2.830J / 6.780J / ESD.63J Control of Manufacturing Processes (SMA 6303)
Spring 2008

Control of Manufacturing Processes

Subject 2.830/6.780/ESD.63

Spring 2008

Lecture #14

Aliasing and Higher Order Models

April 3, 2008

Outline

- Last Time
  - Full Factorial Models
  - Experimental Design
    - Blocks and Confounding
    - Single Replicate Designs

- Today
  - Fractional Factorial Designs
  - Aliasing Patterns
  - Implications for Model Construction
  - Process Optimization using DOE

Fractional Factorial Experiments

- What if we do less than full factorial 2^k?
- Example: run < 2^3 experiments for 3 inputs
  - From regression model for 3 inputs:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_3 x_3 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123z} x_1 x_2 x_3 + \varepsilon$$

  - We will not be able to find all 8 coefficients

2^{3-1} Experiment

- Consider doing 4 experiments instead of 8; e.g.:

	x_1	x_2	$x_1 x_2$
1	-1	-1	$+1$
2	$+1$	-1	-1
3	-1	$+1$	-1
4	$+1$	$+1$	$+1$

- This is a 2^2 array
- Could also be for 3 inputs if we define $x_3 = x_1 x_2$

2^{3-1} Experiment

	x_1	x_2	x_3
1	-1	-1	$+1$
2	$+1$	-1	-1
3	-1	$+1$	-1
4	$+1$	$+1$	$+1$

But now we can only define 4 coefficients in the model: e.g.:

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

i.e. no interaction terms

2^{3-1} Experiment

Or we could choose other terms:

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{13} x_1 x_3$$

or:

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_{12} x_1 x_2 + \beta_3 x_3$$

or:

…

Confounding / Aliasing

- We actually have the following:

$$\hat{y} = \beta_0 + \beta'_1 z_1 + \beta'_2 z_2 + \beta'_3 z_3$$

- where the *z* variable represent sums of the various input terms, e.g.

$$z_1 = x_1 x_2 + x_3$$

$$z_2 = x_1 + x_2 x_3 \ldots$$

- where the specific choice of the experimental array determines what these sums are

Confounding / Aliasing

2^3 Array: (Our **X** matrix)

Test	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

Confounding / Aliasing

Consider upper half:

Test	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

Look at columns for C - no change at all! or C = -I

Also AC = -A and BC = -B, and ABC = -AB

Confounding / Aliasing

Test	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

$\text{Contrast}_A = [\,-(1)+a-b+ab\,]$

AC is an alias of A

$\text{Contrast}_{AC} = [\,(1)-a+b-ab\,]$

Note that alias of A =A*(-C)

Defining Relation I = -C

Choice of Design?

- Aliases
  – Must have one of the pair assumed negligible ("sparsity of effects")

- Balance/Orthogonality
  – Sufficient excitation of inputs
  – Enable short-cut estimation of model effects and model coefficients

Balance and Orthogonality

Test	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

Note: All columns have equal number of + and - signs (Balance)

Sum of product of any two columns = 0 (Orthogonality)

-All combinations occur the same number of times

Balance/Orthogonality in 2^{3-1}

Test	I	A	B	C	AB	AC	BC	ABC
1	1	-1	-1	-1	1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
c	1	1	1	-1	1	-1	-1	-1
ab	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	1	-1	1	-1	-1
bc	1	-1	1	1	-1	-1	1	-1
abc	1	1	1	1	1	1	1	1

- A and B are balanced; C is not
- A, B and C are orthogonal

Better Subset – Balanced/Orthogonal

Test	I	A	B	C	AB	AC	BC	ABC
1	1	-1	-1	-1	1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
c	1	1	1	-1	1	-1	-1	-1
ab	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	1	-1	1	-1	-1
bc	1	-1	1	1	-1	-1	1	-1
abc	1	1	1	1	1	1	1	1

With this array:

- balance for A, B, C

- all but ABC are orthogonal

- defining relation I=ABC

e.g. aliases of A:
A*ABC=A*I
A*A = I
BC aliased with A

Aliases:

A BC

B AC

C AB

I ABC

Design Resolution

- Resolution III
  - No main aliases with other main effects
  - Main - interaction aliases
- Resolution IV
  - No alias between main effects and 2 factor effects, but others exist
- Resolution V
  - No main and no 2 factor aliases ….

Design Resolution

Test	I	A	B	C	AB	AC	BC	ABC
1	1	-1	-1	-1	1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
c	1	1	1	-1	1	-1	-1	-1
ab	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	1	-1	1	-1	-1
bc	1	-1	1	1	-1	-1	1	-1
abc	1	1	1	1	1	1	1	1

With this array:

- balance for A, B, C
- all but A B C are orthogonal
- defining relation I=ABC

2^{3-1}_{III}

e.g. aliases of A:
A*ABC=A*I
A*A = I
BC aliased with A

Main effects aliased with interactions only

Aliases:

A BC

B AC

C AB

I ABC

Smaller Fraction 2^{k-p}

- p = 1 1/2 fraction
- p = 2 1/4 fraction
- p $1/2^p$

A Different Fraction

Consider I = AC

Test	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

A Different Fraction

Consider I = AC

Test	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
b	1	-1	1	-1	-1	1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
abc	1	1	1	1	1	1	1	1

Balance?

Orthogonality?

