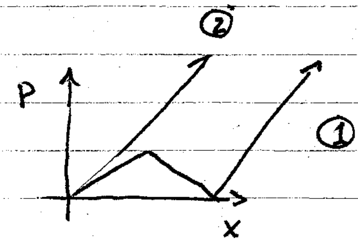
Th F
23
30
7
13 No class

$\boxed{\text{Wed. 12 @ noon}}$

Cancel Fri 7

Pick date for oral presentations

Recall confusion from last time

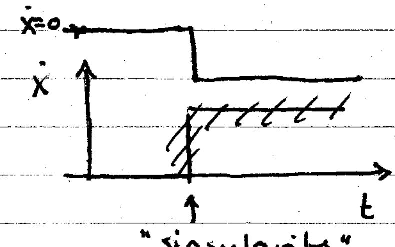


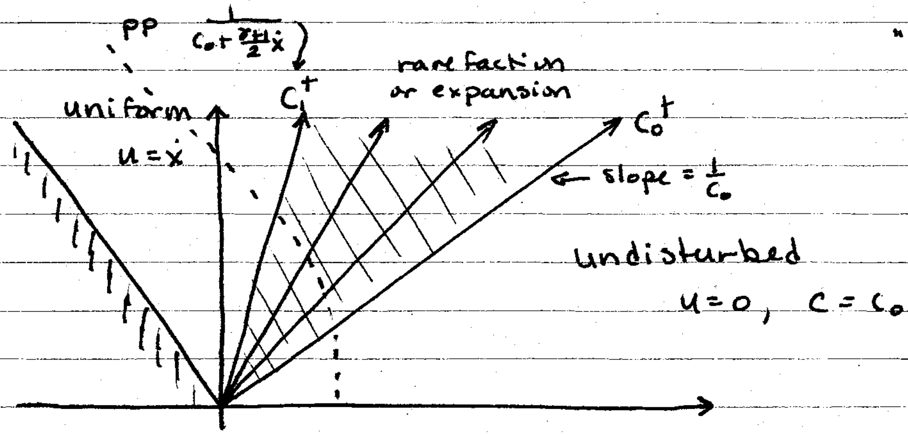
"The portion of a wave that increases the density as it passes is called a compression."

$[P] > 0$ compression shock $(P_2 - P_1) > 0$

Piston Withdrawal

$$\dot{x} = \begin{cases} 0 & t < 0 \\ \text{const} < 0 & t > 0 \end{cases}$$





singular pt.
multiple values for $\dot{x}$ ∴
multiple values for $u+c$

Recall C^+ characteristics are straight lines
Slope $= \frac{1}{u+c}$

∴ slopes of characteristics vary between

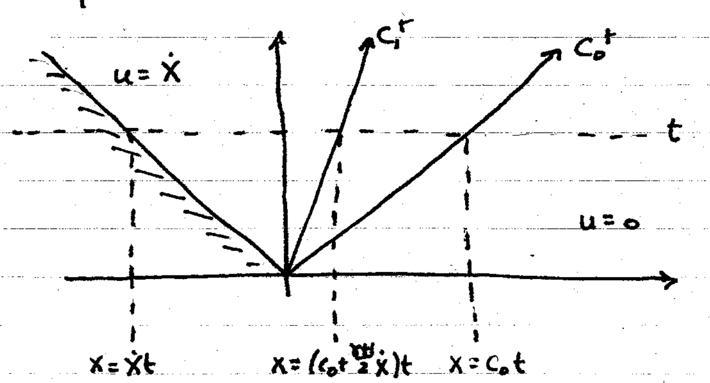
$$\frac{1}{u_0 + c_0} = \frac{1}{c_0} \quad \text{and} \quad \frac{1}{\dot{x} + c_1}$$

Recall from last time (since $J^- = \text{const}$)

$$c = c_0 + \frac{\gamma-1}{2} u$$

$$\Rightarrow \quad \frac{1}{\dot{X}+c_1} = \frac{1}{\dot{X}+c_0+\frac{\gamma-1}{2}\dot{X}} = \frac{1}{c_0 + \frac{\gamma+1}{2}\dot{X}}$$

This is all the info we need to find u and all thermodynamic quantities for all x and t.



