An Application Using Complex Numbers

Example of Programming with Complex Numbers

Conformal Mapping of a Circle into an Airfoil

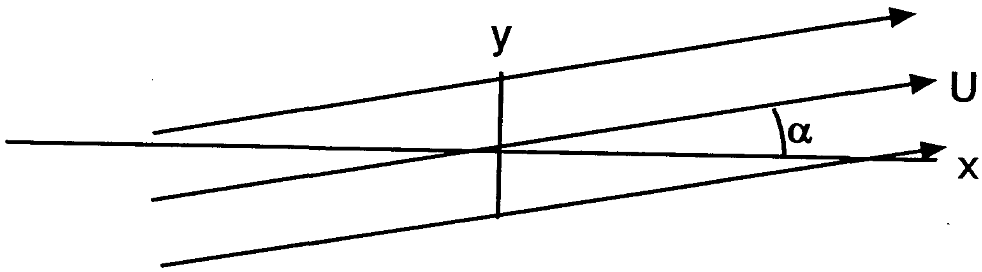
2D Flow: ϕ is velocity potential, ψ is stream function.

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Complex Numbers

$$z = x + iy \qquad \Phi = \phi + i\psi \qquad \frac{d\Phi}{dz} = u - iv$$

Simple Example



$$u = U \cos \alpha \qquad v = U \sin \alpha$$

$$\phi = Ux \cos \alpha + Uy \sin \alpha \qquad \psi = Uy \cos \alpha - Ux \sin \alpha$$

$$\Phi = \phi + i\psi = Ux \cos \alpha + Uy \sin \alpha + iUy \cos \alpha - iUx \sin \alpha$$

$$\frac{\partial \Phi}{\partial x} = U \cos \alpha - iU \sin \alpha = u - iv$$

$$\frac{\partial \Phi}{\partial (iy)} = \frac{1}{i}\frac{\partial \Phi}{\partial y} = -i\frac{\partial \Phi}{\partial y} = -iU \sin \alpha + U \cos \alpha = u - iv$$

Now we map a circle in the z-plane to an airfoil in the ζ-plane.

Streamlines in z-plane map into streamlines in ζ-plane.

The circle is a streamline in the z-plane and the airfoil is a streamline in the ζ-plane.

$$(u - iv)_\zeta = \frac{d\Phi}{d\zeta} = \frac{d\Phi/dz}{d\zeta/dz} = \frac{(u - iv)_z}{d\zeta/dz}$$

The Karman-Trefftz mapping function is:

$$\zeta = \lambda a \frac{(z+a)^\lambda + (z-a)^\lambda}{(z+a)^\lambda - (z-a)^\lambda}$$

λ and a are real numbers and $\lambda > 1$.

$$\frac{d\zeta}{dz} = 4\lambda^2 a^2 \frac{(z-a)^{\lambda-1}(z+a)^{\lambda-1}}{[(z+a)^\lambda - (z-a)^\lambda]^2}$$

For large z,

$$\zeta = \lambda a \frac{(z^\lambda + a\lambda z^{\lambda-1} + ...) + (z^\lambda - a\lambda z^{\lambda-1} + ...)}{(z^\lambda + a\lambda z^{\lambda-1} + ...) - (z^\lambda - a\lambda z^{\lambda-1} + ...)}$$

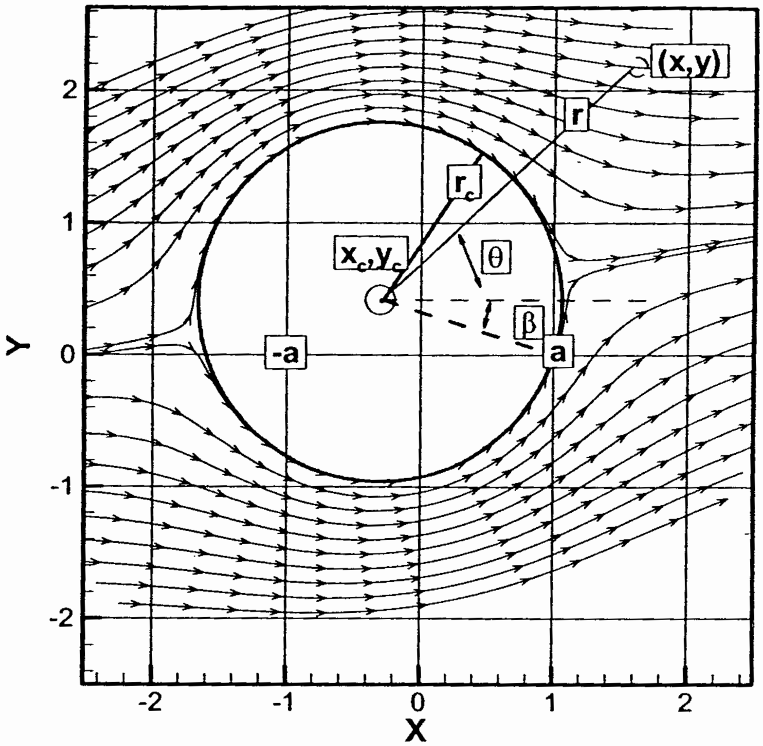
$$\zeta = \frac{\lambda a 2 z^\lambda + ...}{2\lambda z^{\lambda-1} a + ...} = z + ...$$