I=AC Aliases

A with C

B with ABC

AB with BC

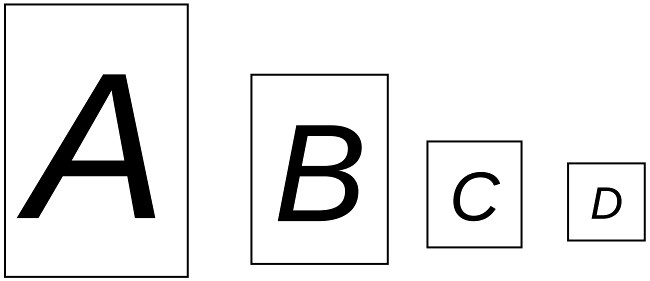
How Decide What Aliasing To Choose?

- Prior knowledge of process

- Rules of thumb
  - Sparsity of effects
  - Hierarchy of effects
  - Inheritance of effects

Sparsity of Effects

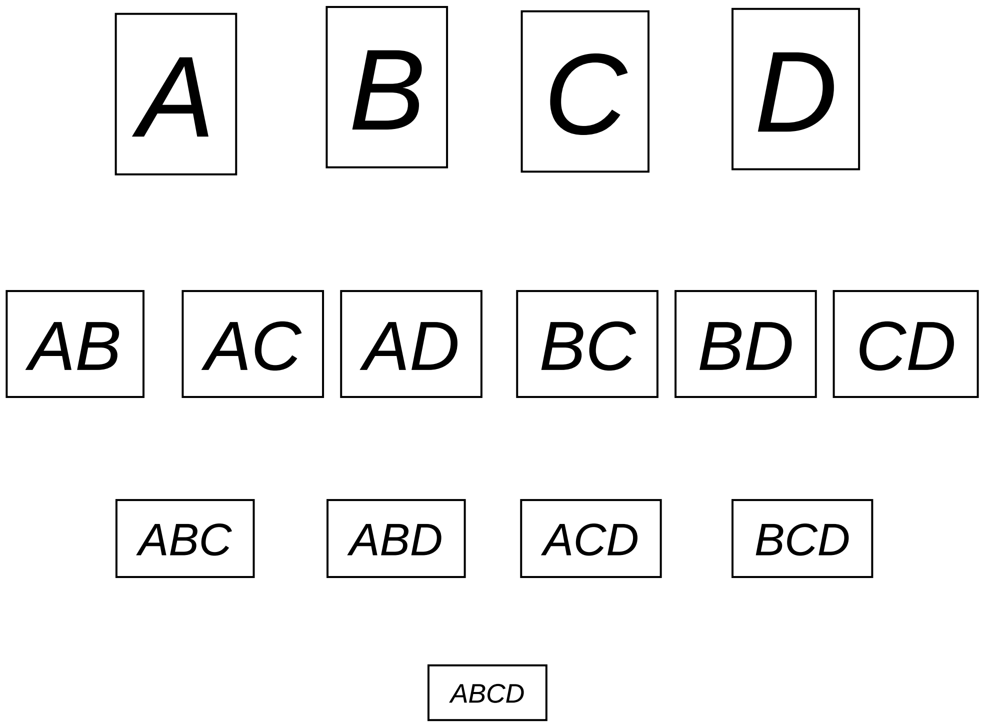
- An experimenter may list a large number of effects for consideration
- A small number of effects usually explain the majority of the variance



Courtesy of Prof. Dan Frey

Hierarchy

- Main effects are usually more important than two-factor interactions
- Two-way interactions are usually more important than three-factor interactions
- And so on



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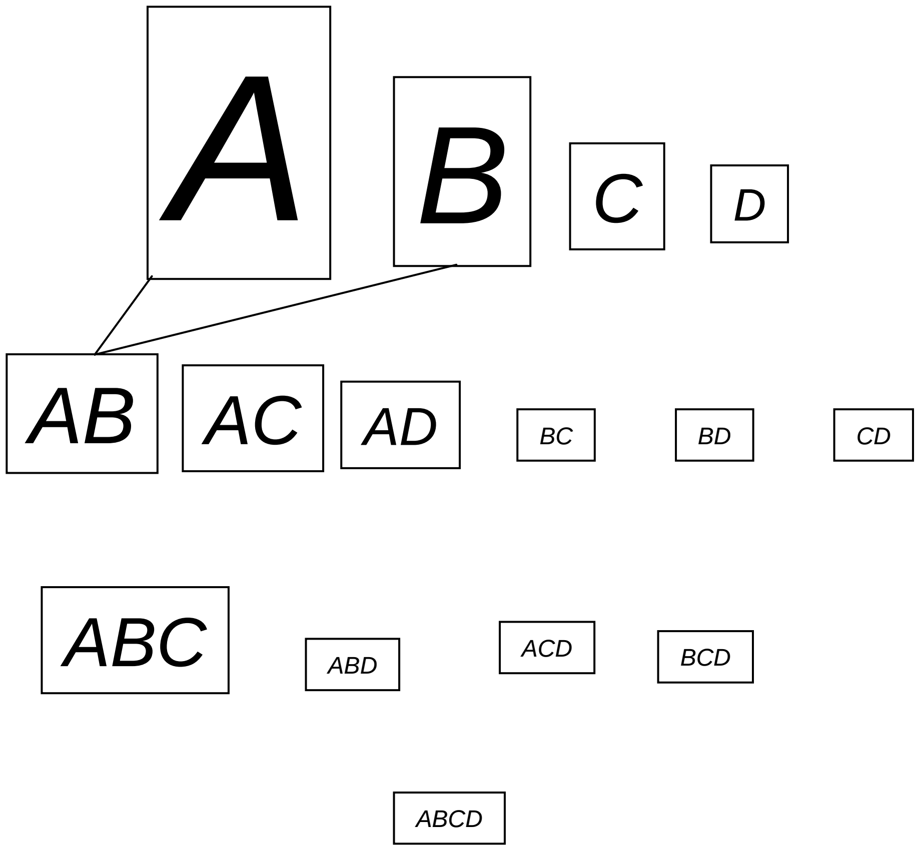
Inheritance

- Two-factor interactions are **most** likely when both participating factors (parents?) are strong

- Two-way interactions are **least** likely when neither parent is strong

- And so on



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Design Resolution

- Resolution III $2^{3-1}_{III} \quad I = ABC$
  – No main aliases with other main effects
  – Main - interaction aliases

- Resolution IV $2^{4-1}_{IV} \quad I = ABCD$
  – No alias between main effects and 2 factor effects, but others exist

- Resolution V $2^{5-1}_{V} \quad I = ABCDE$
  – No main and no 2 factor aliases ….