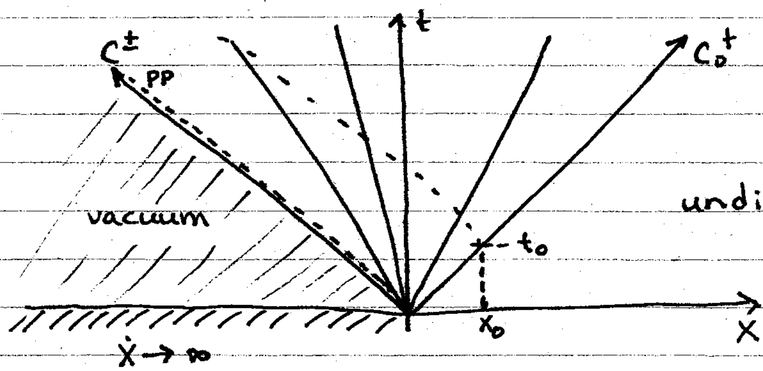
$$\boxed{\begin{array}{lll} \infty > x \geq c_0 t & \begin{cases} u = 0 \\ c = c_0 \end{cases} & \text{undisturbed} \\ c_0 t \geq x \geq (c_0 + \frac{\gamma+1}{2}\dot{X})t & \begin{cases} u = \frac{2}{\gamma+1}\left(\frac{x}{t} - c_0\right) \\ c = c_0 + \frac{\gamma-1}{\gamma+1}\left(\frac{x}{t} - c_0\right) \end{cases} & \text{expansion wave} \\ (c_0 + \frac{\gamma+1}{2}\dot{X})t \geq x \geq \dot{X}t & \begin{cases} u = \dot{X} \\ c = c_0 + \frac{\gamma-1}{2}\dot{X} \end{cases} & \text{uniform} \end{array}}$$

$$\frac{1}{\text{slope}} = \frac{x}{t} = u + c = u + c_0 + \frac{\gamma-1}{2} u \Rightarrow \frac{x}{t} = c_0 + \frac{\gamma+1}{2} u$$

$$\Rightarrow \quad u = \frac{2}{\gamma+1}\left(\frac{x}{t} - c_0\right)$$

$$c = c_0 + \frac{\gamma-1}{\cancel{2}} \frac{\cancel{2}}{\gamma+1}\left(\frac{x}{t} - c_0\right)$$

This solution is ok provided piston moves slower than u_{escape}. If $|\dot{X}| > |u_{escape}|$ then gas separates.



Recall $\quad u_{escape} = -2c_0/(\gamma-1)$

$$\frac{x}{t} = u + \cancel{c}^{\,0} = -2c_0/(\gamma-1)$$

Solution as above except:

$$\infty > x \geq c_0 t \qquad \begin{cases} u = 0 \\ c = c_0 \end{cases} \qquad \text{undisturbed}$$

$$c_0 t \geq x \geq -2c_0/(\gamma-1) \qquad \begin{cases} \text{same as expansion} \\ \text{wave above} \end{cases}$$

$$-2c_0/(\gamma-1) \geq x \geq -\infty \qquad \underline{\text{vacuum}} \quad (\rho = 0,\ P = 0)$$

Particle path follows: $\quad u = \boxed{\frac{dx}{dt} = \frac{2}{\gamma+1}\left(\frac{x}{t} - c_0\right)}$ First order ODE for $x(t)$

$$\Rightarrow \quad x = -\frac{2}{\gamma-1} c_0 t + A t^{2/(\gamma+1)}$$

need to find this const. through b.c.

Apply b.c. @ $\quad c_0 t_0 = x_0$

$$x_0 = -\frac{2}{\gamma-1} c_0 t_0 + A t_0^{2/(\gamma+1)}$$

$$x_0 + \frac{2}{\gamma-1} c_0 t_0 = A t_0^{2/(\gamma+1)} \Rightarrow A = t_0^{-2/(\gamma+1)} \left(x_0 + \frac{2}{\gamma-1} c_0 t_0 \right)$$

