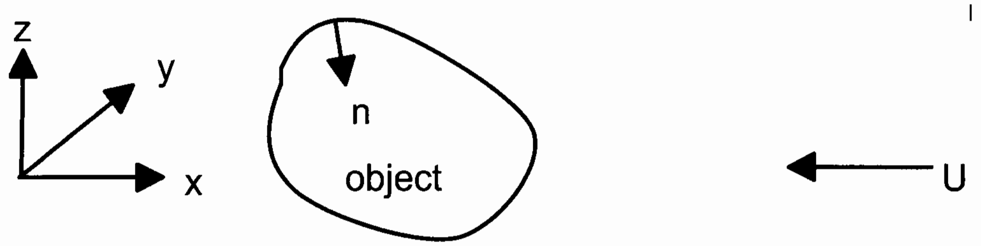
P1

PANEL METHODS



Sketch of an Object in a Uniform Stream

$$\Phi = -Ux + \phi$$

Boundary Condition on Perturbation Potential

$$\frac{\partial \Phi}{\partial n} = 0 \qquad \longrightarrow \qquad \frac{\partial \phi}{\partial n} = U\hat{i} \cdot \mathbf{n}$$

Three Dimensional Flows

$$\int\int_S \left[\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}\right] dS = \begin{cases} 0 & (x,y,z) \text{ outside } S \\ -2\pi\phi(x,y,z) & (x,y,z) \text{ on } S \\ -4\pi\phi(x,y,z) & (x,y,z) \text{ inside } S \end{cases}$$

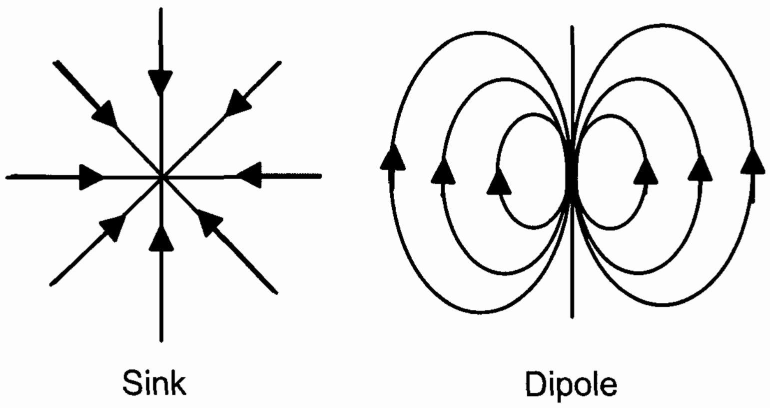
$$G(x,y,z,\xi,\eta,\zeta) = \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} = \frac{1}{r}$$

$$\text{where: } r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

G can be taken as: $G = \frac{1}{4\pi r} + H(x,y,z,\xi,\eta,\zeta)$ where H is an *analytic function* ($\nabla^2 H = 0$ in both (x,y,z) and in (ξ,η,ζ) coordinates. It is used sometimes when a particular H makes the integrand zero on flow boundaries external to an object, thereby removing the necessity of integrating over them.

Interpretation of Green's Theorem

Sketch of Streamlines from Sink and from Dipole



Sketch of Sink and Dipole Streamlines

The velocity potential inside the fluid domain and on object surface expressed as distributions of sources (or sinks for the Green function we have chosen) and dipoles on the surface of the object.

$G(x, y, z, \xi, \eta, \zeta)$ is the velocity potential at (x, y, z) due to a point sink of unit strength at (ξ, η, ζ)

A sheet of of sinks with strength σ per unit area causes the normal velocity to jump by $4\pi\sigma$ when crossing the surface from inside the object out into the fluid.

$\frac{\partial G}{\partial n}$ is the velocity potential at (x, y, z) due to a point dipole of unit dipole moment at (ξ, η, ζ) with the axis of the dipole normal to the object and pointing out of the fluid and into the interior of the object.

A sheet of dipoles with strength μ per unit area causes the velocity potential to jump by $-4\pi\mu$ when crossing the surface from inside the object out to the fluid. That's why the dipole moment per unit area needs to be ϕ to generate a velocity potential of $-4\pi\phi$ in the fluid just outside the surface and in the fluid.