2^{4-2}

	A	B	C	D
1	-1	-1	-1	-1
2	1	-1	-1	-1
3	-1	1	-1	-1
4	1	1	-1	-1
5	-1	-1	1	-1
6	1	-1	1	-1
7	-1	1	1	-1
8	1	1	-1	-1
9	-1	-1	-1	1
10	1	-1	-1	1
11	-1	1	-1	1
12	1	1	-1	1
13	-1	-1	1	1
14	1	-1	1	1
15	-1	1	1	1
16	1	1	1	1

Four Main Effects
Four tests?

Suppose we want to alias A with BCD and ABC

What are the defining relations?

2^{4-2}

Suppose we want to alias A with BCD and ABC

	I	A	B	AB	C	AC	BC	ABC	D	AD	BD	CD	ABD	ACD	BCD	ABCD
-1	1	-1	-1	1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1
a	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1
b	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1
ab	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	1	1	1
c	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
ac	1	1	-1	-1	1	1	-1	-1	-1	-1	1	-1	1	-1	1	1
bc	1	-1	1	-1	1	-1	1	-1	-1	1	-1	-1	1	1	-1	1
abc	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	1	1	1
d	1	-1	-1	1	-1	1	1	-1	1	-1	-1	-1	1	1	1	-1
ad	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
bd	1	-1	1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	-1	1
cd	1	1	1	1	-1	-1	-1	-1	1	1	1	-1	1	-1	-1	-1
abd	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	-1	-1	1
acd	1	1	-1	-1	1	1	-1	-1	1	1	-1	1	-1	1	-1	-1
bcd	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1
abcd	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

A BCD = I

A ABC = BC = I

Run only (1), bc, ad and abcd

2^{4-2}

Suppose we want to alias A with BCD and ABC

	I	A	B	AB	C	AC	BC	ABC	D	AD	BD	CD	ABD	ACD	BCD	ABCD
(1)	1	-1	-1	1	-1	1	1	-1	-1	1	1	1	-1	-1	-1	1
bc	1	-1	1	-1	1	-1	1	-1	-1	1	-1	-1	1	1	-1	1
ad	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1
abcd	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

A BCD = I

A ABC = BC = I (NB AD = I also)

Aliases?

Defining Relations

I=BC

I=AD

I=ABCD

A - ABC	B - C	C - ABD	D - ABC
A - D	B - ABD	C - ACD	D - BCD
A - BCD	B - ACD		

Outline

- Fractional Factorial Designs
- Aliasing Patterns

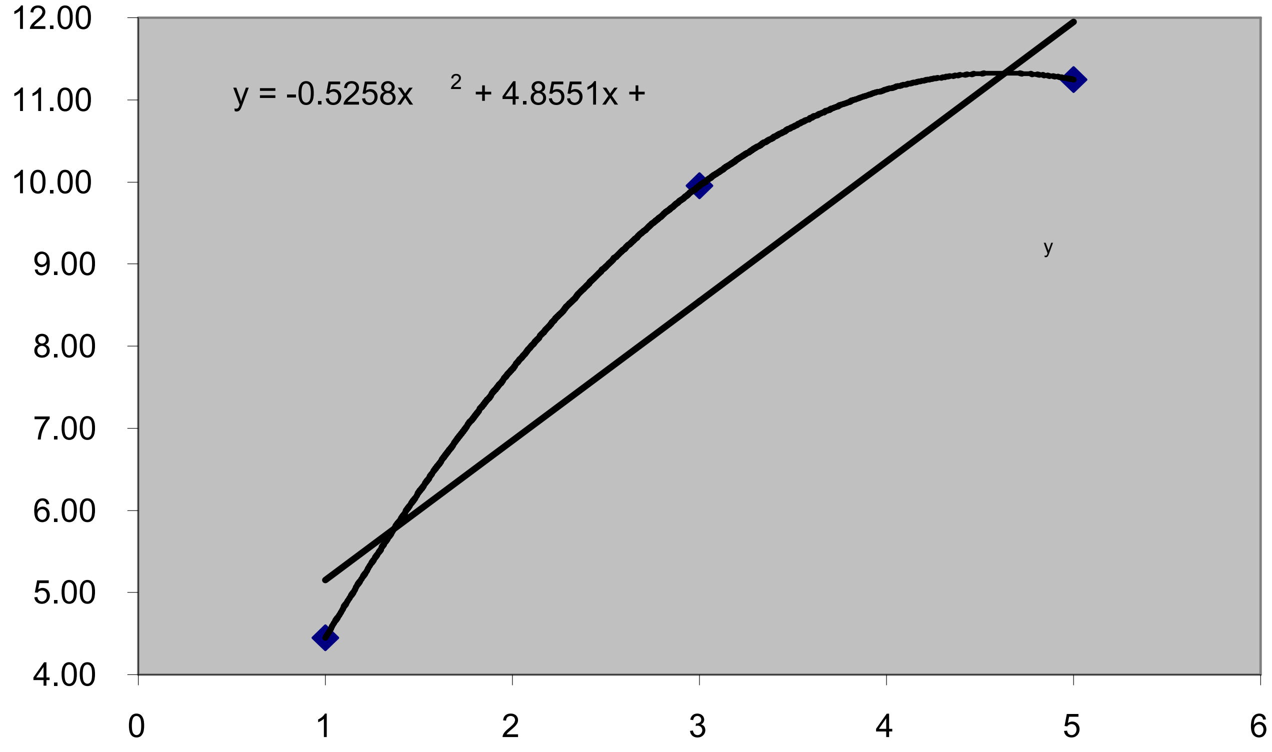
- Implications for Model Construction
- Process Optimization using DOE