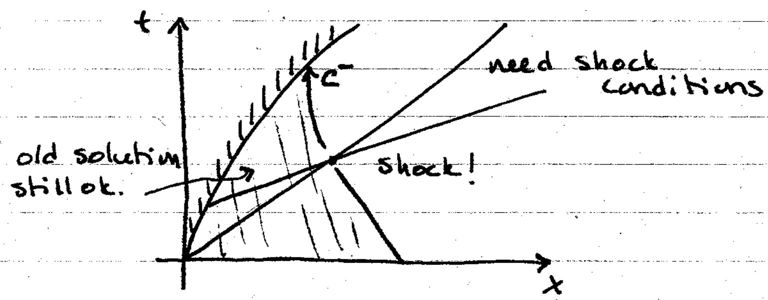
$$\Rightarrow x = -\frac{2}{\gamma-1} c_0 t + t_0^{-2/(\gamma+1)} \left(x_0 + \frac{2}{\gamma-1} c_0 t_0 \right) t^{2/(\gamma+1)}$$

$$= -\frac{2}{\gamma-1} c_0 t + \left(\frac{t}{t_0}\right)^{2/(\gamma+1)} x_0 \left(1 + \frac{2}{\gamma-1} \right)$$

$$\boxed{x = -\frac{2}{\gamma-1} c_0 t + \frac{\gamma+1}{\gamma-1} x_0 \left(\frac{c_0 t}{x_0} \right)^{2/(\gamma+1)}}$$

Continuous Piston Advance

Now $\dot{X} > 0$



Again characteristics end @ piston: $u + c = c_0 + \frac{\gamma+1}{2} \dot{X}$

$\Rightarrow$ slope decreases as $t \uparrow$ if piston is accelerating

After shock forms, need ~~shk~~ shock conditions

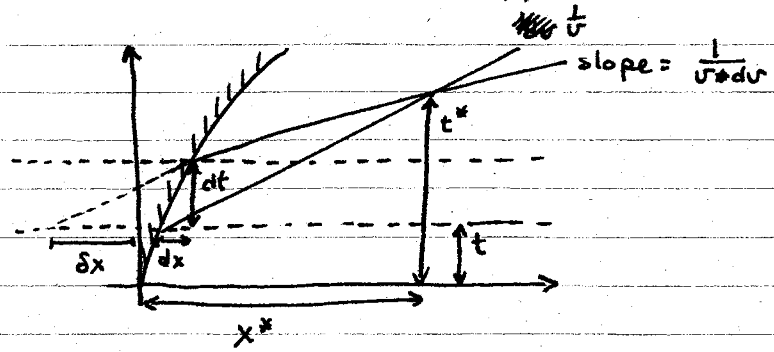
Solution plan: (1) Find if shock occurs

(2) Find when + where shock occurs

(3) Apply characteristic before shock

(4) Apply shock conditions after

Time of shock formation.



$$v = \frac{x^*}{t^*-t} \qquad v + dv = \frac{\delta x + dx}{dt} = \frac{\delta x + x^*}{t^*-t}$$

$$\cancel{v} + dv = \frac{\delta x}{t^*-t} + \cancel{v}$$

$$\Rightarrow \quad t^* - t = \frac{\delta x}{d(u+c)}$$

Need δx and $d(u+c)$

$$u + c = c_0 + \frac{\gamma+1}{2}\dot{X} \quad \Rightarrow \quad d(u+c) = \frac{\gamma+1}{2}\ddot{X}\,dt$$

$$dt\,(v + \overset{\text{small}}{\cancel{dv}}) = \delta x + dx$$

$$\delta x = v\,dt - dx = v\,dt - \dot{X}dt$$

$$= \left(c_0 + \frac{\gamma+1}{2}\dot{X} - \dot{X}\right)dt$$

$$= \left(c_0 + \frac{\gamma-1}{2}\dot{X}\right)dt$$

$$\therefore \quad t^* - t = \frac{\left(c_0 + \frac{\gamma-1}{2}\dot{X}\right)\cancel{dt}}{\frac{\gamma+1}{2}\ddot{X}\,\cancel{dt}}$$

$$\boxed{t^* = t + \frac{1}{\ddot{X}}\left(\frac{2}{\gamma+1}c_0 + \frac{\gamma-1}{\gamma+1}\dot{X}\right)}$$

Minimum @ $\frac{dt^*}{dt} = 0$

$$0 = 1 + \frac{\ddot{x}\left(\frac{\gamma-1}{\gamma+1}\ddot{x}\,\cancel{dx}\right) - \left(\frac{2}{\gamma+1}c_0 + \frac{\gamma-1}{\gamma+1}\dot{x}\right)\dddot{x}\,\cancel{dx}}{\ddot{x}^2}$$

$$1 = \frac{\left(\frac{2}{\gamma+1}c_0 + \frac{\gamma-1}{\gamma+1}\dot{x}\right)\dddot{x}}{\ddot{x}^2} - \frac{\gamma-1}{\gamma+1}$$