Arrangement of the Integral Equation

- arrange the equation for ϕ in the form of an integral equation with unknowns on the left and known quantities on the right.
- The equation will be applied on the surface of the object where the boundary conditions specify part of the equation.
- Excluding an infinitesimal surface around the point $(\xi, \eta, \zeta) = (x, y, z)$ from the region of integration makes the constant on the right hand side of the equation equal to 2π instead of 4π.

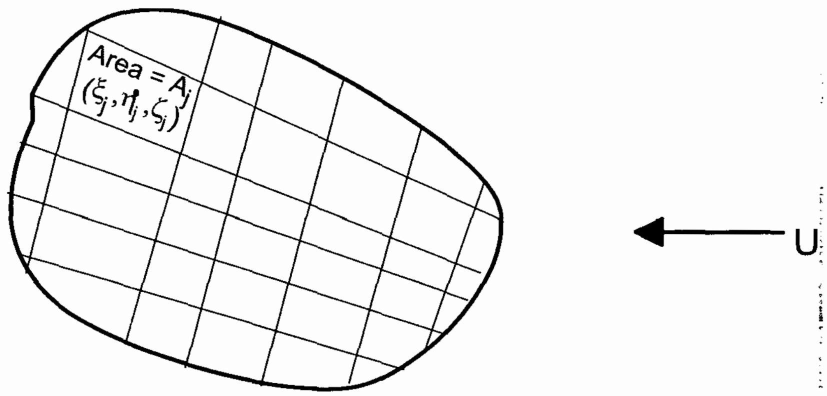
For problems where $\frac{\partial \phi}{\partial n}$ is known on the fluid side of the surface, we write the equation for the unknown ϕ with a known right hand side.

$$\int\int_S \phi \frac{\partial G}{\partial n} \, dS_{\xi,\eta,\zeta} + 2\pi\phi(x,y,z) = \int\int_S G \frac{\partial \phi}{\partial n} \, dS_{\xi,\eta,\zeta}$$

When we substitute $U\hat{\imath} \cdot \mathbf{n}$ for $\frac{\partial \phi}{\partial n}$ the integral equation for the unknown velocity potential ϕ is:

$$\int\int_S \phi \frac{\partial G}{\partial n} \, dS_{\xi,\eta,\zeta} + 2\pi\phi(x,y,z) = \int\int_S G U \hat{\imath} \cdot \mathbf{n}(\xi,\eta,\zeta) \, dS_{\xi,\eta,\zeta}$$

Numerical Form of the Integral Equation



Sketch of an Object Surface Divided into Quadrilateral Panels

- A quadrilateral panel with four corners on the surface will not necessarily be planar.
- The simplest approach is to use approximating planar panels with ϕ constant on each panel.
- For an approximating planar panel, j, place panel centroid, $c_j = (\xi_j, \eta_j, \zeta_j)$ on the actual object surface and orient the panel such that its normal is in the direction of the cross-product of the two diagonal vectors of the non-planar panel.
- This leaves gaps between the panels which are sources of error. However, the smaller the panels, the smaller the gaps. A less erroneous procedure uses non-planar panels, but the integrations for the discretized equation becomes more complicated.

Making the Numerical Equations

- Write a separate equation for values of (x, y, z) at the centroid of each panel.
- There are N centroids where ϕ will be calculated so there will be N equations.
- The calculation (field) points, which are the panel centroids will be labeled by the index i and the value of ϕ on the i^{th} panel is called ϕ_i These are the unknowns.
- For each equation (one for each of the N valies of i), the integrals on the right hand side are done over each panel (j) individually and the results are summed together.

$$\sum_{j=1}^{N} \int\int_{S_j} \phi(\xi, \eta, \zeta) \frac{\partial G_{ij}}{\partial n_j} \, dS_{\xi,\eta,\zeta} + 2\pi\phi_j \, \delta_{ij} = \sum_{j=1}^{N} \int\int_{S_j} G_{ij} U \, \hat{\imath} \cdot \mathbf{n_j}(\xi, \eta, \zeta) \, dS_{\xi,\eta,\zeta}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The symbol G_{ij} means the Green function with the field point at the centroid of the i^{th} panel and with the source point varying over all locations in the j^{th} panel as the integration is carried out. Since, in the approximation being used, ϕ is a constant,

$$\sum_{j=1}^{N} \phi_j \int\int_{S_j} \frac{\partial G_{ij}}{\partial n_j} \, dS_{\xi,\eta,\zeta} + 2\pi\phi_j \, \delta_{ij} = \sum_{j=1}^{N} \int\int_{S_j} G_{ij} U \, \hat{\imath} \cdot \mathbf{n_j}(\xi, \eta, \zeta) \, dS_{\xi,\eta,\zeta}$$

