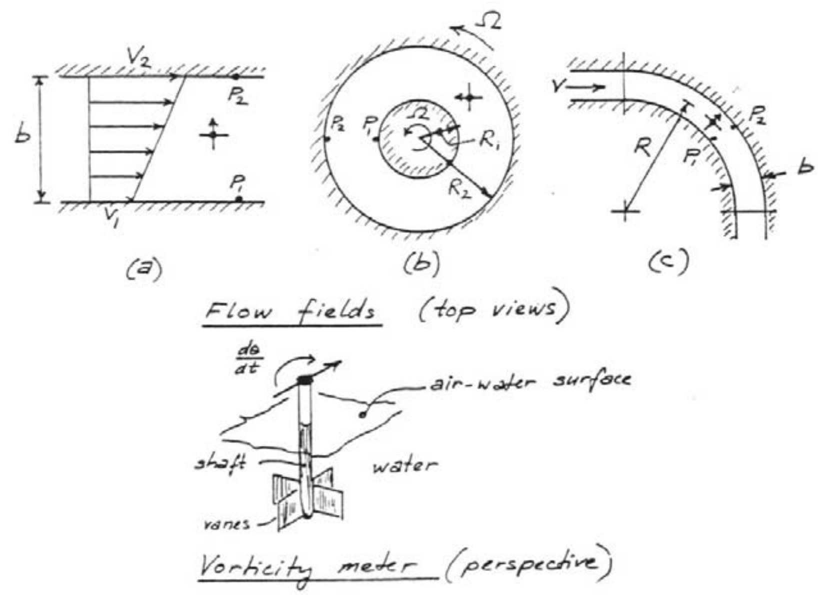
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 10.3

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



Consider the three different, steady, two dimensional incompressible flow fields illustrated in Figures (a), (b), and (c) above. The flows represent

- (a) a parallel flow in a channel, with a linear velocity profile,
- (b) a flow in an annulus, with the liquid rotating like a solid body in unison with its bounding walls at an angular speed Ω, and
- (c) an inviscid flow in a bend of width b small compared with the mean radius of curvature R. the flow being uniform at a speed V prior to entering the bend.

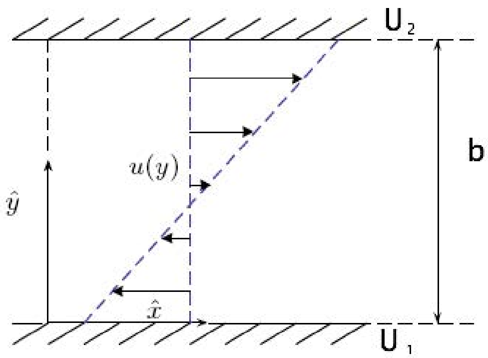
A primitive 'vorticity meter' made of two perpendicular vanes mounted on a shaft, as sketched, is inserted into each of these flows, oriented so that its shaft with the arrow on top is pointing upwards from the paper. The 'vorticity meter' is neutrally buoyant and its axis moves with the bulk motion of the fluid in which it is immersed.

You are to estimate, for each of the cases (a) to (c),

- (i) the clockwise angular rate of rotation $\frac{d\theta}{dt}$ of the 'vorticity meter' and
- (ii) the pressure difference $p_2 - p_1$

In case (c) do this for a point in the middle of the bend.

Solution:



- (a) First, let's calculate the vorticity, in general it is

$$\underline{\omega} = \underline{\hat{x}}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \underline{\hat{y}}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \underline{\hat{z}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{10.3a}$$

which in the $2D$ case is,

$$\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{10.3b}$$

which can be interpreted as twice the averaged rotation of the local axis,

$$\omega = \frac{d\theta_1}{dt} + \frac{d\theta_2}{dt} = 2\Omega_{local}, \tag{10.3c}$$

or using streamlines

$$\omega_z = \left(\frac{\partial V}{\partial n} + \frac{V}{R}\right). \tag{10.3d}$$

In this case,

$$\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\frac{U_2 - U_1}{b}, \tag{10.3e}$$

then

$$\boxed{\frac{d\theta}{dt} = -\frac{U_2 - U_1}{2b}.} \tag{10.3f}$$

Now, the Euler's equations of motion in streamline coordinates are:

$$\frac{\partial}{\partial s}\left(p + \frac{1}{2}\rho v^2 + \rho g z\right) = 0, \qquad s - direction, \tag{10.3g}$$

$$\frac{\partial}{\partial r}(p + \rho g z) = \frac{\rho v^2}{r}, \qquad r - direction, \tag{10.3h}$$

$$\frac{\partial}{\partial z}(p + \rho g z) = 0, \qquad z - direction, \tag{10.3i}$$

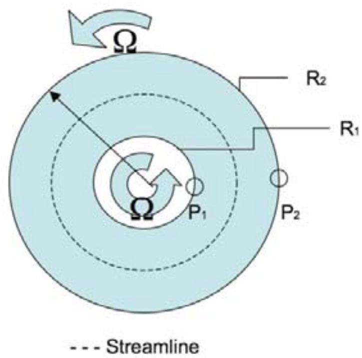
(10.3j)

then by Euler in the r direction, (in this problem $r = y$ direction),

$$\frac{\partial}{\partial y}(p) = 0, \tag{10.3k}$$

then $\boxed{p \text{ is a constant}}$ along the cross section. Notice that 'simply' using Bernoullis' equation to obtain the pressure difference is not possible, because the points do not form part of the same streamline, and therefore they have a different Bernoulli constant, B.

