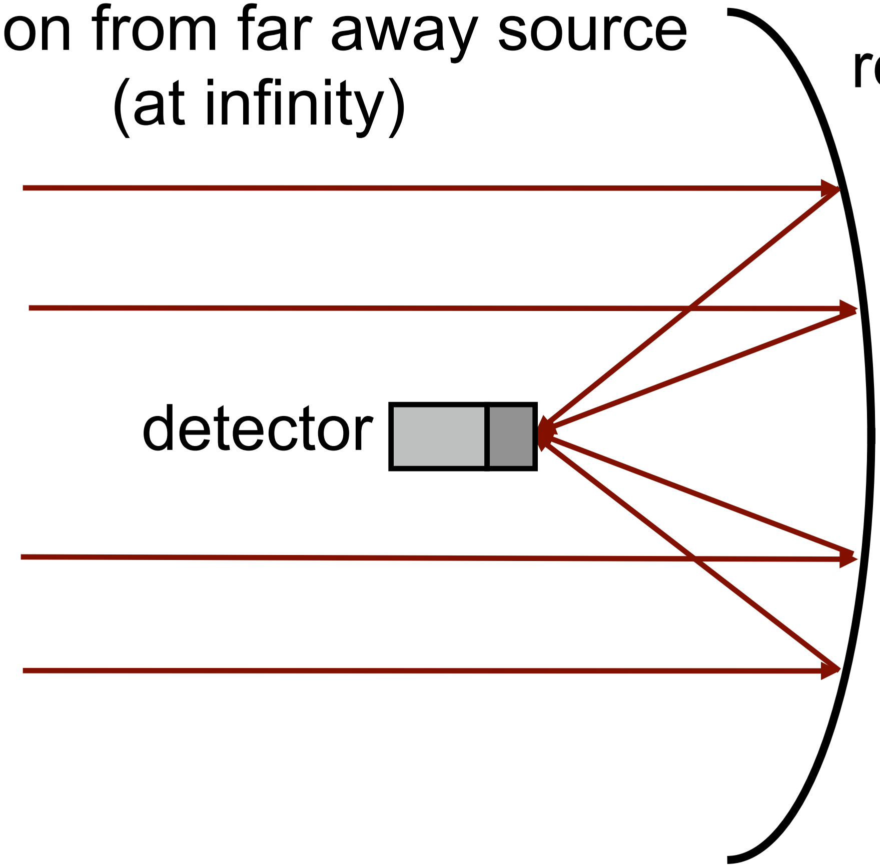
Overview

- Last lecture:
  - Wavefronts and rays
  - Fermat's principle
  - Reflection
  - Refraction
- Today: two applications of Fermat's principle to the problem of *perfectly* focusing a plane wave to a point:
  - paraboloidal reflector
  - ellipsoidal refractor

Curved reflecting surfaces



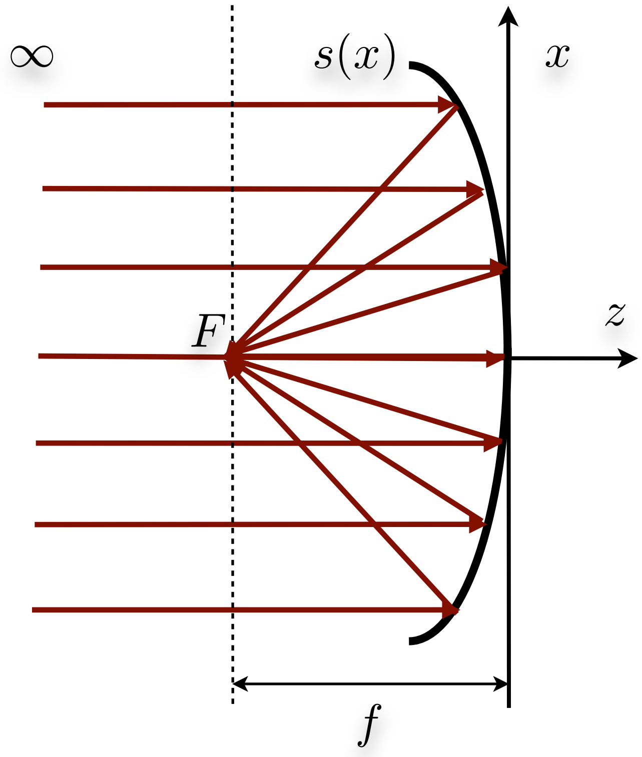
Courtesy of NASA/JPL-Caltech.

Image by hyperborea at Flickr.

Applications:

solar concentrators, satellite dishes, radio telescopes

Paraboloidal reflector: *perfect* focusing



focus at F

What should the shape function $s(x)$ be in order for the incoming parallel ray bundle to come to perfect focus?

The easiest way to find the answer is to invoke Fermat's principle: since the rays from infinity follow the *minimum* path before they meet at P, it follows that they must follow the *same* path.

$$2f = f - s + \sqrt{x^2 + (f-s)^2}$$

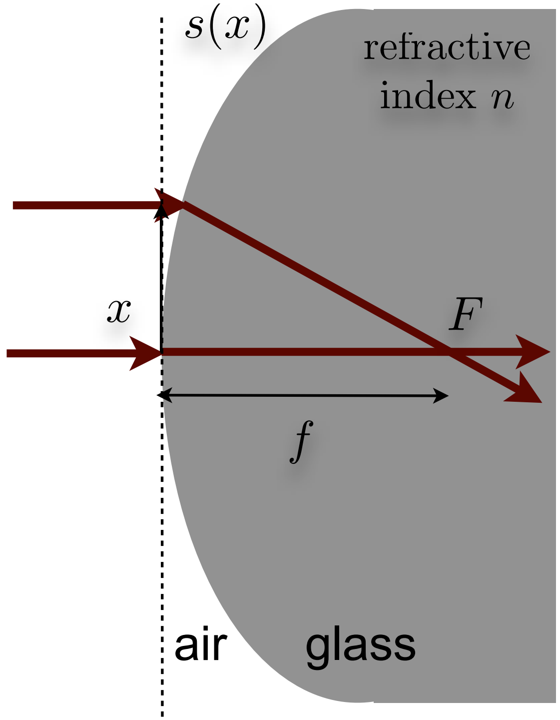
$$f + s = \sqrt{x^2 + (f-s)^2}$$

$$\begin{aligned} x^2 &= (f+s)^2 - (f-s)^2 \Rightarrow \\ &= 4sf \Rightarrow \end{aligned}$$

A paraboloidal reflector focuses a normally incident plane wave to a point

$$\boxed{s(x) = \frac{x^2}{4f}}$$

Ellipsoidal refractor: *perfect* focusing



What should the shape function $s(x)$ be in order for the incoming parallel ray bundle to come to perfect focus?