Define:

$$\int\int_{S_j} \frac{\partial G_{ij}}{\partial n_j} \, dS_{\xi,\eta,\zeta} + 2\pi \, \delta_{ij} \equiv A_{ij}$$

and

$$\sum_{j=1}^{N} \int\int_{S_j} G_{ij} U \, \hat{\imath} \cdot \mathbf{n_j}(\xi, \eta, \zeta) \, dS_{\xi,\eta,\zeta} \equiv B_i$$

For planar panels, $\mathbf{n_j}(\xi, \eta, \zeta)$ is a constant on each panel and we call it $\mathbf{n_j}$ and then $U\,\hat{\imath} \cdot \mathbf{n_j}(\xi, \eta, \zeta)$ is a constant on each panel so that:

$$B_i = \sum_{j=1}^{N} U\,\hat{\imath} \cdot \mathbf{n_j} \int\int_{S_j} G_{ij}\, dS_{\xi,\eta,\zeta}$$

The final set of equations for the N values of ϕ is:

$$\sum_{j=1}^{N} A_{ij}\phi_j = B_i \quad \text{equivalently} \quad \mathbf{A}\phi = \mathbf{B}$$

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Solution Steps

- Do the numerical integrals to generate the matrix **A** and the vector **B** which are integrals of G and $\frac{\partial G}{\partial n}$.
- For each field point i integrals must be done for all N values of j. .
- Once **A** and **B** are determined, solve the set of linear equations is solved for ϕ.
- When the number of panels is less than a few thousand, it is practical to solve the equations by Gaussian Elimination, or an **LU** decomposition.
- The burden of carrying out these procedures is removed from the MATLAB user by obtaining the entire solution for all the values of ϕ with the MATLAB statement: $\phi = \mathbf{A}\backslash\mathbf{B}$.
- After ϕ and $\frac{\partial \phi}{\partial n}$ have been determined on the surface of the object the numerical approximation to the left hand side of Green's Theorem can be used to compute ϕ at any point in the fluid.
- The usual goal of a panel method in fluid mechanics is to find the pressure distribution on an object from which the forces and moments can be computed. With inviscid fluid mechanics for which the panel method was developed, the local pressure, P is given by Bernoulli's equation:

$$P = \rho\left[-\frac{\partial \phi}{\partial t} - |\nabla\phi|^2 - gz\right]$$

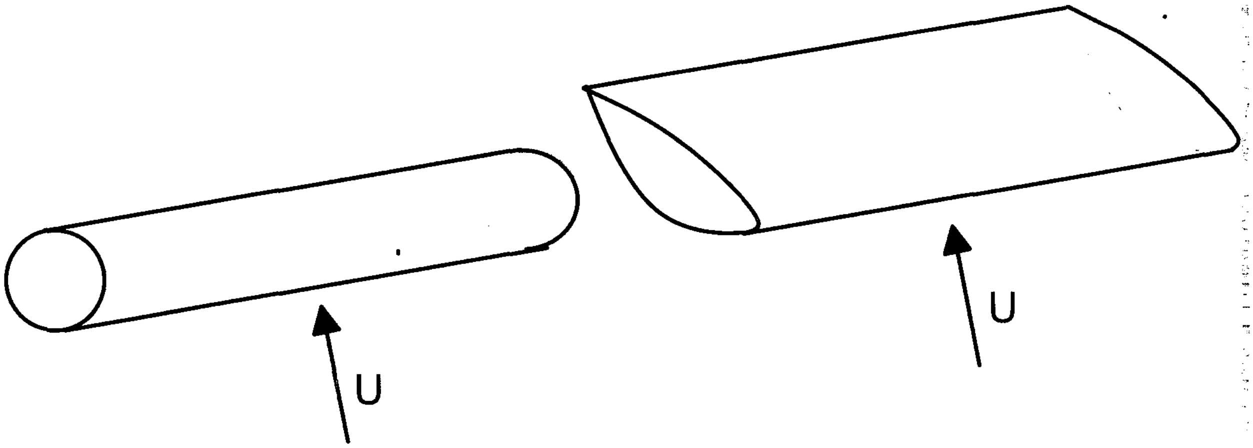
ρ is the fluid density and g is the acceleration due to gravity.