Consider Higher Order Model

$$y = \beta_0 + \beta_1 x_1 + \beta_{21} x_1^{\ 2}$$ Quadratic Model

Now we need all 3 tests

$SS_{Residuals}$
$=SS_{within}$



General Quadratic Equation

$$\eta_m = \beta_0 + \sum_{i=1}^{k} \beta_i x_{i\,m} + \sum_{i=1}^{k} \beta_{2i} x_{im}^2 + \sum_{\substack{j=1 \\ j<i}}^{k} \sum_{i=1}^{k} \beta_{ij} x_{im} x_{j\,m} + h.o.t. + \varepsilon_m$$

3^2 Problem

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1{}^2 + \beta_{22} x_2{}^2$$
$$+ \beta_{21} x_1{}^2 x_2 + \beta_{12} x_1 x_2{}^2 + \beta_{222} x_1{}^2 x_2{}^2$$

- How many levels for each input?

Quadratic Solution

- Same as before with matrix equation: $\underline{\eta}=\mathbf{X}\underline{\beta}+\underline{\varepsilon}$

$$
\begin{vmatrix} \eta_1 \\ \eta_2 \\ \eta_2 \\ \vdots \\ \eta_N \end{vmatrix}
=
\begin{vmatrix}
1 & x_{11} & x_{21} & x_{11}^2 & x_{11}^2 & x_{11}x_{21} & \cdots \\
1 & x_{12} & x_{22} & x_{12}^2 & x_{22}^2 & x_{12}x_{22} & \cdots \\
1 & x_{13} & x_{23} & x_{13}^2 & x_{23}^2 & x_{11}x_{21} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
1 & x_{11} & x_{11} & x_{11}^2 & x_{11}^2 & x_{11}x_{21} & \cdots
\end{vmatrix}
\begin{vmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_{11} \\ \beta_{22} \\ \beta_{12} \\ \vdots \end{vmatrix}
+
\begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{vmatrix}
$$

Experimental Design for Quadratic:

- Full factorial 3^k
  - Three levels per test
- Central Composite Design
  - adding to 2x2 design
- Partial Factorials and Aliases

Consider a Quadratic Model w/Interaction

- Includes linear terms, quadratic terms and all first and second-order interactions
- $=3^k$

	N	
k	No Interactions	Full Model
1	3	3
2	5	9
3	7	27
4	9	81
5	11	243

3^2 Full Factorial – Quadratic Model

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{21} x_1^2 x_2 + \beta_{12} x_1 x_2^2 + \beta_{222} x_1^2 x_2^2$$

	(1)	A	B	AB	A2	B2	A2B	B2A	A2B2
y1	1	-1	-1	1	1	1	-1	-1	1
y2	1	0	-1	0	0	1	0	0	0
y3	1	1	-1	-1	1	1	-1	1	1
y4	1	-1	0	0	1	0	0	0	0
y5	1	0	0	0	0	0	0	0	0
y6	1	1	0	0	1	0	0	0	0
y7	1	-1	1	-1	1	1	1	-1	1
y8	1	0	1	0	0	1	0	0	0
y9	1	1	1	1	1	1	1	1	1

Which Partial Fraction?

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^{\ 2} + \beta_{22} x_2^{\ 2}$$

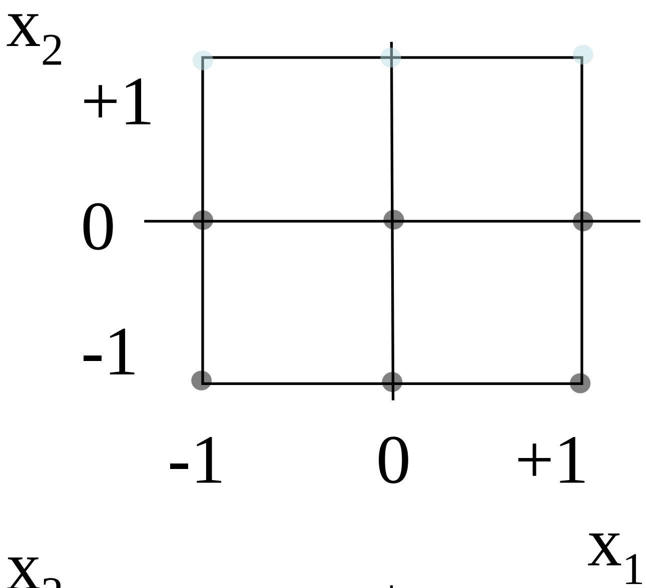
	(1)	A	B	AB	A2	B2	A2B	B2A	A2B2
y1	1	-1	-1	1	1	1	-1	-1	1
y2	1	0	-1	0	0	1	0	0	0
y3	1	1	-1	-1	1	1	-1	1	1
y4	1	-1	0	0	1	0	0	0	0
y5	1	0	0	0	0	0	0	0	0
y6	1	1	0	0	1	0	0	0	0
y7	1	-1	1	-1	1	1	1	-1	1
y8	1	0	1	0	0	1	0	0	0
y9	1	1	1	1	1	1	1	1	1

Which Partial Fraction?