Once again, we invoke Fermat's principle: since the rays from infinity follow the *minimum* path before they meet at P, it follows that they must follow the *same* path.

$$nf = s + n\sqrt{x^2 + (f-s)^2}$$

$$\Rightarrow \ldots \Rightarrow$$

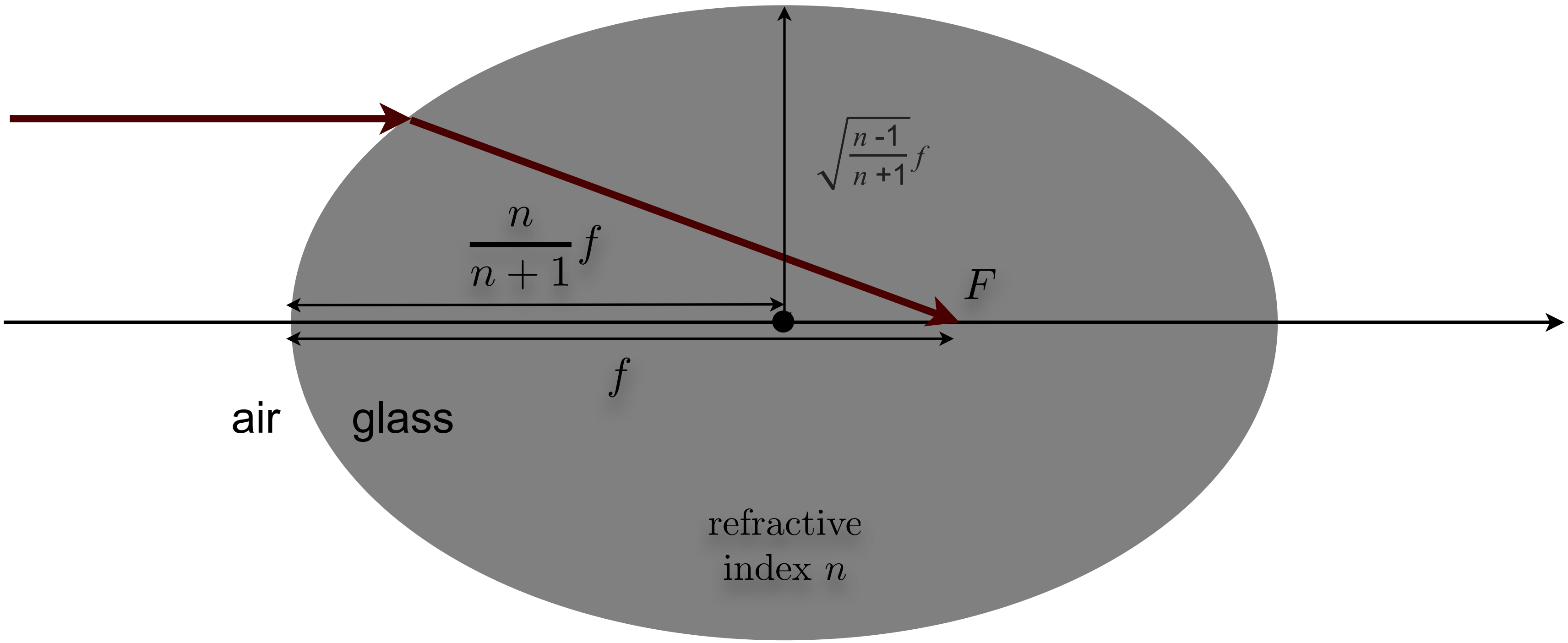
$$\left(n^2 - 1\right) s^2 - 2n\left(n-1\right) fs + n^2 x^2 = 0$$

$$\Rightarrow \ldots \Rightarrow$$

A ellipsoidal refractor focuses a normally incident plane wave to a point

$$\left(s - \frac{n}{n+1} f\right)^2 + \frac{n^2}{n^2-1} x^2 = \left(\frac{n}{n+1} f\right)^2$$

Ellipsoidal refractive concentrator



Surface shape $s(x)$:

$$\left(s-\frac{n}{n+1}f\right)^2+\frac{n^2}{n^2-1}x^2=\left(\frac{n}{n+1}f\right)^2$$

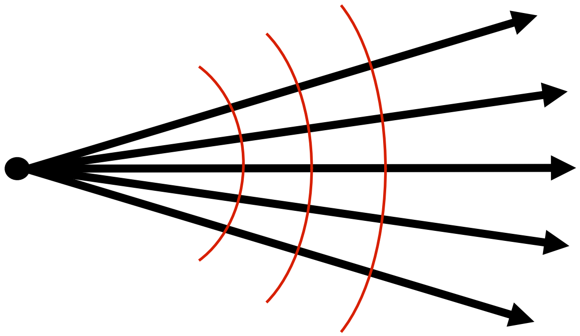
Overview

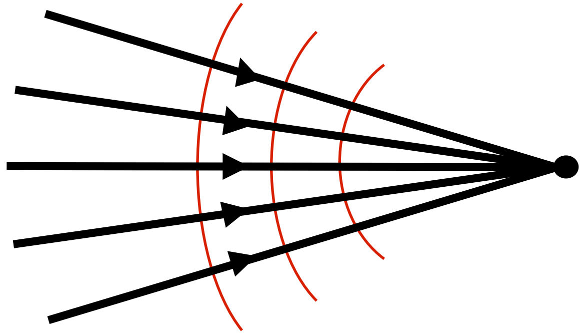
- Last lecture:
    - focusing ray bundles coming from infinity (plane waves)
  - paraboloidal reflector
  - ellipsoidal refractor
- Today:
  - spherical and plane waves
  - perfect focusing and collimation elements:
    - paraboloidal mirrors, ellipsoidal and hyperboloidal refractors
  - imperfect focusing: spherical elements
  - the paraxial approximation
  - ray transfer matrices
- Next lecture:
  - ray tracing using the matrix approach