The first term on the right hand side applies to time-dependent motion. Although this has not been considered explicitely here, one can imagine an object moving sinusoidally so that in a reference frame attached to the object U is sinusoidal and the solution has time dependence. The third term is simply the hydrostatic pressure. The most difficult term on the right hand side to compute is generally the second. It is the square of the velocity at the object surface.

Two Dimensional Panel Methods

The development for two-dimensional flows is similar to the 3D case, except that two dimensional source functions are involved and the dimensionality of some integrals and associated constants are different.

Two dimensional flows are either mathematical abstractions with all flow directions in a two dimensional plane or physical approximations for long prismatic objects with an inflow that is perpendicular to the long axis.



Objects for which the Flow is Nearly Two-Dimensional

For the two-dimensional case, Green's Theorem, is:

$$\int_L \left[\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}\right] d\ell = \begin{cases} 0 & (x,y) \text{ outside } S \\ -\pi\phi(x,y) & (x,y) \text{ on } S \\ -2\pi\phi(x,y) & (x,y) \text{ inside } S \end{cases}$$

and the Green function is:

$$G(x,y,\xi,\eta) = -\ln\sqrt{(x-\xi)^2 + (y-\eta)^2)} = -\ln r$$

$$\text{where: } r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

As in the 3D case, in 2D flows, sometimes the Green function is taken as $-\ln r + h(x,y)$ where h is an analytic function which means $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$. Green's Theorem, still holds with this Green function. It is used for problems with boundaries on which the integral in Green's Theorem vanishes to simplify the integrations that are necessary.

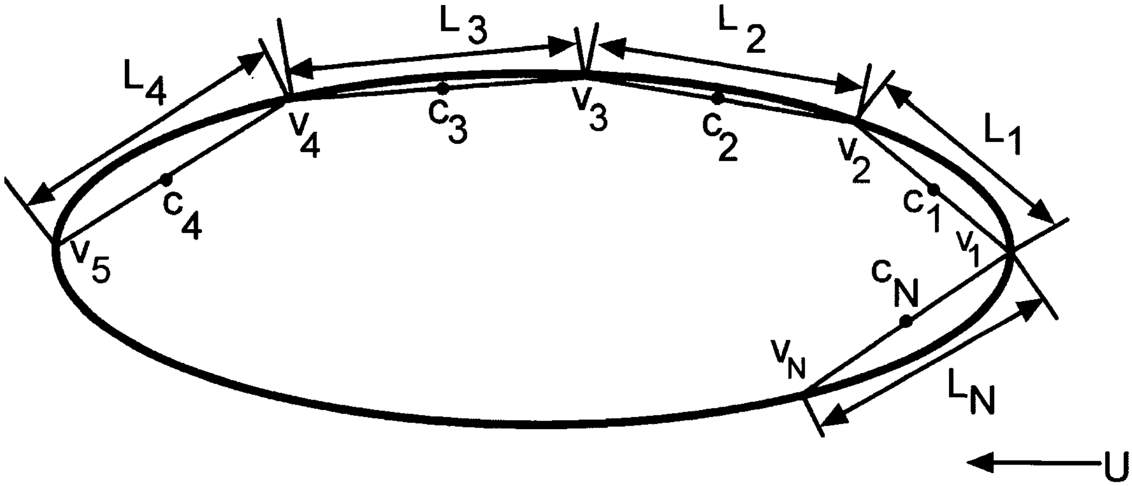
For an incomming stream in the $-x$ direction the total potential is: $\Phi = -Ux + \phi$, the boundary condition for the perturbation potential, ϕ on the surface of the vehicle is $\frac{\partial \phi}{\partial n} = U\,\hat{\imath} \cdot \mathbf{n}$ on the object surface.