- (b)



Since the fluid behaves as in solid body rotation, $\underline{V} = \Omega r \hat{i}_\theta$, and,

$$\boxed{\frac{d\theta}{dt} = \Omega.} \tag{10.3l}$$

Now, using streamlines

$$\omega_z = \left(\frac{\partial V}{\partial n} + \frac{V}{R}\right) = \left(\frac{\partial V}{\partial r} + \frac{V}{r}\right) = \Omega + \Omega = 2\Omega. \tag{10.3m}$$

then, using Euler-n in cylindrical coordinates,

$$\frac{\partial p}{\partial r} = \rho\frac{V^2}{r} = \rho\Omega^2\frac{r^2}{r} = \rho\Omega^2 r, \tag{10.3n}$$

then, integrating,

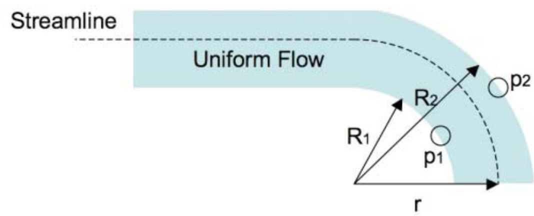
$$\int_1^2 \frac{\partial p}{\partial r} = \rho\frac{\Omega^2}{2}(R_2^2 - R_1^2), \tag{10.3o}$$

then

$$\boxed{p_2 - p_1 = \rho\frac{\Omega^2}{2}(R_2^2 - R_1^2).} \tag{10.3p}$$

Notice that even when the solution 'looks like' Bernoullis' equation, the points do not form part of the same streamline, and therefore the use of Bernoulli is unjustified.

- (c)



First, let's assume circular streamlines. Then, since $\omega_z = 0$ at the entrance, from Helmholtz or Kelvin, $\omega = 0$ at the channel bend, (neglecting the effects of the wall/viscosity). Then,

$$\boxed{\frac{d\theta}{dt} = 0.} \tag{10.3q}$$

Now, for circular streamlines,

$$\omega = \frac{\partial V}{\partial r} + \frac{V}{r} = 0, \tag{10.3r}$$

then integrating,

$$V = \frac{A}{r}, \tag{10.3s}$$

where A is an integration constant. A can be determined matching the flows across the bend and at the entrance, then,

$$Ub = \int_{R_1}^{R_2} \frac{A}{r} dr = A \ln \frac{R_2}{R_1}, \tag{10.3t}$$

then, after simplifying, $A = \frac{Ub}{\ln \frac{R_2}{R_1}}$, then,

$$V = \frac{Ub}{\ln \frac{R_2}{R_1}} \frac{1}{r}. \tag{10.3u}$$

Now, that we know more about the velocity difference in the bend, we can calculate the pressure distribution too. Using Euler-n (cylindrical coordinates).

$$\frac{\partial p}{\partial r} = \rho \frac{V^2}{r} = \rho \frac{A^2}{r^3}, \tag{10.3v}$$

then, integrating,

$$\int_1^2 \frac{\partial p}{\partial r} = \rho \frac{V^2}{r} = -\rho \frac{A^2}{2r^2}\bigg|_{R-\frac{b}{2}}^{R+\frac{b}{2}} = \frac{\rho A^2}{2}\left(\frac{1}{(R-\frac{b}{2})^2} - \frac{1}{(R+\frac{b}{2})^2}\right) \sim \frac{\rho A^2}{R^3} b, \tag{10.3w}$$

then,

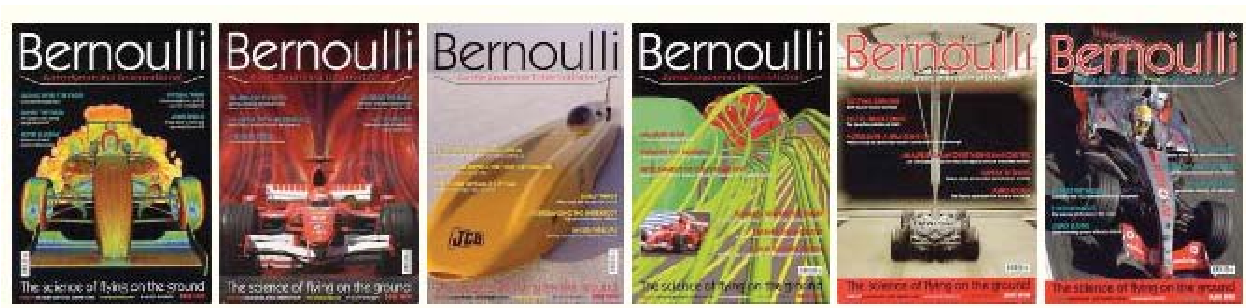
$$\boxed{p_2 - p_1 = \frac{\rho A^2}{R^3} b.} \tag{10.3x}$$

In the limit when $R \gg b$, $A \approx UR$ and then,

$$\boxed{p_2 - p_1 = \frac{\rho U^2}{R} b.} \tag{10.3y}$$

Alternatively, since the flow is inviscid and started as a uniform flow, each streamline has the same Bernoulli constant, B, and the use of Bernoulli's equation is possible in this case.

Note: Bernoulli is probably the most well known name in Fluid Mechanics (even magazines have his name), and his equation is among the most used equations in Fluid Mechanics, but is also one of the most abused. Careful thought must be performed each time the equation is used; it simplifies many problems, and gives information that otherwise might be quite hard to obtain, but quite often too, it's use is unjustified.



Bernoulli Magazine^{TM} is an specialized magazine in Car's Fluid Mechanics.

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Problem Solution by ATP/MC, Fall 2008

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