2.58 HW#2 Solutions

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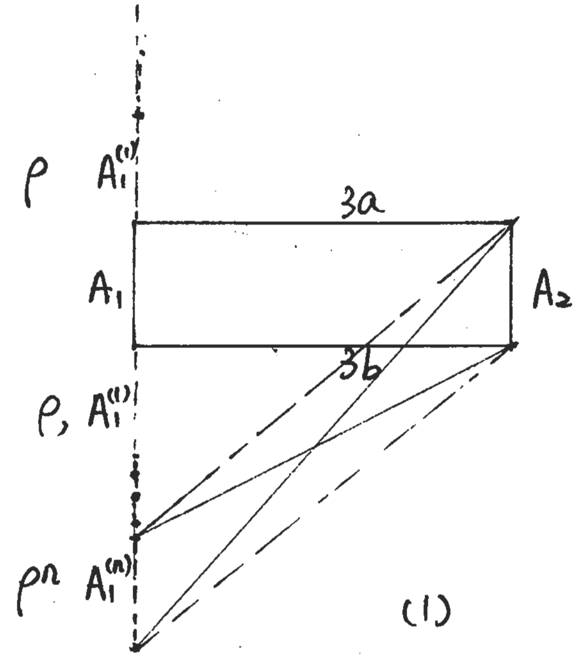
Prob 6.1



(1)

(a) Surface A_3 is specular and Surfaces A_1 & A_2 are black:

$\Rightarrow$ $\varepsilon_1 = \varepsilon_2 = 1$, $\rho_1^s = \rho_2^s = 0$

Applying Eq.(6.20), we obtain

$$E_{b1} - F^s_{1-1} E_{b1} - F^s_{1-2} E_{b2} - (1-\rho^s) F^s_{1-3} E_{b3} = q_1 \quad \cdots ①$$

$$E_{b2} - F^s_{2-1} E_{b1} - F^s_{2-2} E_{b2} - (1-\rho^s) F^s_{2-3} E_{b3} = q_2 \quad \cdots ②$$

Neither A_1 nor its images can see itself such that

$$F^s_{1-1} = 0$$

By symmetry, $F^s_{2-2} = F^s_{1-1} = 0$, $F^s_{13} = F^s_{23}$, $F^s_{12} = F^s_{21}$

By energy conservation, $q_1 = -q_2$

Subtract eq.① from eq.② to yield

$$(1+F^s_{21}) E_{b1} - (1+F^s_{12}) E_{b2} = 2 q_1$$

$$\Rightarrow \quad q_1 = \frac{1+F^s_{12}}{2} \cdot \sigma (T_1^4 - T_2^4) \quad \cdots ③$$

The specular view factor F_{1-2} is given by

$$F^s_{12} = F^d_{12} + 2 \sum_n (\rho_s)^n F_{(n)-2}$$

Using the crossed-string method, we obtain where $F_{(n)-2} = \dfrac{\text{diagonals} - \text{sides}}{2D} = \dfrac{\sqrt{(n-1)^2D^2+L^2} + \sqrt{(n+1)^2D^2+L^2} - 2\sqrt{n^2D^2+L^2}}{2D}$

$$= \frac{1}{2}\cdot\frac{L}{D}\left[\sqrt{(n-1)^2(\tfrac{D}{L})^2+1} + \sqrt{(n+1)^2(\tfrac{D}{L})^2+1} - 2\sqrt{n^2(\tfrac{D}{L})^2+1}\right]$$

$$F^d_{1-2} = \sqrt{1+(\tfrac{L}{D})^2} - \frac{L}{D} = \frac{L}{D}\left[\sqrt{1+(\tfrac{D}{L})^2} - 1\right]$$

When $\frac{D}{L} \ll 1$, we can use the approximation:

$$\sqrt{1+x^2} \approx 1 + \frac{x^2}{2} \qquad (x \ll 1)$$

$$\Rightarrow \quad F^d_{1-2} \approx \frac{1}{2}\frac{D}{L}$$

$$F_{(n)-2} \approx \frac{1}{2}\cdot\frac{L}{D}\left[1 + \frac{(n-1)^2}{2}(\tfrac{D}{L})^2 + 1 + \frac{(n+1)^2}{2}(\tfrac{D}{L})^2 - 2 - n^2(\tfrac{D}{L})^2\right]$$

$$= \frac{1}{2}\frac{D}{L}$$

$$\Rightarrow \quad F^s_{1-2} = \frac{1}{2}\frac{D}{L} + 2\sum_n^\infty (\rho_s)^n \frac{1}{2}\frac{D}{L} = \frac{D}{2L}\left(\frac{2}{\varepsilon} - 1\right)$$