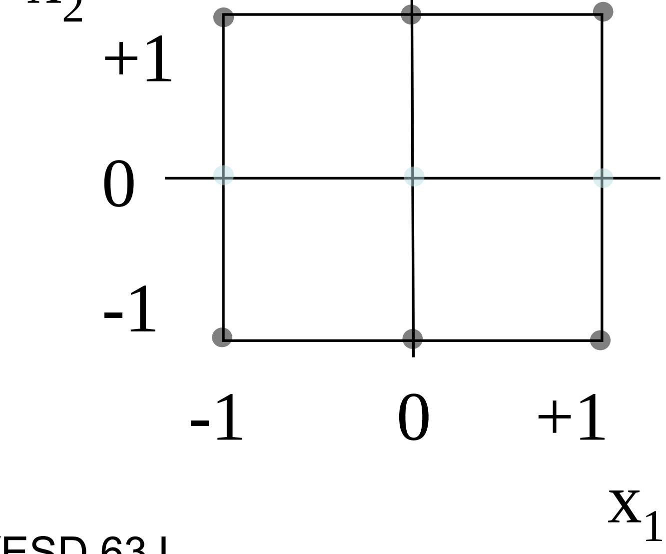
	(1)	A	B	AB	A2	B2	A21	B21	AB22
y1	1	-1	-1	1	1	1	-1	-1	1
y2	1	0	-1	0	0	1	0	0	0
y3	1	1	-1	-1	1	1	-1	1	1
y4	1	-1	0	0	1	0	0	0	0
y5	1	0	0	0	0	0	0	0	0
y6	1	1	0	0	1	0	0	0	0
y7	1	-1	1	-1	1	1	1	-1	1
y8	1	0	1	0	0	1	0	0	0
y9	1	1	1	1	1	1	1	1	1

Which Partial Fraction?

	(1)	A	B	AB	A2	B2	A21	B21	AB22
y1	1	-1	-1	1	1	1	-1	-1	1
y2	1	0	-1	0	0	1	0	0	0
y3	1	1	-1	-1	1	1	-1	1	1
y4	1	-1	0	0	1	0	0	0	0
y5	1	0	0	0	0	0	0	0	0
y6	1	1	0	0	1	0	0	0	0
y7	1	-1	1	-1	1	1	1	-1	1
y8	1	0	1	0	0	1	0	0	0
y9	1	1	1	1	1	1	1	1	1

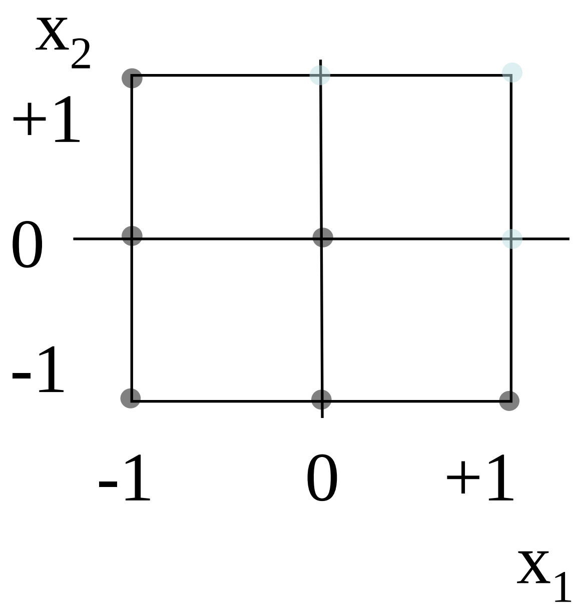


	(1)	A	B	AB	A2	B2	A21	B21	AB22
y1	1	-1	-1	1	1	1	-1	-1	1
y2	1	0	-1	0	0	1	0	0	0
y3	1	1	-1	-1	1	1	-1	1	1
y4	1	-1	0	0	1	0	0	0	0
y5	1	0	0	0	0	0	0	0	0
y6	1	1	0	0	1	0	0	0	0
y7	1	-1	1	-1	1	1	1	-1	1
y8	1	0	1	0	0	1	0	0	0
y9	1	1	1	1	1	1	1	1	1



Which Partial Fraction?

	(1)	A	B	AB	A2	B2	A21	B21	AB22
y1	1	-1	-1	1	1	1	-1	-1	1
y2	1	0	-1	0	0	1	0	0	0
y3	1	1	-1	-1	1	1	-1	1	1
y4	1	-1	0	0	1	0	0	0	0
y5	1	0	0	0	0	0	0	0	0
y6	1	1	0	0	1	0	0	0	0
y7	1	-1	1	-1	1	1	1	-1	1
y8	1	0	1	0	0	1	0	0	0
y9	1	1	1	1	1	1	1	1	1



	(1)	A	B	AB	A2	B2
y1	1	-1	-1	1	1	1
y2	1	0	-1	0	0	1
y3	1	1	-1	-1	1	1
y4	1	-1	0	0	1	0
y5	1	0	0	0	0	0
y7	1	-1	1	-1	1	1