The integral equation for the unknown function, $\phi(x, y)$ on the object surface, having unknown quantities on the left and known quantities on the right hand is:

$$\int_L \phi(\xi, \eta) \frac{\partial G}{\partial n}\, d\ell_{\xi,\eta} + \pi \phi(x, y) = \int_L G\, U\, \hat{\imath} \cdot \mathbf{n}\, d\ell_{\xi,\eta}$$

Numerical Form of the Two Dimensional Integral equation

The 2D curve is divided into panels which become segments of a curve:



Panelization of a Two Dimensional Object

- Each panel(actually a line) is defined by two vertices on the object curve. Panel 1 extends from vertex 1 (labeled v_1 in the figure) to vertex 2, panel 2 extends from vertex 2 to vertex 3, etc.
- The centroids of each panel, are at the mid-points of the arc length of each panel. For flat panels, straight lines are drawn between adjacent vertices and the centroids, called c_i are moved to the midpoints of each straight line panel.
- The unit normal vector, $\mathbf{n}_i$ on panel i, is a constant along the length of the panel. The panels are defined by the (x, y) values of the two vertices at the panel ends.
- For the simplest implementation the potential, ϕ, is approximated as being a constant on each panel.

With these approximations and definitions, the numerical form of the integral equation is:

$$\sum_{i=1}^{N} \phi_j \int_{L_j} \frac{\partial G_{ij}}{\partial n_j} d\ell_j + \pi \phi_j \, \delta_{ij} = \sum_{j=1}^{N} U \, \hat{\imath} \cdot \mathbf{n_j} \int_{L_j} G_{ij} d\ell_j$$

Analogous to the 3D case, we define:

$$\int_{L_j} \frac{\partial G_{ij}}{\partial n_j} d\ell_j + \pi\, \delta_{ij} \equiv A_i j$$

and

$$\sum_{j=1}^{N} U\, \hat{i} \cdot \mathbf{n_j} \int_{L_j} G_{ij} d\ell_j \equiv B_i$$

Again, the final set of equations for the N unknown values of ϕ is:

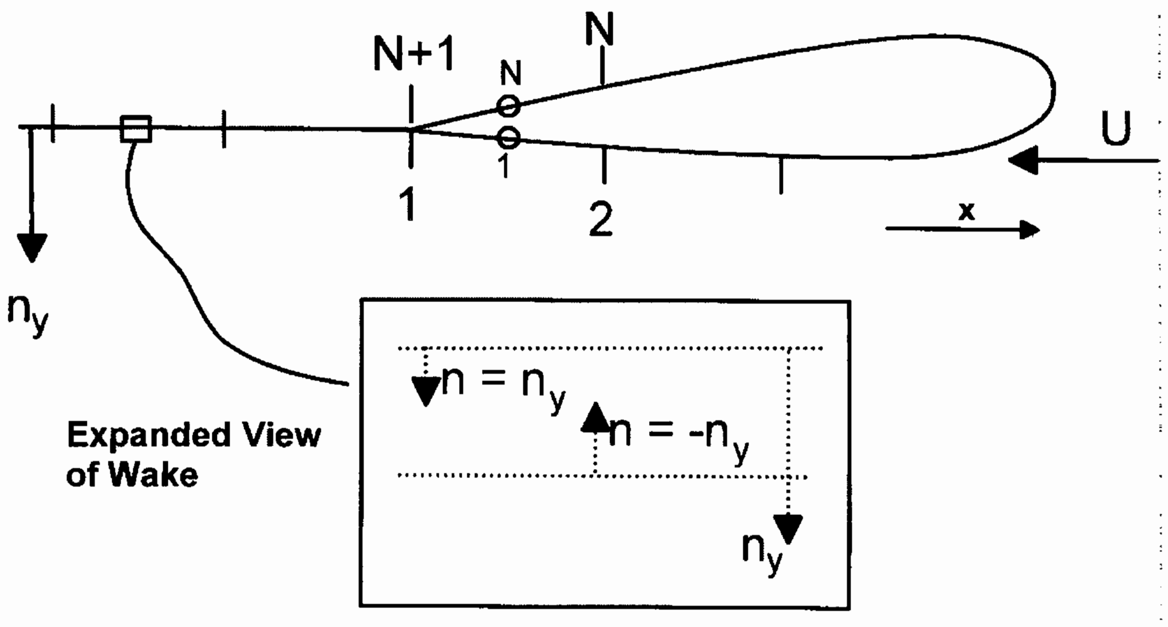
$$\sum_{j=1}^{N} A_{ij}\, \phi_j = B_i \quad \text{or} \quad \mathbf{A}\, \phi = \mathbf{B}$$

Calculation of A_{ij} and B_i

For a field point i located at the midpoint of the $i'th$ panel we need to determine for each panel j the value of A_{ij} by integrating $\frac{\partial G_{ij}}{\partial n_j}$ over the j'th panel. We also need to determine B_i by summing up the terms in the equation for B_i. General purpose MATLAB m-functions for doing these integrals exist and will be provided to students.