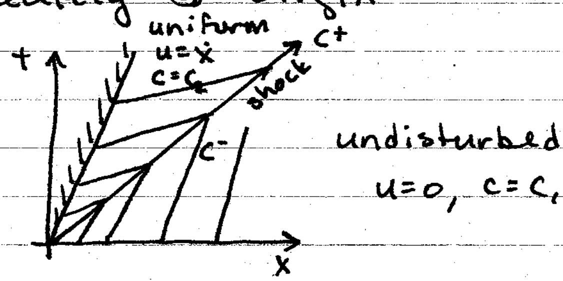
$$\frac{2\gamma}{\cancel{\gamma+1}}\ddot{x}^2 = \left(\frac{2}{\cancel{\gamma+1}}c_0 + \frac{\gamma-1}{\cancel{\gamma+1}}\dot{x}\right)\dddot{x}$$

$$\boxed{\ddot{x}^2 = \frac{1}{\gamma}\left(c_0 + \frac{\gamma-1}{2}\dot{x}\right)\dddot{x}}$$ Implicit eq. for t^*_{min}

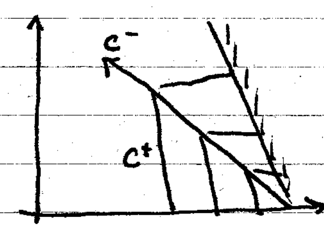
(Note: $x(t)$ is a given fn of t)

Impulsively started piston

As before, singularity (shock this time) forms immediately @ origin



(or)



Step (4) Weak Shocks

Recall $[s] \sim [p]^3 \Rightarrow \approx$ isentropic

Assume J^- (or J^+) is const across shock (then go back and check if this is true.

$$J_1^- = J_2^- \qquad J^- = u - \int dP/\rho c$$

$$u_2 - u_1 = \int_{P_1}^{P_2} \frac{dP}{\rho c} \qquad \left(\frac{\partial v}{\partial P}\right)_s = -\frac{1}{\rho^2 c^2}$$

$$u_2 - u_1 = \int_{P_1}^{P_2} \sqrt{-\left(\frac{\partial v}{\partial P}\right)_s}\, dP \qquad (*)$$

skip

Taylor series:

$$\underset{@\,2}{\left(\frac{\partial v}{\partial P}\right)_s} = \underset{@\,1}{\left(\frac{\partial v}{\partial P}\right)_s} + \left(\frac{\partial^2 v}{\partial P^2}\right)_s (P-P_1) + \frac{1}{2}\left(\frac{\partial^3 v}{\partial P^3}\right)_s (P-P_1)^2 + \ldots$$

Put this in $(*)$ (approx $\sqrt{\ }$) and integrate

$$\frac{u_2-u_1}{c_1} = \underset{\frac{[P]}{\rho_1 c_1}}{\Pi} - \frac{\Gamma_1}{2}\Pi^2 - \frac{1}{6}\left[\Gamma_1^2 + \frac{c_1^6}{2v_1^4}\left(\frac{\partial^3 v}{\partial P^3}\right)_s\right]\Pi^3 + \ldots$$