Quadratic Solution

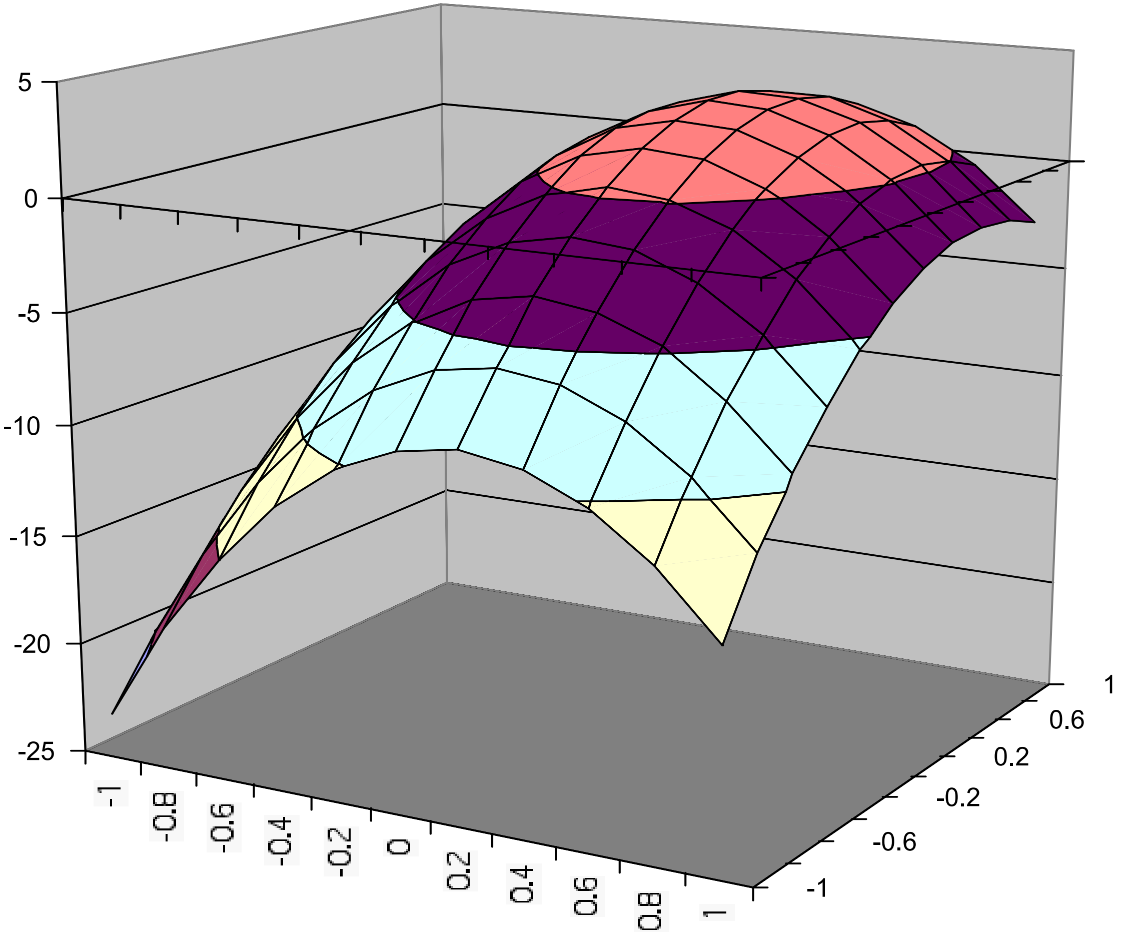
$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^{\ 2} + \beta_{22} x_2^{\ 2} + \beta_{12} x_1 x_2$$

$$\begin{vmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \\ \bar{y}_4 \\ \bar{y}_5 \\ \bar{y}_6 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & x_2 & x_1^{\ 2} & x_2^{\ 2} & x_1 x_2 \\ 1 & x_1 & x_2 & x_1^{\ 2} & x_2^{\ 2} & x_1 x_2 \\ 1 & x_1 & x_2 & x_1^{\ 2} & x_2^{\ 2} & x_1 x_2 \\ 1 & x_1 & x_2 & x_1^{\ 2} & x_2^{\ 2} & x_1 x_2 \\ 1 & x_1 & x_2 & x_1^{\ 2} & x_2^{\ 2} & x_1 x_2 \\ 1 & x_1 & x_2 & x_1^{\ 2} & x_2^{\ 2} & x_1 x_2 \end{vmatrix} \begin{vmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{vmatrix}$$