$$\Rightarrow \quad q_1 = \frac{1+F^s_{1-2}}{2}\sigma(T_1^4 - T_2^4)$$

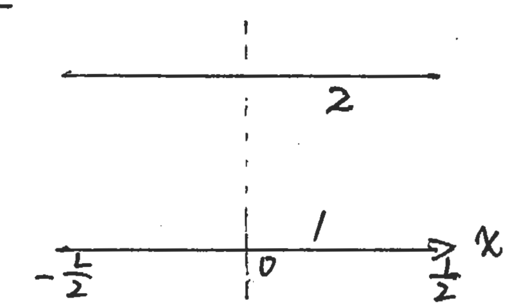
$$= \frac{1}{2}\left[\frac{D}{L}\left(\frac{1}{\varepsilon} - \frac{1}{2}\right) + 1\right]\sigma(T_1^4 - T_2^4)$$

(b) If surface A_3 is diffuse,

$$q_1 = \frac{1+F^d_{12}}{2}\sigma(T_1^4 - T_2^4)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\frac{D}{L}\right)\sigma(T_1^4 - T_2^4)$$

5.34



(a) For surface 1:

$$q_1 = E_{b1}(x_1) - \int_{A_2} E_{b2}(x_2)\, dF_{dA_1-dA_2} \qquad \text{--- (1)}$$

For surface 2:

$$q_2 = 0 = E_{b2}(x_2) - \int_{A_1} E_{b1}(x_1)\, dF_{dA_2-dA_1} \qquad \text{--- (2)}$$

Where $dx_1 dF_{dA_1-dA_2} = dx_2 dF_{dA_2-dA_1} = \frac{1}{2}\frac{h^2 dx_1 dx_2}{[h^2+(x_1-x_2)^2]^{3/2}}$

We can nondimensionize the equations by introducing

$$e_1 = \frac{E_{b1}}{q_1}, \quad e_2 = \frac{E_{b2}}{q_2}, \quad \xi_1 = \frac{x_1}{h}, \quad \xi_2 = \frac{x_2}{h}, \quad \eta = \frac{L}{h}$$

$$\Rightarrow \quad 1 = e_1(\xi_1) - \frac{1}{2}\int_{-\eta/2}^{\eta/2} e_2(\xi_2)\frac{1}{[1+(\xi_1-\xi_2)^2]^{3/2}}\, d\xi_2 \quad \text{--- (1a)}$$

$$0 = e_2(\xi_2) - \frac{1}{2}\int_{-\eta/2}^{\eta/2} e_1(\xi_1)\frac{1}{[1+(\xi_1-\xi_2)^2]^{3/2}}\, d\xi_1 \quad \text{--- (2a)}$$

(b) Replace the kernal with $e^{-|\xi'-\xi|}$:

$$e_1(\xi) = 1 + \frac{1}{2}\int_{-\eta/2}^{\xi} e_2(\xi')e^{-(\xi-\xi')}d\xi' + \frac{1}{2}\int_{\xi}^{\eta/2} e_2(\xi')e^{-(\xi'-\xi)}d\xi' \quad \text{--- (1b)}$$

$$e_2(\xi) = \frac{1}{2}\int_{-\eta/2}^{\xi} e_1(\xi')e^{-(\xi-\xi')}d\xi' + \frac{1}{2}\int_{\xi}^{\eta/2} e_1(\xi')e^{-(\xi'-\xi)}d\xi' \quad \text{--- (2b)}$$

Take the derivative with respect to ξ $\Rightarrow$

$$e_1''(\xi) = -e_2(\xi) + e_1(\xi) - 1 \qquad \text{--- (3)}$$

$$e_2''(\xi) = -e_1(\xi) + e_2(\xi) \qquad \text{--- (4)}$$

(3) + (4) $\Rightarrow$ $(e_1+e_2)'' = -1 \Rightarrow e_1 + e_2 = -\frac{\xi^2}{2} + A\xi + B$

(3) − (4) $\Rightarrow$ $(e_1-e_2)'' = 2(e_1-e_2) - 1 \Rightarrow e_1 - e_2 = C\sinh\sqrt{2}\xi + D\cosh\sqrt{2}\xi + \frac{1}{2}$

By symmetry, $e_1(\xi) = e_1(-\xi)$, $e_2(\xi) = e_2(-\xi)$

$\Rightarrow \quad A = C = 0$

$$\Rightarrow \quad e_1 + e_2 = -\frac{\xi^2}{2} + B \qquad \text{--- (5)}$$

$$e_1 - e_2 = D\cosh\sqrt{2}\xi + \frac{1}{2} \qquad \text{--- (6)}$$

We still need 2 equations to determine B and D.