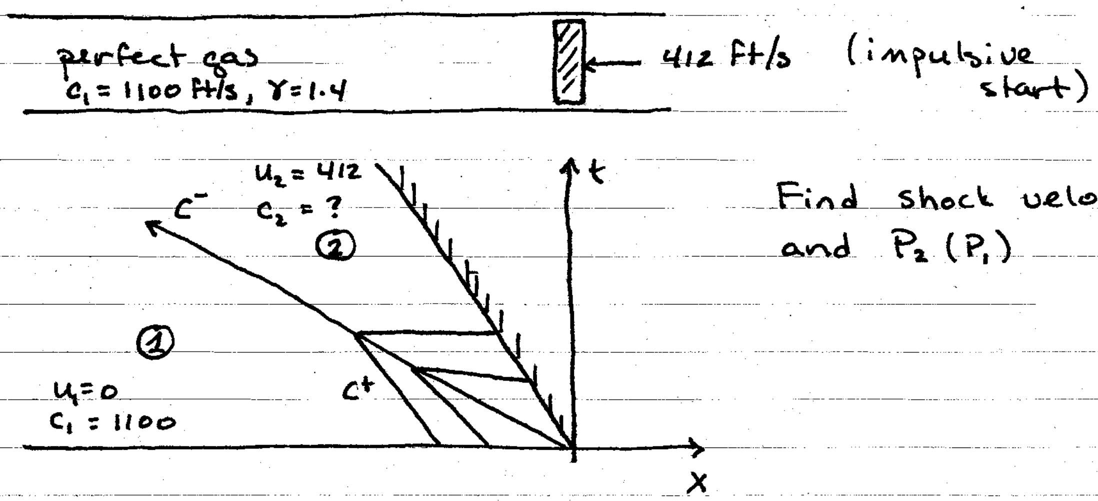
Using Taylor series, we can show that

$$[J^-] = \mathcal{O}(\Pi^3)$$ (same $\mathcal{O}$ as entropy! Treat as const across shock.)

Also follows that the shock velocity is approximately the average of the upstream and downstream wave velocities

$$V_{shock} = \frac{1}{2}\left[(u_1+c_1)+(u_2+c_2)\right] + \underbrace{c_1 E}_{\text{error}}\ \mathcal{O}(\Pi^2)$$

Example



Find shock velocity and $P_2(P_1)$

Assume shock is weak (and check later that this is not violated.)

$$[J^+] = 0$$

$$u_2 + \frac{2}{\gamma-1} c_2 = \cancelto{0}{u_1} + \frac{2}{\gamma-1} c_1$$

$u_2 = -412$ ft/s

$$c_2 = 1100 \text{ f/s} + 412 \text{ ft/s} \left(\frac{\gamma-1}{2}\right) = \boxed{1182 \text{ ft/s}}$$

Shock vel. is (approximately) average of $(u-c)$ on both sides

$$V_{shock} = \frac{1}{2}\left[(-412-1182)+(0-1100)\right] = \boxed{-1347 \text{ ft/s}}$$

Isentropic perfect gas:

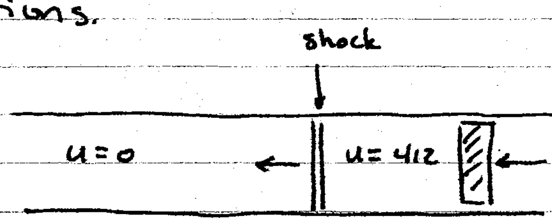
$$\frac{P_2}{P_1} = \left(\frac{v_1}{v_2}\right)^{\gamma} \qquad c^2 = \gamma P v$$

$$\frac{P_2}{P_1} = \left(\frac{c_1^2 \, \cancel{\gamma_2} P_2}{\cancel{\gamma_1} P_1 \, c_2^2}\right)^{\gamma} \Rightarrow \left(\frac{P_2}{P_1}\right)^{1-\gamma} = \left(\frac{c_1}{c_2}\right)^{2\gamma}$$

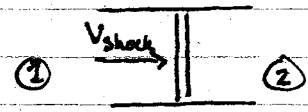
$$\Rightarrow \left(\frac{P_2}{P_1}\right) = \left(\frac{c_2}{c_1}\right)^{2\gamma/(\gamma-1)}$$

$$P_2 = P_1 \left(\frac{1182}{1100}\right)^{2.8/0.4} = \boxed{1.654 \, P_1}$$

But we could have also found this using normal shock relations.



Move to frame where shock is stationary



$$M_{in} = \frac{w_1}{c_1} = \frac{V_{shock}}{c_1}$$

$$-\frac{[w]}{c_1} = \frac{412}{1100} = 0.37455 \Rightarrow M_{in} = 1.25$$

from tables

$$V = -1.25\,(1100\ \text{ft/s}) = \boxed{-1375\ \text{ft/s}}$$

$$\frac{P_2}{P_1} = \boxed{1.656}$$

from tables

Is weak approx ok? check that $\Pi \ll 1$

$$\Pi = \frac{[P]}{\gamma P_1} = \frac{(1.656-1)\cancel{P_1}}{1.4\,\cancel{P_1}} = 0.469$$ (approx. is borderline ok...)