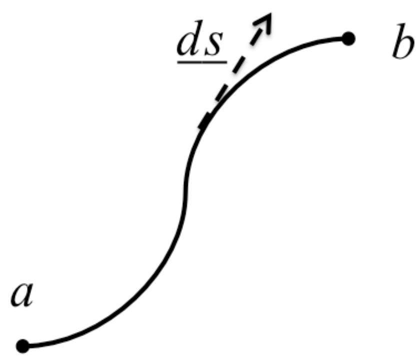


Replacing $\underline{u}$ by $\underline{\nabla\phi}$ will lead to:

$$\int_A (\underline{\nabla} \times (\underline{\nabla}\phi)) \cdot \underline{dA} = \oint_C \underline{\nabla}\phi.\underline{ds} \tag{12}$$

Now considering the right hand side of (12), for the line integral of a gradient vector we have the following:

$$\int_a^b \underline{\nabla\phi} \cdot \underline{ds} = \phi(b) - \phi(a) \tag{13}$$



Using (13) for the closed integral over the curve C, we will have:

$$\oint_C \underline{\nabla}\phi.\underline{ds} = \phi(a) - \phi(a) \tag{14}$$

In which a can be any arbitrary point on the curve C.
Using (14) and also having (12) for any arbitrary area, A, one can conclude that $\nabla \times (\nabla\phi) = 0$

□

Problem Solution by Bavand, Fall 2012

MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Panton 3.12

This problem is from "Incompressible Flow" by Ronald L. Panton

Write the following formulas in Gibbs's notation using the symbol ∇. Convert the expressions to Cartesian notation and prove that the equations are correct.

$$div(\phi \underline{v}) = \phi \, div \, \underline{v} + \underline{v} \, . \, grad \, \phi$$

$$div(\underline{u} \times \underline{v}) = \underline{v} \cdot curl \, \underline{u} - \underline{u} \cdot curl \, \underline{v}$$

$$curl(\underline{u} \times \underline{v}) = \underline{v} \cdot grad \, \underline{u} - \underline{u} \cdot grad \, \underline{v} + \underline{u} \, div \, \underline{v} - \underline{v} \, div \, \underline{u}$$

Solution:

a.

$$div(\phi \mathbf{v}) = \phi\, div\underline{v} + \underline{v} \cdot grad\phi \tag{1}$$

Using Gibbs Notation we can rewrite equation 1 as:

$$\underline{\nabla} \cdot (\phi\ \underline{v}) = \phi\ \underline{\nabla} \cdot \underline{v} + \underline{v} \cdot \underline{\nabla}\phi \tag{2}$$

In order to write the equation in index notation, starting from left hand side we have:

$$\underline{\nabla} \cdot (\phi \underline{v}) = (\frac{\partial}{\partial x_i}\ \underline{e}_i) \cdot (\phi\ v_j \underline{e}_j) = \frac{\partial\ \phi v_j}{\partial x_i}(\underline{e}_i \cdot \underline{e}_j) = \frac{\partial\ \phi v_j}{\partial x_i}\delta_{ij} \tag{3}$$

We know that δ_{ij} is only non zero when $i = j$, therefore:

$$\frac{\partial \phi v_j}{\partial x_i}\delta_{ij} = \frac{\partial \phi v_i}{\partial x_i} \tag{4}$$

where i is the summation variable. Then for the first term on the right hand side, following the same method as above:

$$\phi\ \underline{\nabla} \cdot \underline{v} = \phi\ (\frac{\partial}{\partial x_i}\underline{e}_i) \cdot v_j \underline{e}_j = \phi\ \frac{\partial v_j}{\partial x_i}(\underline{e}_i \cdot \underline{e}_j) = \phi\ \frac{\partial v_j}{\partial x_i}\delta_{ij} = \phi\ \frac{\partial v_i}{\partial x_i} \tag{5}$$