Spherical and plane waves

diverging spherical wave | converging spherical wave

point source / point object





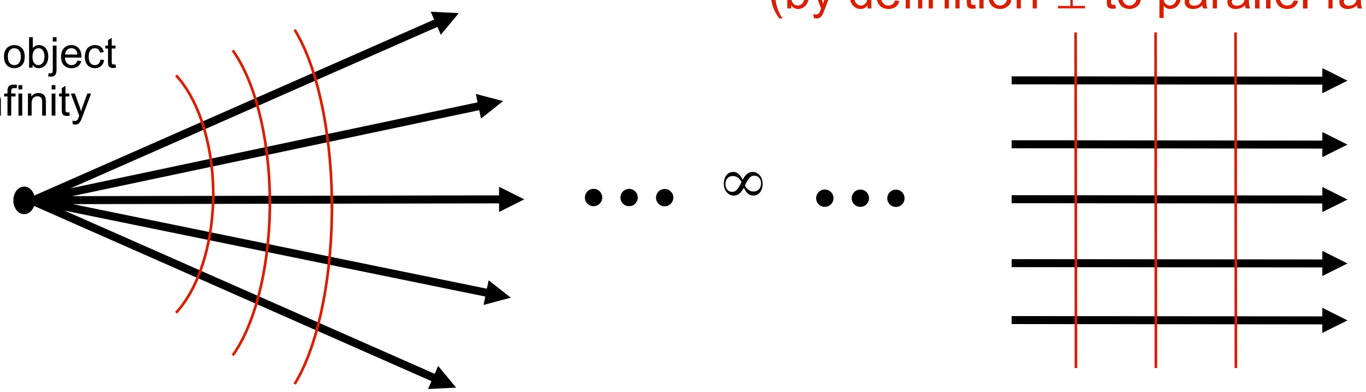
point image

spherical wave-fronts
(by definition ⊥ to divergent fan of rays)

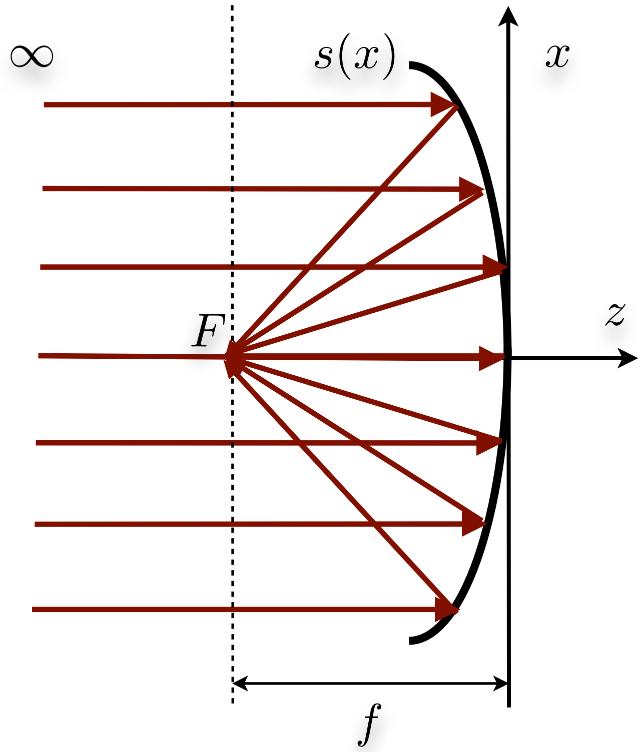
spherical wave at infinity ⇔ plane wave

planar wave-fronts
(by definition ⊥ to parallel fan of rays)

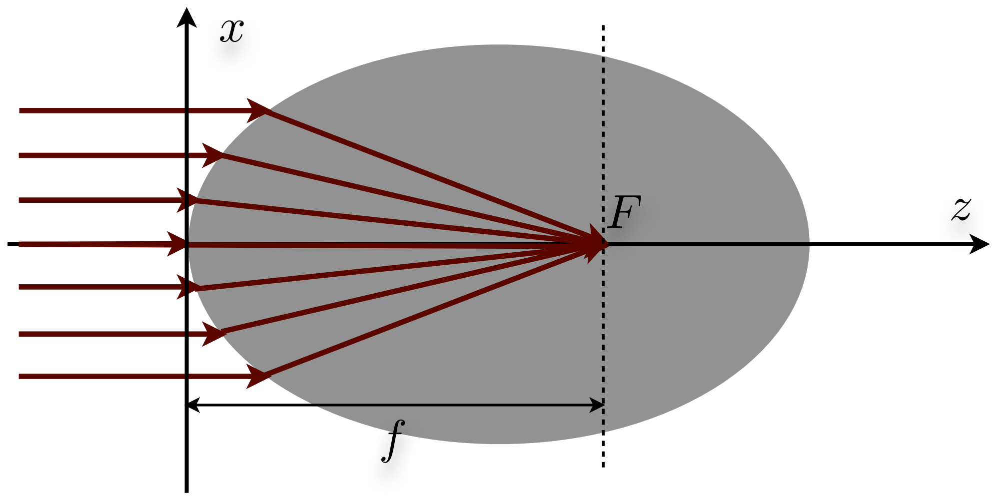
point object at infinity



Perfect imaging of point sources at infinity



focus at F



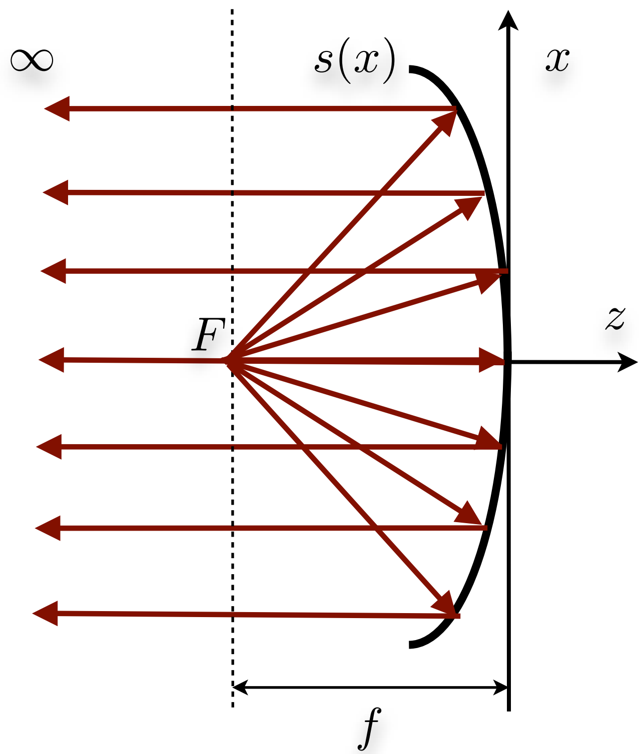
$$s = \frac{x^2}{4f}$$

$$\left(s - \frac{n}{n+1}f\right)^2 + \frac{n^2}{n^2-1}x^2 = \left(\frac{n}{n+1}f\right)^2$$