Situations With the Generation of Lift

- If the object in a streaminf flow has a sharp trailing edge, it will usually generate lift.
- The lift is related to circulation around the object so that ϕ jumps acorss the trailing edge. This results in a "wake" across which the potential jumps by the same amount as at the trailing edge.
- To obtain a flow domain in which the velocity potential is analytic (satisfies $\nabla^2\phi = 0$) the wake must be excluded from the domain. The line of the wake can be treated as part of the object upon which there is a uniform strength dipole sheet which makes the velocity potential jump when crossing it.
- Here we will use an approximation in which the wake is presumed to follow the free stream direction. Theoretically, the wake is infinitely long. For practical purposes, we can model the wake as being about two airfoil chord lengths long since dipoles further than this from the object (airfoil) will have negligible effect on the flow on the object.



A Two-Dimensional Lifting Airfoil With a Wake

- Green's theorem is applied to a curve which starts at the far end of the wake at the bottom, goes along the bottom of the wake, goes around the foil, and finally goes along the top of the wake. On the top and the bottom of the wake, on any wake panel, $\frac{\partial \phi}{\partial n}$ are equal and opposite For any control point on the foil where the potential is to be determined and a source panel on the wake, G is the same for the two elements of the integration path, one on the wake top and one exactly under it on the wake bottom. Therefore, for the control points on the foil, $\int_{wake} G \frac{\partial \phi}{\partial n} d\ell = 0$.

- At the start of the wake at its junction with the airfoil, there is a jump in potential in going from the bottom to the top.

$$\phi_{wake\ top} - \phi_{wake\ bottom} \equiv \Delta\phi = \phi_N - \phi_1$$

This jump in potential is maintained all along the wake because as one moves aft along the wake the potential changes the same amount on the top and on the bottom by $\int_{tr\ edge}^{x} u(x)dx$.

- Thus, for control (field) points on the foil, with L being the path around the foil, W being the single line along the wake from its aft most considered point to the trailing edge of the foil, and $\mathbf{n}$ on the wake pointing downward for the configuration shown in the Figure , Green's theorem takes the form:

$$\int_L \phi(\xi,\eta) \frac{\partial G}{\partial n}\, d\ell_{\xi,\eta} + \int_W \Delta\phi \frac{\partial G}{\partial n} d\ell_{\xi,\eta} + \pi\phi(x,y) = \int_L G\, U\, \hat{\imath} \cdot \mathbf{n}\, d\ell_{\xi,\eta}$$

- If the wake panels are labeled $N+1$ to M with lengths ds_{j_w}, the discretized form of the equations appropriate for numerical solution for values of ϕ at control points, i, on the foil are:

$$\sum_{j=1}^{N} \phi_j \int_{L_j} \frac{\partial G_{ij}}{\partial n_j} d\ell_j + (\phi_N - \phi_1) \sum_{j_w=N+1}^{M} \int_{L_{j_w}} \frac{\partial G_{ij_w}}{\partial n_{j_w}} d\ell_{j_w} + \pi\phi_i = \sum_{j=1}^{N} U\, \hat{\imath} \cdot n_j \int_{L_j} G_{ij} d\ell_j$$

To put the preceding equation in the same format that was used for the non-lifting case, we define:

$$\int_{L_j} \frac{\partial G_{ij}}{\partial n_j} d\ell_j + \pi\,\delta_{ij} + \int_{Ljw} \frac{\partial G_{ij_w}}{\partial n_{j_w}} d\ell_{jw}\,(\delta_{jN} - \delta_{j1}) \equiv Q_{ij}$$

Then, the final set of equations to be solved is:

$$\sum_{j=1}^{N} Q_{ij}\,\phi_j = B_i$$

which has the matrix notation:

$$\mathbf{Q}\,\phi = \mathbf{B}$$

Computation of pressures and forces

For steady flow, Bernoulli's equation for dynamic pressure is: $P = -\rho|V|^2$ This presumes that in the very far field, the pressure is hydrostatic and there is no dynamic pressure there. In unsteady flows, there is an additional contribution to the dynamic pressure equal to $-\rho\frac{\partial\phi}{\partial t}$ which is straightforward to compute at each control point.