And for the second term:

$$\underline{v} \cdot \underline{\nabla}\phi = v_i\ \underline{e}_i \cdot \frac{\partial \phi}{\partial x_j}\underline{e}_j = v_i\ \frac{\partial \phi}{\partial x_j}\underline{e}_i \cdot \underline{e}_j = v_i\ \frac{\partial \phi}{\partial x_j}\delta_{ij} = v_i\ \frac{\partial \phi}{\partial x_i} \tag{6}$$

And thus equation 1 in index notation has a form of:

$$\frac{\partial \phi v_i}{\partial x_i} = \phi \frac{\partial v_i}{\partial x_i} + v_i \frac{\partial \phi}{\partial x_i} \tag{7}$$

Now in order to prove equation 1, we start from the left hand side of equation 7 and use the chain rule to open the derivative:

$$\frac{\partial \phi v_i}{\partial x_i} = \phi \frac{\partial v_i}{\partial x_i} + v_i \frac{\partial \phi}{\partial x_i} \tag{8}$$

Multiplying the last term by $\delta_{ii} = 1$:

$$\phi \frac{\partial v_i}{\partial x_i} + v_i \frac{\partial \phi}{\partial x_i}\delta_{ii} = \phi \frac{\partial v_i}{\partial x_i} + v_i \frac{\partial \phi}{\partial x_i}(\underline{e}_i \cdot \underline{e}_i) = \phi \frac{\partial v_i}{\partial x_i} + v_i \underline{e}_i \cdot \frac{\partial \phi}{\partial x_i}\underline{e}_i \tag{9}$$

By multiplying the last term with $\delta_{ii} = 1$ we are able to get to the dot product of two vector quantities. Also we know that in index notation:

$$\frac{\partial v_i}{\partial x_i} = \underline{\nabla} \cdot \underline{v} \tag{10}$$

$$\frac{\partial \phi}{\partial x_i}\underline{e}_i = \underline{\nabla}\phi \tag{11}$$

Therefore substituting back from equations 10 and 11 into 9 we get:

$$\underline{\nabla} \cdot (\phi \underline{v}) = \frac{\partial \phi v_i}{\partial x_i} = \phi \frac{\partial v_i}{\partial x_i} + v_i \underline{e}_i \cdot \frac{\partial \phi}{\partial x_i}\underline{e}_i = \phi \underline{\nabla} \cdot \underline{v} + \underline{v} \cdot \underline{\nabla}\phi \tag{12}$$

and equation 1 is proved.

b.

$$div(\underline{u} \times \underline{v}) = \underline{v} \cdot curl\underline{u} - \underline{u} \cdot curl\underline{v} \tag{13}$$

Equation 13 can be written in Gibbs notation as:

$$\underline{\nabla} \cdot (\underline{u} \times \underline{v}) = \underline{v} \cdot \underline{\nabla} \times \underline{u} - \underline{u} \cdot \underline{\nabla} \times \underline{v} \tag{14}$$

In index notation, the left hand side can be written as:

$$\underline{\nabla} \cdot (\underline{u} \times \underline{v}) = (\frac{\partial}{\partial x_i}\underline{e}_i) \cdot u_j v_k \varepsilon_{jkl}\ \underline{e}_l = (\frac{\partial u_j v_k}{\partial x_i})\ \varepsilon_{jkl}\ \underline{e}_i \cdot \underline{e}_l = \frac{\partial u_j v_k}{\partial x_i}\ \varepsilon_{jkl}\ \delta_{il} \tag{15}$$

δ_{il} is only non zero when $i = l$. Thus:

$$\frac{\partial u_j v_k}{\partial x_i}\varepsilon_{jkl}\delta_{il} = \frac{\partial u_j v_k}{\partial x_i}\varepsilon_{jki} \tag{16}$$

Therefore:

$$\underline{\nabla} \cdot (\underline{u} \times \underline{v}) = \frac{\partial u_j v_k}{\partial x_i}\varepsilon_{jki} \tag{17}$$