Paraboloidal reflector

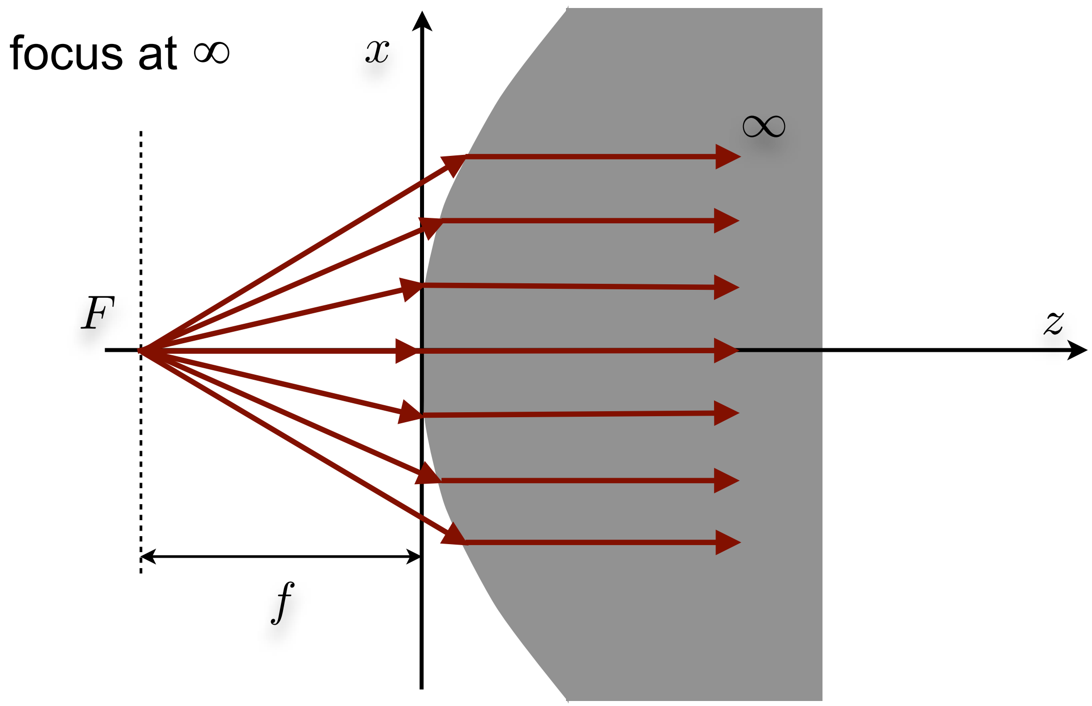
Ellipsoidal refractor

Perfect imaging of point sources *to* infinity



$$s = \frac{x^2}{4f}$$

Paraboloidal reflector

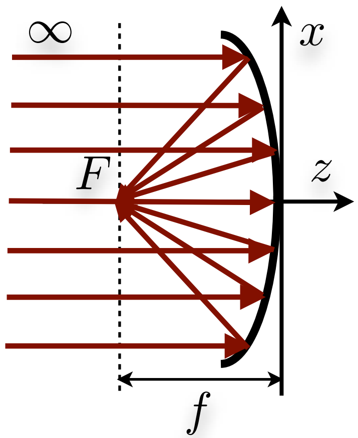


$$\left(s + \frac{1}{n+1}f\right)^2 - \frac{1}{n^2-1}x^2 = \left(\frac{1}{n+1}f\right)^2$$

Hyperboloidal refractor

Summary: objects and images at infinity

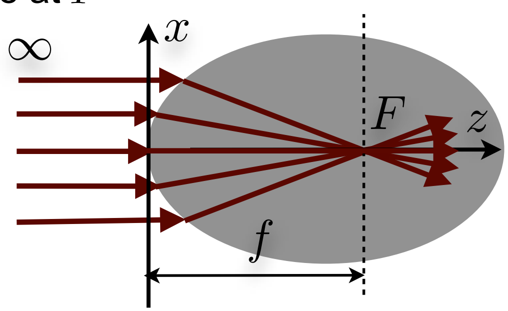
object at ∞
image at F



paraboloidal reflector

$$s=\frac{x^2}{4f}$$

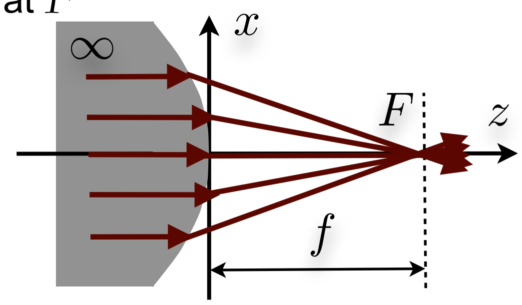
object at ∞
image at F



ellipsoidal refractor

$$\left(s-\frac{n}{n+1}f\right)^2+\frac{n^2}{n^2-1}x^2=\left(\frac{n}{n+1}f\right)^2$$

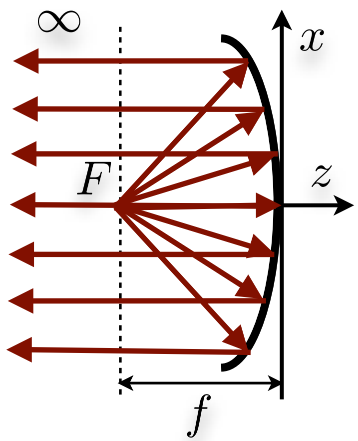
object at ∞
image at F



hyperboloidal refractor

$$\left(s-\frac{1}{n+1}f\right)^2-\frac{1}{n^2-1}x^2=\left(\frac{1}{n+1}f\right)^2$$

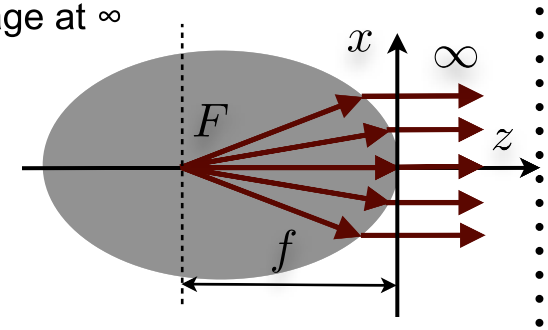
object at F
image at ∞



paraboloidal reflector

$$s=\frac{x^2}{4f}$$

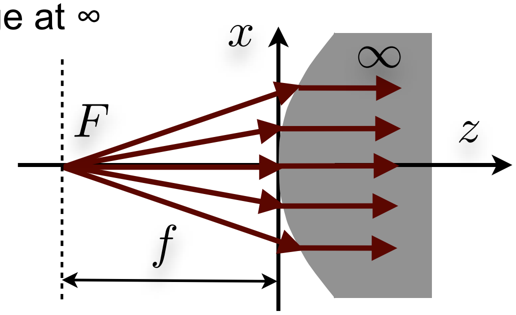
object at F
image at ∞



ellipsoidal refractor

$$\left(s+\frac{n}{n+1}f\right)^2+\frac{n^2}{n^2-1}x^2=\left(\frac{n}{n+1}f\right)^2$$