Here we concentrate on computation of the $-rho|V|^2$ term in the dynamic pressure. It is most convenient to know this pressure, P_i at each control point. Then the force is: $\mathbf{F} = \sum_{i=1}^{N} P_i \mathbf{n}_i d\ell_i$

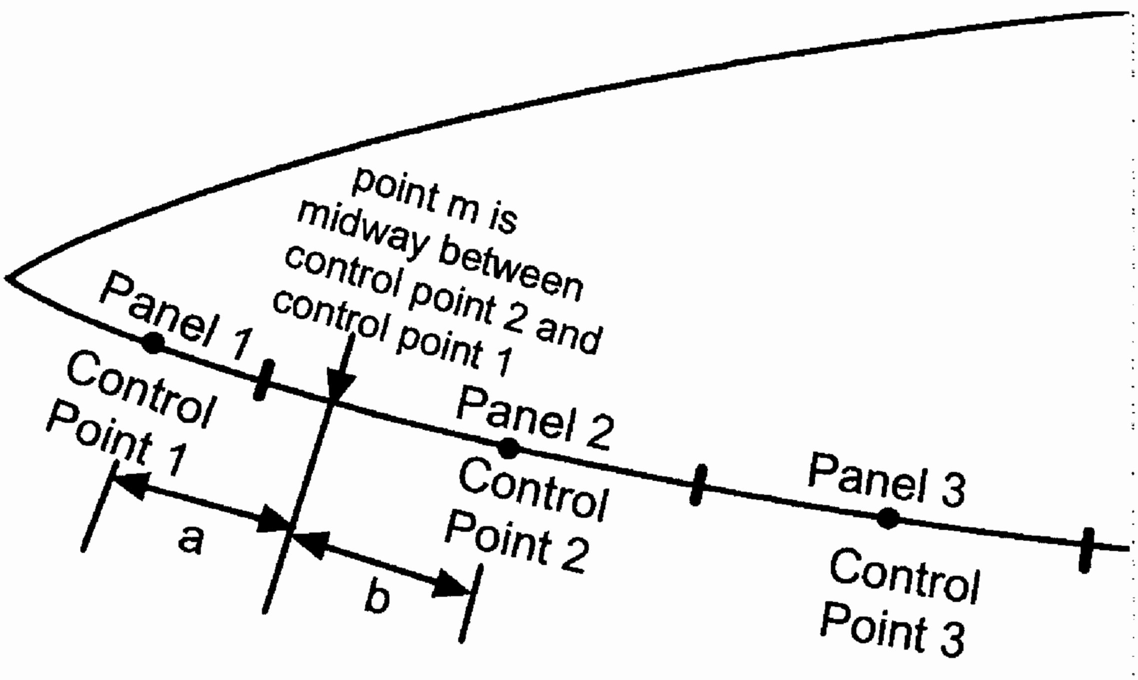
The central problem is calculation of the velocity at the surface of the object. At each control point, i, the total flow is tangent to the object surface so we need only find tangential velocity. The tangential velocity is the tangential derivative of the total potential, Φ (perturbation potential plus any exterior potential such as $-Ux$ for the steady flow problems we have been considering). For two-dimensional flows the tangential velocity component has a single direction at each control point. We know how to determine the tangential derivative at a point using a modified central difference procedure for smooth objects. Even when the object has a sharp edge, this procedure can be used for all control points except for those adjacent to a trailing edge. In addition, the numerical derivative can be determined at a point that is midway between the first and second control points from the trailing edge. Then the tangential derivative can be approximated at a control point nearest the trailing edge by numerical extrapolation. An example follows.

We know how to calculate the numerical tangential Derivative, called d_2 at-Control point 2. The lengths of the panels are called $L_1, L_2, ...$ The tangential derivative at point m, called d_m is given by:

$$d_m = \frac{\Phi_2 - \Phi_1}{0.5(L_1 + L_2)}$$

Then, by extrapolation, the derivative at control point 1, called d_1 is numerically approximated as:

$$d_1 = d_m - a\frac{d_2 - d_m}{b}$$



Tangential Derivative at a Control Point Near a Sharp Edge