As for the right hand side:

$$\begin{gathered} \underline{v} \cdot \underline{\nabla} \times \underline{u} = \\ v_i \underline{e}_i \cdot \frac{\partial}{\partial x_j}\underline{e}_j \times u_k \underline{e}_k = \\ v_i \frac{\partial u_k}{\partial x_j}\ \underline{e}_i \cdot (\underline{e}_j \times \underline{e}_k) = \\ v_i \frac{\partial u_k}{\partial x_j}\ \underline{e}_i \cdot \varepsilon_{jkl}\underline{e}_l = \\ v_i \frac{\partial u_k}{\partial x_j}\ \varepsilon_{jkl}\ \delta_{il} = \\ v_i \frac{\partial u_k}{\partial x_j}\ \varepsilon_{jki} \end{gathered} \tag{18}$$

$$\begin{gathered} \underline{u} \cdot \underline{\nabla} \times \underline{v} = \\ u_i \underline{e}_i \cdot \frac{\partial}{\partial x_j}\underline{e}_j \times v_k \underline{e}_k = \\ u_i \underline{e}_i \cdot \frac{\partial v_k}{\partial x_j}(\underline{e}_j \times \underline{e}_k) = \\ u_i \frac{\partial v_k}{\partial x_j}\ \varepsilon_{jkl}\underline{e}_i \cdot \underline{e}_l = \\ u_i \frac{\partial v_k}{\partial x_j}\ \varepsilon_{jkl}\ \delta il = \\ u_i \frac{\partial v_k}{\partial x_j}\ \varepsilon_{jki} \end{gathered} \tag{19}$$

Therefore, equation 13 in index notation is written as:

$$\frac{\partial u_j v_k}{\partial x_i}\varepsilon_{jki} = v_i \frac{\partial u_k}{\partial x_j}\ \varepsilon_{jki} - u_i \frac{\partial v_k}{\partial x_j}\ \varepsilon_{jki} \tag{20}$$

In order to prove equation 13, we start from the left hand side and use the chain rule to open the derivatives:

$$\underline{\nabla} \cdot (\underline{u} \times \underline{v}) = \frac{\partial u_j v_k}{\partial x_i}\ \varepsilon_{jki} = \frac{\partial u_j}{\partial x_i}\ \varepsilon_{jki} v_k + \frac{\partial v_k}{\partial x_i}\ \varepsilon_{jki} u_j \tag{21}$$

Then multiplying the first term by δ_{kk} and the second term by δ_{jj} ($\delta_{kk} = \delta_{jj} = 1$ does not add anything to the equation, however it helps in creating the dot product needed to prove the equations) :

$$\frac{\partial u_j}{\partial x_i}\ \varepsilon_{jki} v_k + \frac{\partial v_k}{\partial x_i}\ \varepsilon_{jki} u_j = \frac{\partial u_j}{\partial x_i}\ \varepsilon_{jki} v_k \delta_{kk} + \frac{\partial v_k}{\partial x_i}\ \varepsilon_{jki} u_j \delta_{jj} \tag{22}$$

We know that $\varepsilon_{jki} = -\varepsilon_{ikj}$. Thus:

$$\begin{aligned} \frac{\partial u_j}{\partial x_i}\ \varepsilon_{jki} v_k \delta_{kk} + \frac{\partial v_k}{\partial x_i}\ \varepsilon_{jki} u_j \delta_{jj} &= \\ \frac{\partial u_j}{\partial x_i}\ \varepsilon_{ijk} v_k \delta_{kk} - \frac{\partial v_k}{\partial x_i}\ \varepsilon_{ikj} u_j \delta_{jj} &= \\ \frac{\partial u_j}{\partial x_i}\ \varepsilon_{jki} v_k\ \underline{e}_k \cdot \underline{e}_k - \frac{\partial v_k}{\partial x_i}\ \varepsilon_{ikj} u_j\ \underline{e}_j \cdot \underline{e}_j &= \\ \frac{\partial u_j}{\partial x_i}\ \varepsilon_{jki}\ \underline{e}_k \cdot v_k \underline{e}_k - \frac{\partial v_k}{\partial x_i}\ \varepsilon_{ikj} \underline{e}_j \cdot u_j \underline{e}_j & \end{aligned} \tag{23}$$