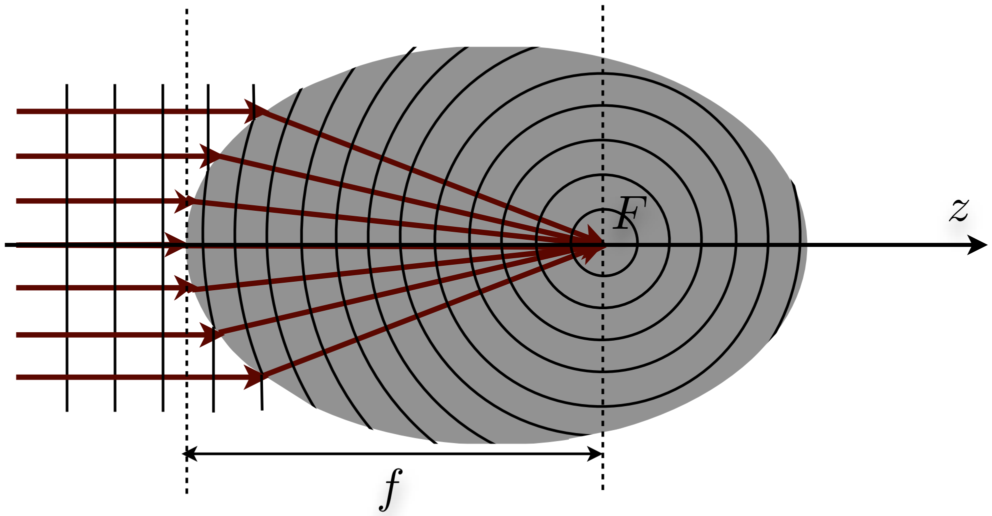
object at F
image at ∞



hyperboloidal refractor

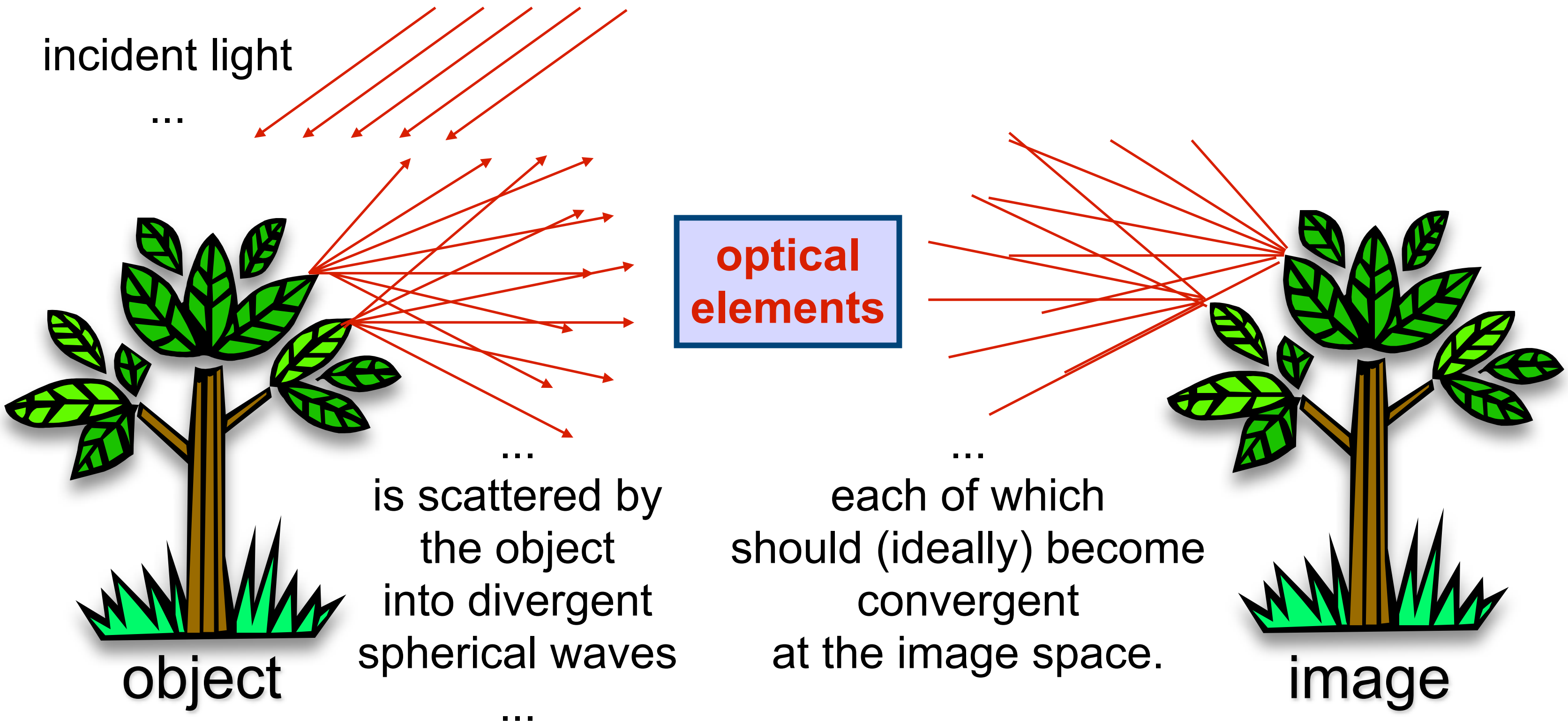
$$\left(s+\frac{1}{n+1}f\right)^2-\frac{1}{n^2-1}x^2=\left(\frac{1}{n+1}f\right)^2$$

Focusing: from planar to spherical wavefronts



- The wavefronts are spaced by λ in air, by λ/n in the dielectric medium
- The wavefronts remain continuous at the interface
- Refraction at the curved interface causes the wavefronts to bend
- The elliptical shape of the refractive interface at on-axis incidence works out exactly so the planar wavefronts become spherical inside the dielectric medium
  $\Rightarrow$ perfect focusing results (within the approximations of geometrical optics)
- Any shape other than elliptical or off-axis incidence would have resulted in a non-spherical wavefront, therefore imperfect focusing
  - such imperfectly focusing wavefronts are called **aberrated**