$$\underline{y} = X\underline{\beta}$$

$$\underline{\beta} = X^{-1}\underline{y}$$

A Quadratic Surface



$$y = 1 + 5x_1 + 5x_2 + x_1 x_2 - 10x_1^2 - 5x_2^2$$

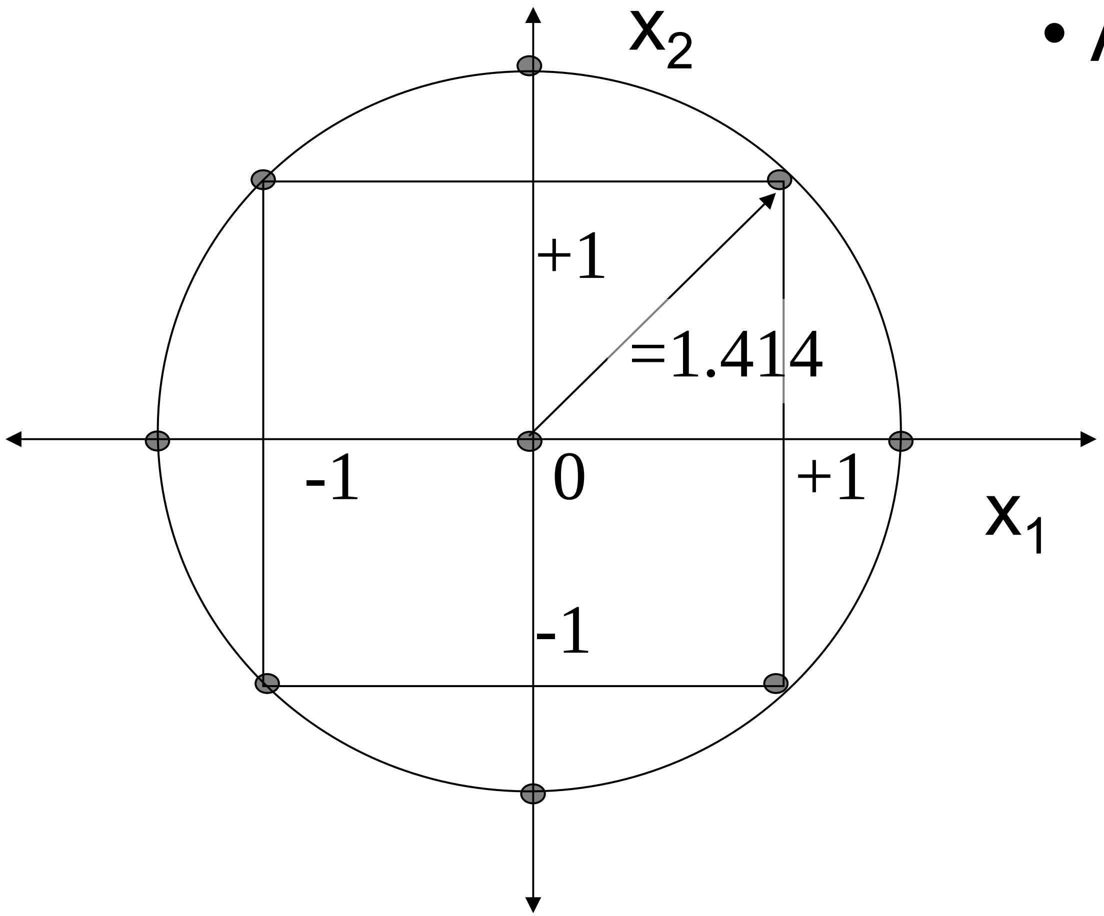
A "Standard" 3^2 Full Factorial Design

Test	x1	X2
1	-1	-1
2	0	-1
3	1	-1
4	-1	0
5	0	0
6	1	0
7	-1	1
8	0	1
9	1	1

Central Composite Design

- Consider the case:
  - First Experiment is 2^2 with 4 tests
  - Model is shown to have poor fit
    - High SS_{Quad} for intermediate point
  - Decide to go to Quadratic
  - Not Sure of Shape of Surface

Central Composite Design



- Add 5 additional points:
  - One at center
  - One equidistant from center along each axis

Central Composite

Test	x1	X2
1	-1	-1
2	+1	-1
3	-1	+1
4	+1	+1
5	0	0
6	0	1.414
7	1.414	0
8	0	-1.414
9	-1.414	0

original tests (tests 1–4)

additional tests (tests 5–9)

Outline

- Fractional Factorial Designs
- Aliasing Patterns
- Implications for Model Construction

- Process Optimization using DOE

Process Optimization

- Create an Objective Function "J"

  Minimize or Maximize

$$\max_{\underline{x}} J \qquad \min_{\underline{x}} J$$

J=J(factors) ; J($\underline{x}$); J(α)