Far field flow in z-plane is equal to far field flow in ζ-plane.

$d\zeta/dz = 0$ at $z = a$ and at $z = -a$. If either of these points are in the flow field, $u - iv$ must equal zero there to avoid infinite velocity in ζ-plane.

Approach

Locate circle so that $z = -a$ is inside it.
Locate circle so that $z = a$ is on circle and $u - iv$ there is zero. $z = a$ maps into the trailing edge of the airfoil and since $d\zeta/dz = 0$ there it can be sharp.



Flow around a circle with zero circulation. The center of the circle is located at $x = -.3, y = 0.4$. The circle passes through $x = a = 1.0$. The flow angle of attack is 10 degrees.

The inflow angle is $\alpha = 10$ degrees, the circle radius is $r_c = \sqrt{1.3^2 + 0.4^2} = 1.3602$ and the flow is:

$$u = U\cos\alpha - U\left(\frac{r_c}{r}\right)^2 \cos(2\theta - \alpha)$$

$$v = U\sin\alpha - U\left(\frac{r_c}{r}\right)^2 \sin(2\theta - \alpha)$$

This flow is not zero at $z = a$.

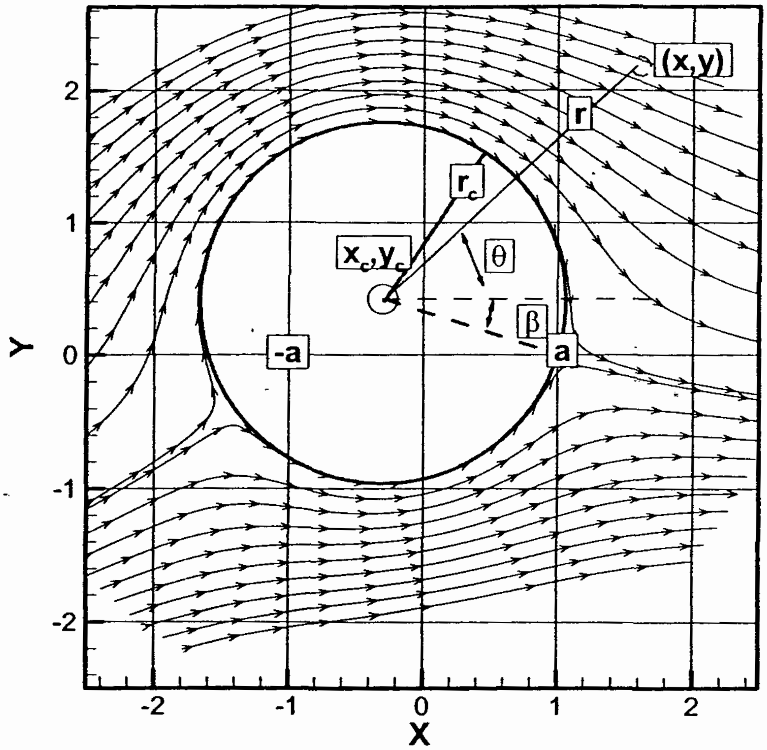
To make the flow zero at $z = a$ add circulation Γ

$$\Gamma = 4\pi r_c U \sin(-\beta - \alpha) \qquad \beta = \sin^{-1} \frac{y_c}{r_c}$$

Then:

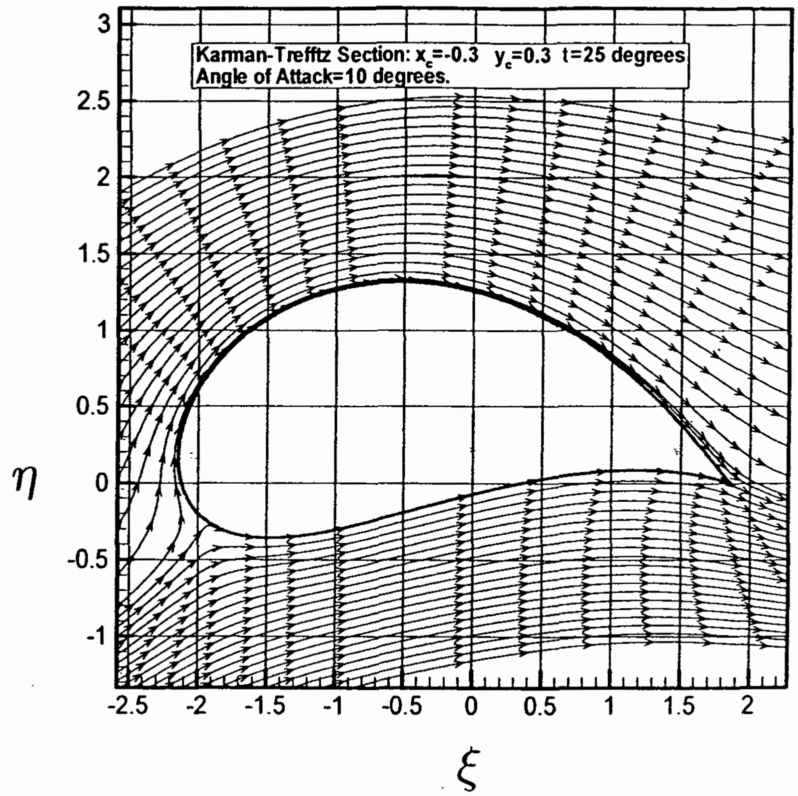
$$u = U \cos\alpha - U \left(\frac{r_c}{r}\right)^2 \cos(2\theta - \alpha) - \frac{\Gamma}{2\pi r} \sin\theta$$

$$v = U \sin\alpha - U \left(\frac{r_c}{r}\right)^2 \sin(2\theta - \alpha) + \frac{\Gamma}{2\pi r} \cos\theta$$



Flow around a circle with circulation. The center of the circle is located at $x = -.3, y = 0.4$. The circle passes through $x = a = 1.0$. Note that the rear stagnation point has moved to $x = a$.

cvle



The circle maps into an airfoil shape. The included angle , τ (in degrees) at the tail is:

$$\tau = 180(2 - \lambda)$$

The Pressure Distribution

$$P - P_\infty = \frac{1}{2}\rho U^2 - \frac{1}{2}\rho q^2 \qquad q^2 = u^2 + v^2$$