The need for "perfect imagers"

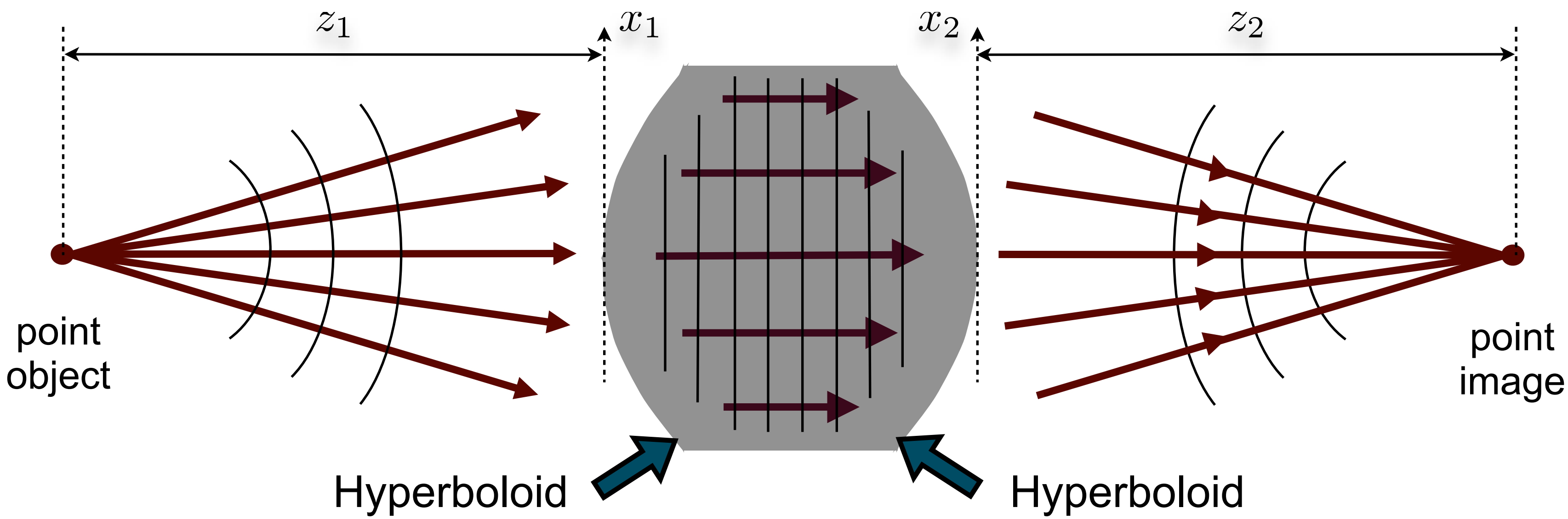


We can think of the job of the imaging system as "**mapping**" **point sources** emanating from scattering of the incident light by object points **to point images**.
Ideally, <u>each</u> object point should be mapped onto a <u>single</u> point image.

However, we saw that even the asphere cannot do a perfect imaging job except for object points on or near the axis. Therefore, imaging can be achieved only approximately.

Perfect imaging on-axis

The purpose of the simplest imaging system is to convert a diverging spherical wave to a converging spherical wave, *i.e.* to image a point object to a point image.



$$\left(s+\frac{z_1}{n+1}\right)^2-\frac{x_1^2}{n^2-1}=\left(\frac{z_1}{n+1}\right)^2 \qquad \left(s-\frac{z_2}{n+1}\right)^2-\frac{x_2^2}{n^2-1}=\left(\frac{z_2}{n+1}\right)^2$$

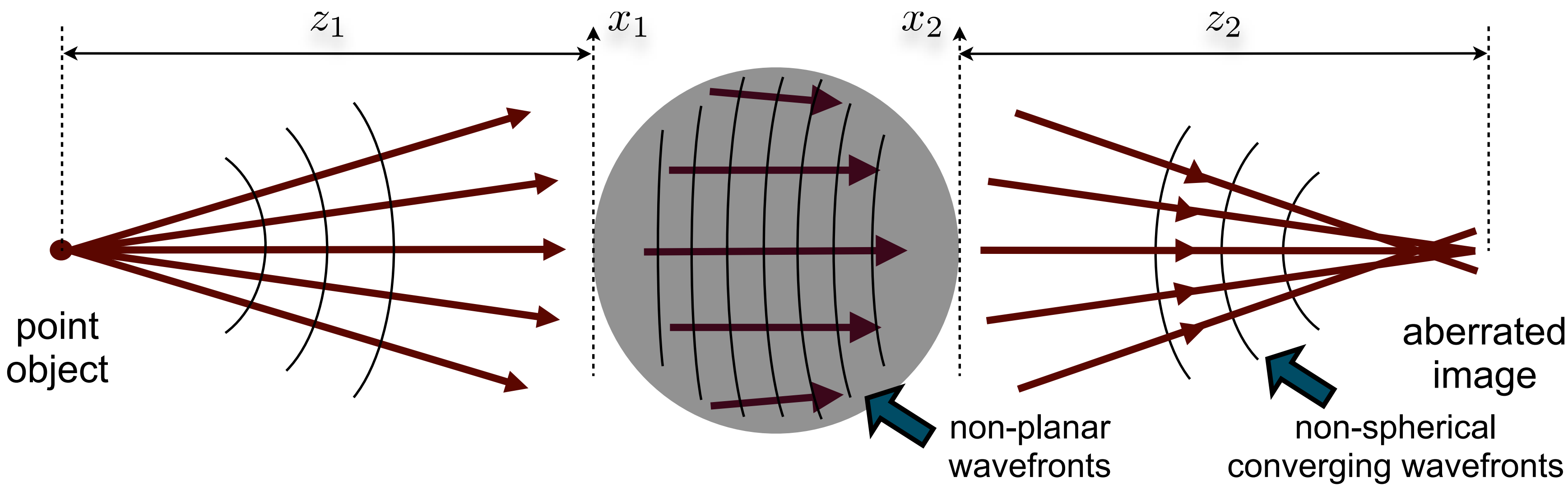
This ideal imaging element is referred to as **asphere**, ("not a sphere" in Greek) or as **aspheric lens**.

It works perfectly on axis and reasonably well in a limited range of angles off-axis.

Manufacturing constraints usually limit refractive elements to spherical surfaces.

Aberrated imaging with spherical elements

If the asphere is replaced by a sphere,
the refracted wavefront inside the sphere is not planar;
neither is the refracted wavefront after the sphere spherical.