From the definition of curl in index notation we know:

$$\underline{\nabla} \times \underline{u} = \frac{\partial u_j}{\partial x_i}\ \varepsilon_{ijk} \underline{e}_k \tag{24}$$

and

$$\underline{\nabla} \times \underline{v} = \frac{\partial v_k}{\partial x_i}\ \varepsilon_{ikj} \underline{e}_j \tag{25}$$

Therefore,

$$\frac{\partial u_j}{\partial x_i}\ \varepsilon_{jki}\ \underline{e}_k \cdot v_k \underline{e}_k - \frac{\partial v_k}{\partial x_i}\ \varepsilon_{ikj} \underline{e}_j \cdot u_j \underline{e}_j = \tag{26}$$

$$(\underline{\nabla} \times \underline{u}) \cdot \underline{v} - (\underline{\nabla} \times \underline{v}) \cdot \underline{u} \tag{27}$$

Since dot product is commutative, we can rewrite equation 27 as:

$$(\underline{\nabla} \times \underline{u}) \cdot \underline{v} - (\underline{\nabla} \times \underline{v}) \cdot \underline{u} = \underline{v} \cdot (\underline{\nabla} \times \underline{u}) - \underline{u} \cdot (\underline{\nabla} \times \underline{v}) \tag{28}$$

And thus equation 13 is proved.

c.

$$curl(\underline{u} \times \underline{v}) = \underline{v} \cdot grad\ \underline{u} - \underline{u} \cdot grad\ \underline{v} + \underline{u} \cdot div\ \underline{v} - \underline{v} \cdot div\ \underline{u} \tag{29}$$

Equation 29 in Gibbs notation is presented as:

$$\underline{\nabla} \times (\underline{u} \times \underline{v}) = \underline{v} \cdot \underline{\nabla}\ \underline{u} - \underline{u} \cdot \underline{\nabla}\ \underline{v} + \underline{u}\ \underline{\nabla} \cdot \underline{v} - \underline{v}\ \underline{\nabla} \cdot \underline{u} \tag{30}$$

For the index notation, starting from the left hand side of equation 29:

$$\begin{aligned} \underline{\nabla} \times (\underline{u} \times \underline{v}) = \frac{\partial}{\partial x_i} \underline{e}_i \times (u_j \underline{e}_j \times v_k \underline{e}_k) &= \\ \frac{\partial u_j v_k}{\partial x_i}\ \underline{e}_i \times (\underline{e}_j \times \underline{e}_k) &= \\ \frac{\partial u_j v_k}{\partial x_i}\ \underline{e}_i \times (\varepsilon_{jkl}\ \underline{e}_l) &= \\ \frac{\partial u_j v_k}{\partial x_i}\ \varepsilon_{jkl}\ \underline{e}_i \times \underline{e}_l &= \\ \frac{\partial u_j v_k}{\partial x_i}\ \varepsilon_{jkl} \varepsilon_{ilm}\ \underline{e}_m & \end{aligned} \tag{31}$$

For the first term on the right hand side we have:

$$\begin{gathered} \underline{v} \cdot \underline{\nabla}\ \underline{u} = \\ v_i \underline{e}_i \cdot (\frac{\partial u_j}{\partial x_k} \underline{e}_j \underline{e}_k) = \\ v_i\ \partial \frac{\partial u_j}{\partial x_k}\ \underline{e}_i \cdot (\underline{e}_j \underline{e}_k) \end{gathered} \tag{32}$$

Using the identity $\underline{c} \cdot (\underline{a}\ \underline{b}) = (\underline{a} \cdot \underline{b})\underline{c}$:

$$\begin{gathered} v_i\ \partial \frac{\partial u_j}{\partial x_k}\ \underline{e}_i \cdot (\underline{e}_j\ \underline{e}_k) = \\ v_i\ \partial \frac{\partial u_j}{\partial x_k} (\underline{e}_i \cdot \underline{e}_j)\ \underline{e}_k = \\ v_i\ \partial \frac{\partial u_j}{\partial x_k}\ \delta_{ij} \underline{e}_k = \\ v_i\ \partial \frac{\partial u_i}{\partial x_k}\ \underline{e}_k \end{gathered} \tag{33}$$