Let $\xi = 0$ in eqns. (1b) and (2b):

$$e_1(0) = 1 + \int_0^{\eta/2} e_2(\xi') e^{-\xi'} d\xi' \qquad \text{--- (5a)}$$

$$e_2(0) = \int_0^{\eta/2} e_1(\xi') e^{-\xi'} d\xi' \qquad \text{--- (6a)}$$

(5a) + (6a) $\Rightarrow \quad e_1(0) + e_2(0) = B = 1 + \int_0^{\eta/2} \left(-\frac{\xi'^2}{2} + B\right) e^{-\xi'} d\xi'$

$$\Rightarrow B = \frac{\eta^2}{8} + \frac{\eta}{2} + 1$$

(5a) − (6a) $\Rightarrow \quad e_1(0) - e_2(0) = D + \frac{1}{2} = 1 - \int_0^{\eta/2} \left[D\cosh(\sqrt{2}\xi') + \frac{1}{2}\right] \cdot e^{-\xi'} d\xi'$

$$\Rightarrow D = \frac{1}{2} \cdot \frac{1}{\sqrt{2}\sinh\left(\frac{\eta}{\sqrt{2}}\right) + \cosh\left(\frac{\eta}{\sqrt{2}}\right)}$$

From eqns (5) and (6), we can solve for e_1 and e_2:

$$e_1 = -\frac{\xi^2}{4} + \frac{D}{2}\cosh(\sqrt{2}\xi) + \frac{B}{2} + \frac{1}{4}$$

$$e_2 = -\frac{\xi^2}{4} - \frac{D}{2}\cosh(\sqrt{2}\xi) + \frac{B}{2} - \frac{1}{4}$$

where $B = \frac{\eta^2}{8} + \frac{\eta}{2} + 1$

$$D = \frac{1}{2} \frac{1}{\sqrt{2}\sinh\left(\frac{\eta}{\sqrt{2}}\right) + \cosh\left(\frac{\eta}{\sqrt{2}}\right)}$$

(c) If the surfaces are gray, we can obtain the governing equations for the radiosity:

$$\mathcal{E}_1(x_1) = \mathcal{E}_1 = J_1(x_1) - \int_{A_2} J_2(x_2)\, dF_{dA_1-dA_2}$$

$$\mathcal{E}_2(x_2) = 0 = J_2(x_2) - \int_{A_1} J_1(x_1)\, dF_{dA_2-dA_1}$$

Nondimensionise the above equations by using $j_1 = \frac{J_1}{\mathcal{E}_1}$, $j_2 = \frac{J_2}{\mathcal{E}_1}$, $\xi = \frac{X}{h}$, $\eta = \frac{L}{h}$.

$$\Rightarrow \quad 1 = j_1(\xi_1) - \frac{1}{2}\int_{-\eta/2}^{\eta/2} j_2(\xi_2) \frac{1}{[1+(\xi_1-\xi_2)^2]^{3/2}}\, d\xi_2 \quad \text{--- } (7)$$

$$0 = j_2(\xi_2) - \frac{1}{2}\int_{-\eta/2}^{\eta/2} j_1(\xi_1) \frac{1}{[1+(\xi_1-\xi_2)^2]^{3/2}}\, d\xi_1 \quad \text{--- } (8)$$

Equations (7) and (8) are exactly the same as eqns (1a) and (2a)

$$\Rightarrow \quad j_1(\xi_1) = e_1(\xi_1), \quad j_2(\xi_2) = e_2(\xi)$$

Since $E_b = J + (\frac{1}{\varepsilon} - 1)\mathcal{E}$, we have

$$\sigma T_1^4 = E_{b1} = J_1 + (\frac{1}{\varepsilon} - 1)\mathcal{E}_1 = \mathcal{E}_1 (e_1 + \frac{1}{\varepsilon} - 1)$$

$$\sigma T_2^4 = E_{b2} = J_2 = \mathcal{E}_1 e_2$$

$$\Rightarrow \quad T_1 = \sqrt[4]{\frac{\mathcal{E}_1}{\sigma}(e_1 + \frac{1}{\varepsilon} - 1)}$$

$$T_2 = \sqrt[4]{\frac{\mathcal{E}_1}{\sigma} e_2}$$

(d) By symmetry, we can rewrite eqns (1a) and (1b) as

$$e_1(\xi) - \frac{1}{2}\int_0^{\frac{\pi}{2}} e_2(\xi')\left\{\frac{1}{[1+(\xi-\xi')^2]^{3/2}} + \frac{1}{[1+(\xi+\xi')^2]^{3/2}}\right\} d\xi' = 1$$

$$e_2(\xi) - \frac{1}{2}\int_0^{\frac{\pi}{2}} e_1(\xi')\left\{\frac{1}{[1+(\xi-\xi')^2]^{3/2}} + \frac{1}{[1+(\xi+\xi')^2]^{3/2}}\right\} d\xi' = 0$$