The resulting image is imperfect, or **aberrated**,
because the converging rays fail to focus (intersect) at a single point.

Confusingly enough, this type of imperfect imaging is referred to as
spherical aberration.

We will learn about more types of aberrations in detail later.

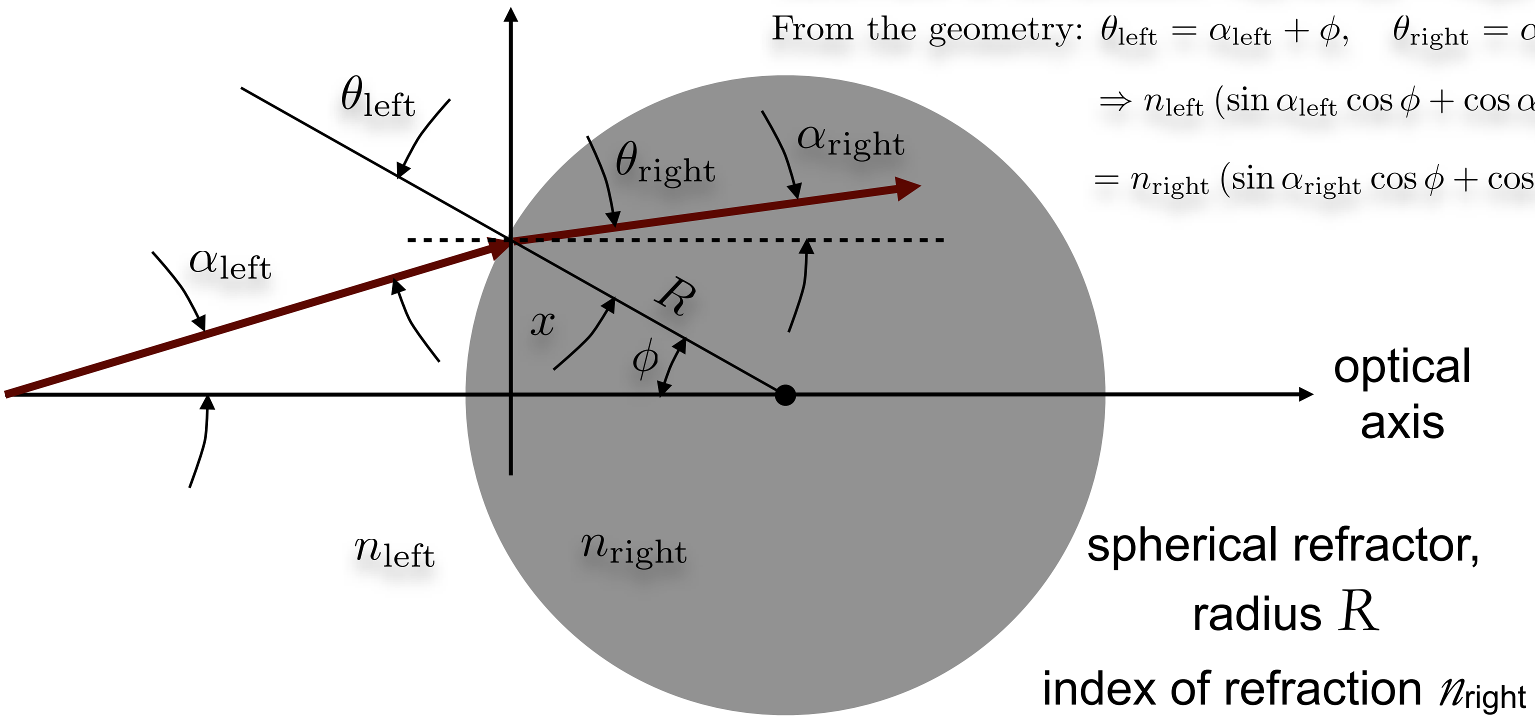
Refraction from a sphere: paraxial approximation

Snell's Law at the interface: $n_{\text{left}} \sin\theta_{\text{left}} = n_{\text{right}} \sin\theta_{\text{right}}$

From the geometry: $\theta_{\text{left}} = \alpha_{\text{left}} + \phi$, $\quad \theta_{\text{right}} = \alpha_{\text{right}} + \phi$

$$\Rightarrow n_{\text{left}} \left(\sin\alpha_{\text{left}}\cos\phi + \cos\alpha_{\text{left}}\sin\phi\right) =$$

$$= n_{\text{right}} \left(\sin\alpha_{\text{right}}\cos\phi + \cos\alpha_{\text{right}}\sin\phi\right)$$



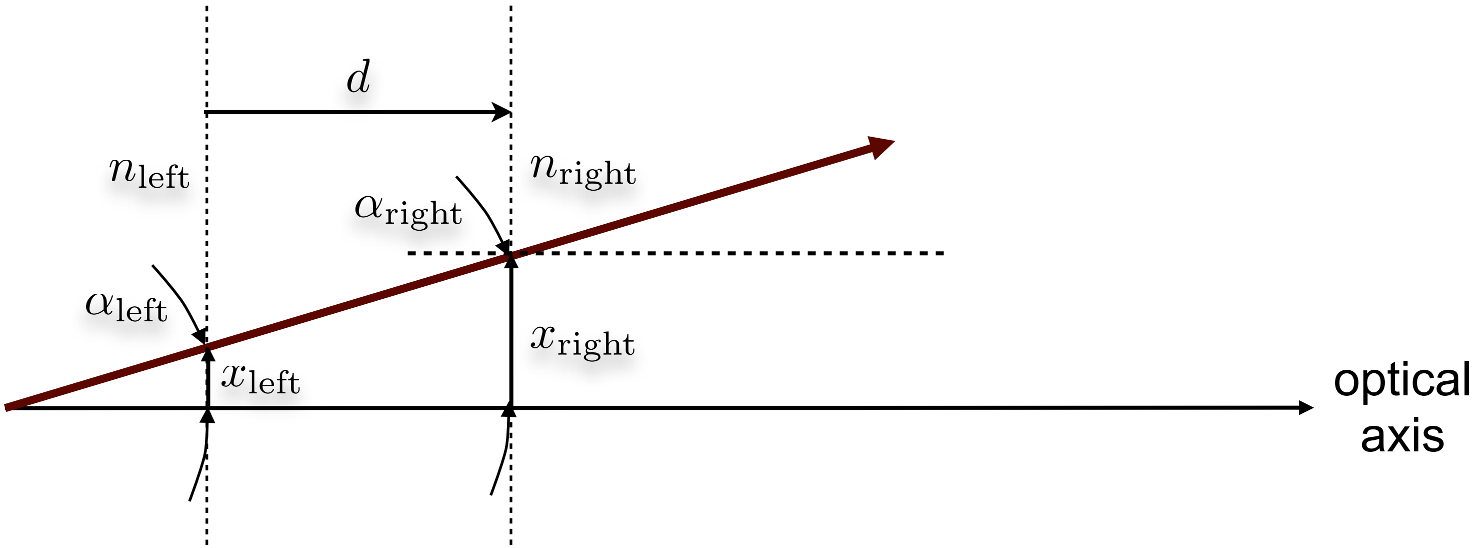
Now assume $x \ll R$, $\alpha_{\text{left}} \ll 1$, $\alpha_{\text{right}} \ll 1$.
This set of assumptions constitutes the **paraxial approximation.**