$$C_p = \frac{P - P_\infty}{1/2\rho U^2} = 1 - \left(\frac{q}{U}\right)^2$$

CVIF

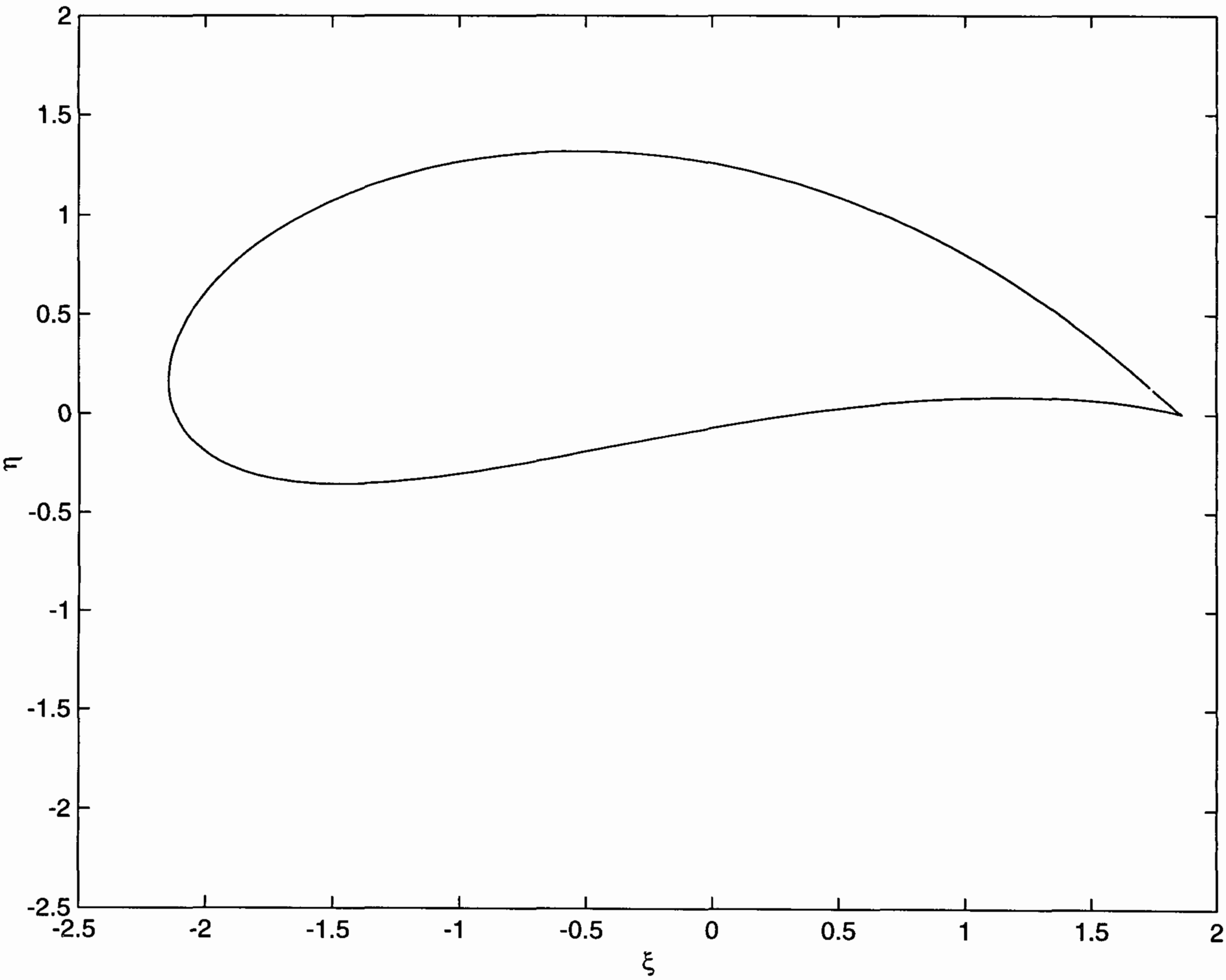
Procedure to Compute Pressure Coefficient

1. Make a sequence of points on the circle.
2. Determine value of z for each point.
3. Use complex number programming to determine the value of z and $d\zeta/dz$ for each point.
4. $(u - iv)_\zeta = (u - iv)_z / \frac{d\zeta}{dz}$.
5. $q^2 = (u - iv)_\zeta \, (u + iv)_\zeta$.
6. $C_p = 1 - (q/U)^2$.

```
% cp1 in matlab

a=1.0;
alpha=0.1745;
lambda=1.8611;
xc = -0.3;
yc =0.4;
UU=1.0;
gamma=-7.779695;
dpr=180./pi;
rc = sqrt((1.0-xc).^2 + yc .^2);
fid = fopen('cpm.dat','w');
degv = (1:1:360);
angv=degv ./dpr;
xv = xc + ( rc .* cos(angv));
yv = yc + ( rc .* sin(angv));
zv = xv + i*yv;
zetav=lambda*a*((zv + a) .^ lambda + (zv-a) .^lambda) ./ ...
    ((zv+a) .^ lambda - (zv-a) .^ lambda);
lm = lambda - 1.0;
dzetadzv = 4.0 * lambda ^2 * a ^2 * (zv-a) .^ lm .*(zv+a) .^lm ./ ...
    (((zv + a) * lambda - (zv -a) .^ lambda) .^ 2);
uv = (UU*cos(alpha)) - UU*cos(2.0 .* angv - alpha) - ...
    (gamma / (2.0*pi*rc)) * sin(angv);
vv = (UU * sin(alpha)) - UU*sin(2.0*angv - alpha) + ...
    (gamma/(2.0*pi*rc)) .* cos(angv);
wz = uv -i*vv;
wzeta = wz ./(dzetadzv + eps);
q = wzeta.*(conj(wzeta));
cp = 1.0 - q / (UU .^2);
cpm = -cp;
for m = 1:360
fprintf(fid,'%7.3f  %7.3f  %7.3f  %7.3f %7.3f  %7.3f  %7.3f\n', ...
  real(zetav(m)), imag(zetav(m)), cpm(m), real(zv(m)),imag(zv(m)),...
    real(wz(m)),imag(wz(m)));
end,
fclose(fid) ,
```

VT PROG 2



KT PROG. 3