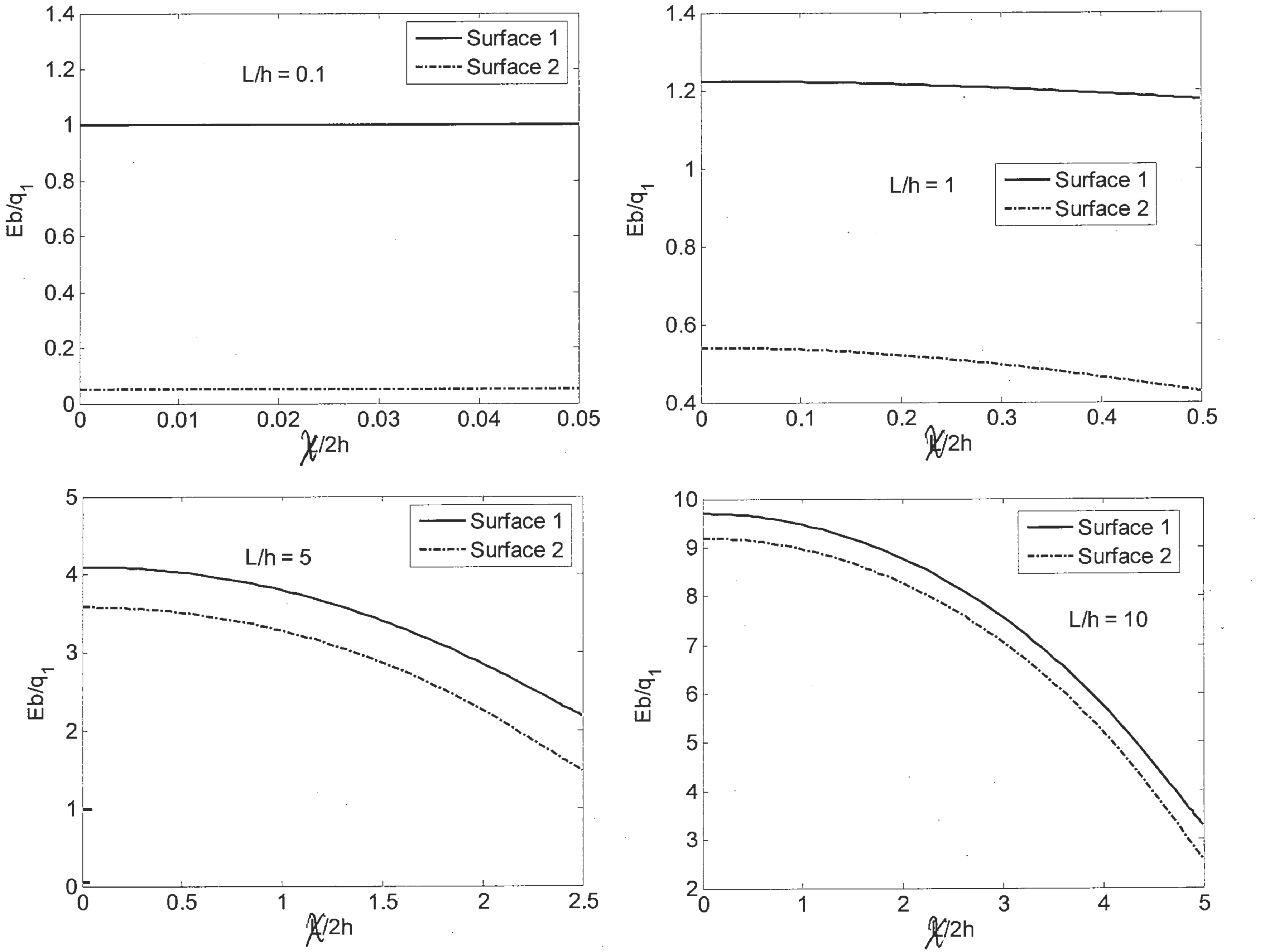
Discretise the integral using quadratures:

$$e_1(\xi_i) - \sum_{j=1}^{n} \frac{1}{2}\omega_j f(\xi_i,\xi_j)\, e_2(\xi_j) = 1$$

$$e_2(\xi_i) - \sum_{j=1}^{n} \frac{1}{2}\omega_j f(\xi_i,\xi_j)\, e_1(\xi_j) = 0$$

The linear equations can be solved by iteration or eliminations. (Guassian or LU)

Monte - Carlo


L/h = 0.1
Surface 1
Surface 2
Eb/q1
X/2h
L/h = 1
Surface 1
Surface 2
Eb/q1
X/2h
L/h = 5
Surface 1
Surface 2
Eb/q1
X/2h
L/h = 10
Surface 1
Surface 2
Eb/q1
X/2h