Same with the rest of the terms:

$$\begin{gathered} \underline{v} \cdot \underline{\nabla}\ \underline{u} = \\ u_i\ \underline{e}_i \cdot (\frac{\partial v_j}{\partial x_k}\ \underline{e}_j \underline{e}_k) = \\ u_i\ \frac{\partial v_j}{\partial x_k}\ \underline{e}_i \cdot (\underline{e}_j \underline{e}_k) = \\ u_i\ \frac{\partial v_j}{\partial x_k} (\underline{e}_i \cdot \underline{e}_j)\ \underline{e}_k = \\ u_i\ \frac{\partial v_i}{\partial x_k}\ \underline{e}_k \end{gathered} \tag{34}$$

$$\begin{gathered} \underline{u}\ \underline{\nabla} \cdot \underline{v} = \\ u_i\ \underline{e}_i (\frac{\partial}{\partial x_j}\ \underline{e}_j \cdot v_k \underline{e}_k) = \\ u_i\ \frac{\partial v_k}{\partial x_j}\ \underline{e}_i (\underline{e}_j \cdot \underline{e}_k) = \\ u_i\ \frac{\partial v_k}{\partial x_j}\ \underline{e}_i \delta jk = \\ u_i\ \frac{\partial v_j}{\partial x_j}\ \underline{e}_i \end{gathered} \tag{35}$$

$$\begin{gathered} \underline{v}\ \underline{\nabla} \cdot \underline{u} = \\ v_i\ \underline{e}_i (\frac{\partial}{\partial x_j}\ \underline{e}_j \cdot v_k \underline{e}_k) = \\ v_i\ \frac{\partial u_k}{\partial x_j}\ \underline{e}_i (\underline{e}_j \cdot \underline{e}_k) = \\ v_i\ \frac{\partial u_k}{\partial x_j}\ \underline{e}_i \delta jk = \\ v_i\ \frac{\partial u_j}{\partial x_j}\ \underline{e}_i \end{gathered} \tag{36}$$

Thus equation 29 in index notation is given as :

$$\frac{\partial u_j v_k}{\partial x_i}\ \varepsilon_{jkl}\varepsilon_{ilm}\ \underline{e}_m = v_i\partial\frac{\partial u_i}{\partial x_k}\ \underline{e}_k - u_i\frac{\partial v_i}{\partial x_k}\ \underline{e}_k + u_i\frac{\partial v_j}{\partial x_j}\underline{e}_i - v_i\frac{\partial u_j}{\partial x_j}\underline{e}_i \tag{37}$$

In order to prove this equation, starting from the left hand side (equation 31), we use the identity shown in equation 3.3.5 in Panton to change the alternating tensor into the Kronecker delta. Also, using the fact that $\varepsilon_{ilm} = \varepsilon_{lmi}$ and $\varepsilon_{jkl} = \varepsilon_{ljk}$ we have:

$$\begin{gathered}\frac{\partial u_j v_k}{\partial x_i}\ \varepsilon_{jkl}\varepsilon_{ilm}\ \underline{e}_m = \\ \frac{\partial u_j v_k}{\partial x_i}\ \varepsilon_{ljk}\varepsilon_{lmi}\ \underline{e}_m = \\ \frac{\partial u_j v_k}{\partial x_i}(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij})\ \underline{e}_m = \\ (u_j\frac{\partial v_k}{\partial x_i} + v_k\frac{\partial u_j}{\partial x_i})(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij})\ \underline{e}_m = \\ (u_j\frac{\partial v_k}{\partial x_i}\delta_{mj}\delta_{ik} - u_j\frac{\partial v_k}{\partial x_i}\delta_{mk}\delta_{ij} + v_k\frac{\partial u_j}{\partial x_i}\delta_{mj}\delta_{ik} - v_k\frac{\partial u_j}{\partial x_i}\delta_{mk}\delta_{ij})\ \underline{e}_m\end{gathered} \tag{38}$$