From this follows:

$$\sin\alpha_{\text{left}} \approx \alpha_{\text{left}}; \quad \cos\alpha_{\text{left}} \approx 1; \quad \sin\alpha_{\text{right}} \approx \alpha_{\text{right}}; \quad \cos\alpha_{\text{right}} \approx 1. \qquad \sin\phi = \frac{x}{R}, \quad \cos\phi \approx 1.$$

$$\Rightarrow n_{\text{left}} \left(\alpha_{\text{left}} + \frac{x}{R}\right) = n_{\text{right}} \left(\alpha_{\text{right}} + \frac{x}{R}\right) \Rightarrow n_{\text{right}}\alpha_{\text{right}} = n_{\text{left}}\alpha_{\text{left}} + \frac{n_{\text{left}} - n_{\text{right}}}{R} x$$

Free space propagation: paraxial approximation

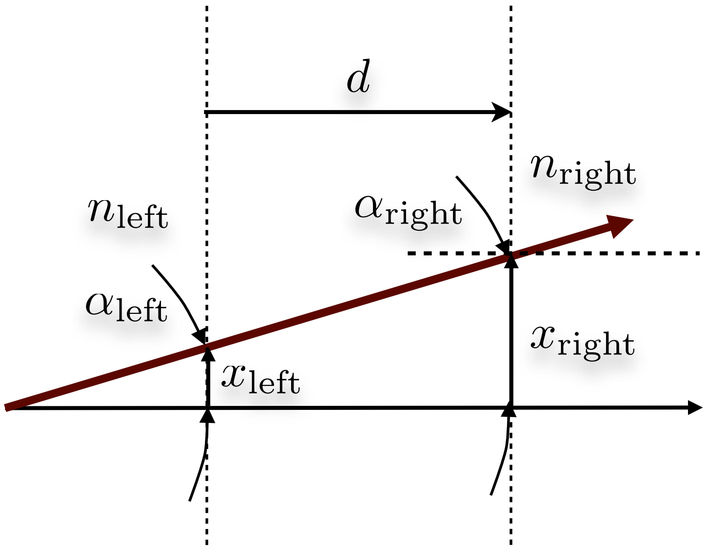


Consider two positions, separated by distance d, along a ray propagating in free space of uniform index of refraction $n_{\text{left}} = n_{\text{right}} \equiv n$.

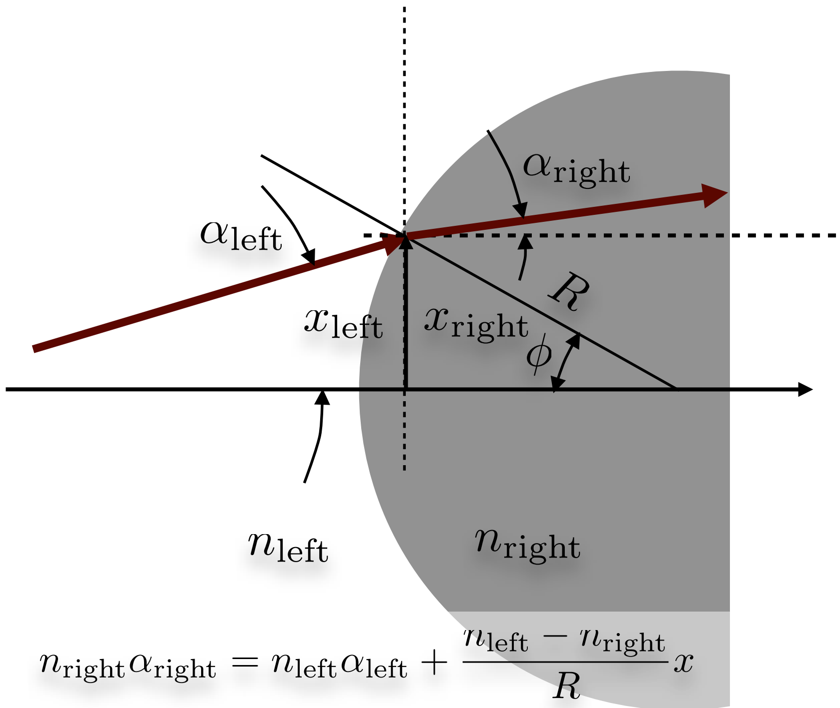
According to the Fermat principle, $n_{\text{left}}\alpha_{\text{left}} = n_{\text{right}}\alpha_{\text{right}}$.

From the geometry we find $x_{\text{right}} = x_{\text{left}} + d \tan\alpha_{\text{left}} \approx x_{\text{left}} + d\alpha_{\text{left}}$, since $\tan\alpha_{\text{left}} \approx \alpha_{\text{left}}$ in the paraxial approximation $\alpha_{\text{left}} \ll 1$.

Ray transfer matrices



optical axis



$$n_{\text{right}}\alpha_{\text{right}} = n_{\text{left}}\alpha_{\text{left}}$$

$$x_{\text{right}} = x_{\text{left}} + d\alpha_{\text{left}}$$

$$n_{\text{right}}\alpha_{\text{right}} = n_{\text{left}}\alpha_{\text{left}} + \frac{n_{\text{left}} - n_{\text{right}}}{R}x$$

$$x_{\text{right}} = x_{\text{left}}$$

or, in matrix form:

$$\begin{pmatrix} n_{\text{right}}\alpha_{\text{right}} \\ x_{\text{right}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{d}{n_{\text{left}}} & 1 \end{pmatrix} \begin{pmatrix} n_{\text{left}}\alpha_{\text{left}} \\ x_{\text{left}} \end{pmatrix} \qquad \begin{pmatrix} n_{\text{right}}\alpha_{\text{right}} \\ x_{\text{right}} \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{n_{\text{right}} - n_{\text{left}}}{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_{\text{left}}\alpha_{\text{left}} \\ x_{\text{left}} \end{pmatrix}$$

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