The first term in equation 38 is only non zero when $m = j$ and $i = k$.

$$\begin{gathered}u_j\frac{\partial v_k}{\partial x_i}\delta_{mj}\delta_{ik}\ \underline{e}_m = \\ u_j\frac{\partial v_i}{\partial x_i}\ \underline{e}_j = \\ u_i\ \underline{e}_j\frac{\partial v_i}{\partial x_i} = \\ \underline{u}\ \underline{\nabla}\cdot\underline{v}\end{gathered} \tag{39}$$

The second term is only non zero when $m = k$ and $i = j$:

$$\begin{gathered}-u_j\frac{\partial v_k}{\partial x_i}\delta_{mk}\delta_{ij}\ \underline{e}_m = \\ -u_i\frac{\partial v_k}{\partial x_i}\ \underline{e}_k\delta_{ii} = \\ -u_i\frac{\partial v_k}{\partial x_i}\ \underline{e}_k(\underline{e}_i\cdot\underline{e}_i) = \\ -u_i\frac{\partial v_k}{\partial x_i}\ \underline{e}_i\cdot(\underline{e}_k\ \underline{e}_i) = \\ -u_i\underline{e}_i\cdot\frac{\partial v_k}{\partial x_i}\underline{e}_k\ \underline{e}_i = \\ -\underline{u}\cdot\underline{\nabla}\ \underline{v}\end{gathered} \tag{40}$$

Here we used the identity $\underline{c}\cdot(\underline{a}\ \underline{b}) = (\underline{a}\cdot\underline{b})\ \underline{c}$ to change $\underline{e}_k\ (\underline{e}_i\cdot\underline{e}_i)$ into the dot product of a vector and a tensor. The third term is only non zero when $m = j$ and $i = k$:

$$
\begin{aligned}
v_k \frac{\partial u_j}{\partial x_i} \delta_{mj} \delta_{ik} \, \underline{e}_m &= \\
v_i \frac{\partial u_j}{\partial x_i} \, \underline{e}_j \delta_{ii} &= \\
v_i \frac{\partial u_j}{\partial x_i} \, \underline{e}_j (\underline{e}_i \cdot \underline{e}_i) &= \\
v_i \frac{\partial u_j}{\partial x_i} \, \underline{e}_i \cdot (\underline{e}_j \, \underline{e}_i) &= \\
v_i \, \underline{e}_i \cdot \frac{\partial u_j}{\partial x_i} \underline{e}_j \, \underline{e}_i &= \\
\underline{v} \cdot \underline{\nabla} \, \underline{u} &
\end{aligned} \tag{41}
$$

And the fourth term is only non zero when $m = k$ and $i = j$:

$$
\begin{aligned}
-v_k \frac{\partial u_j}{\partial x_i} \delta_{mk} \delta_{ij} \, \underline{e}_m &= \\
-v_k \frac{\partial u_i}{\partial x_i} \, \underline{e}_k &= \\
-v_k \, \underline{e}_k \frac{\partial u_i}{\partial x_i} &= \\
-\underline{v} \, \underline{\nabla} \cdot \underline{u} &
\end{aligned} \tag{42}
$$

Putting equations 39, 40, 41 and 42 back into equation 38 and equating it with equation 31, we get:

$$
\underline{\nabla} \times (\underline{u} \times \underline{v}) = \underline{v} \cdot \underline{\nabla} \, \underline{u} - \underline{u} \cdot \underline{\nabla} \, \underline{v} + \underline{u} \, \underline{\nabla} \cdot \underline{v} - \underline{v} \, \underline{\nabla} \cdot \underline{u} \tag{43}
$$

Therefore, equation 29 is proved.

□

Problem Solution by Shabnam Raayai, Fall 2013