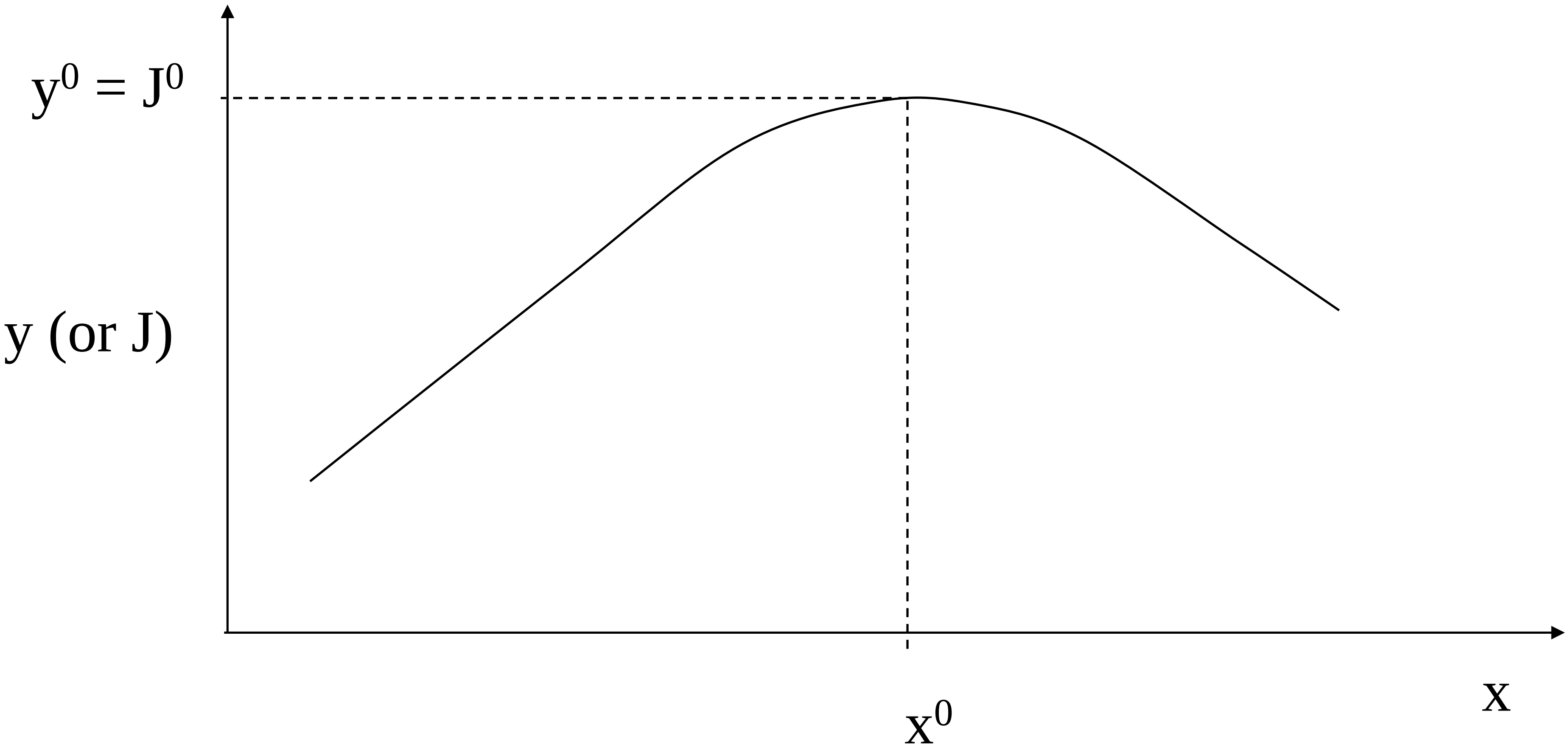
Adjust J via factors

with constraints, such as….

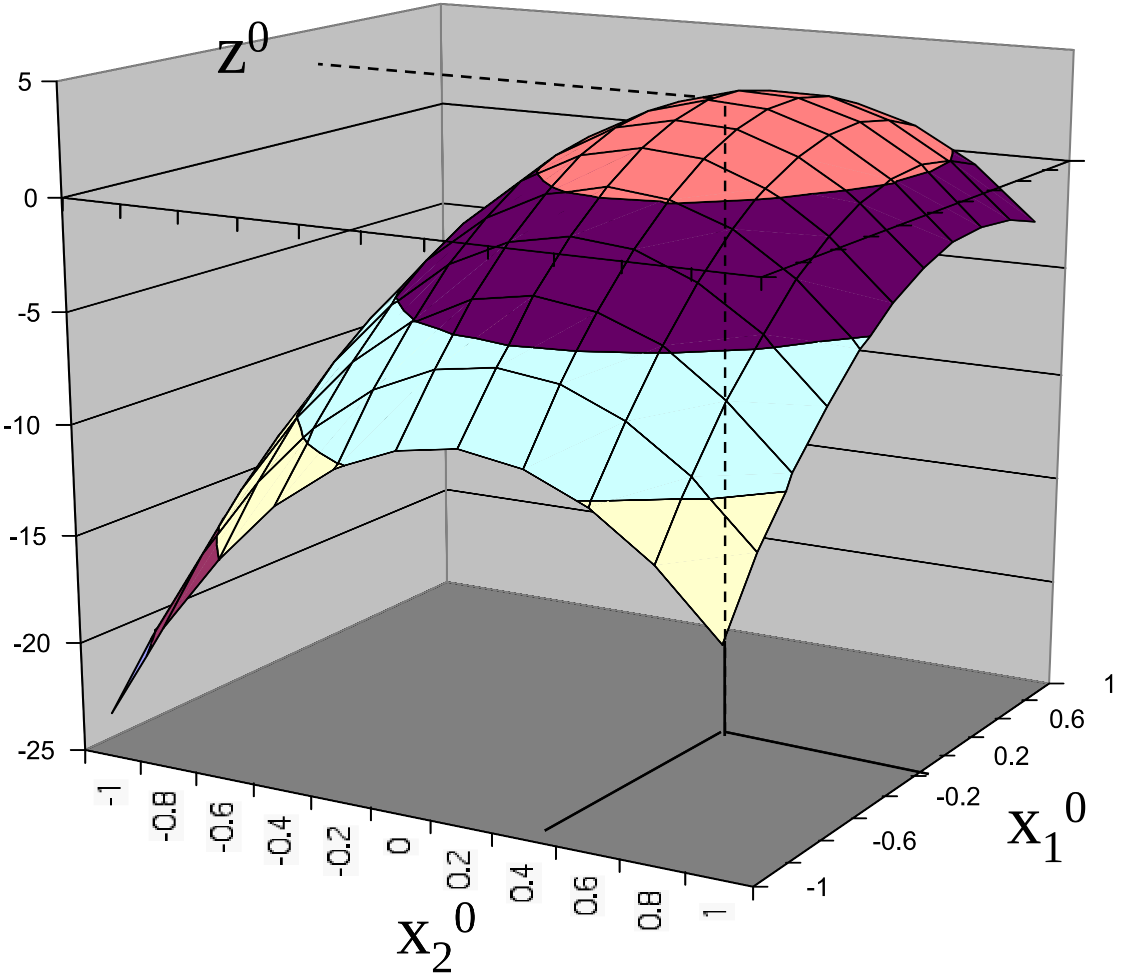
Methods for Optimization

- Analytical Solutions
  - $\partial y / \partial x = 0$
- Gradient Searches
  - Hill climbing (steepest ascent/descent)
  - Local min or max problem
  - Excel solver given a convex function

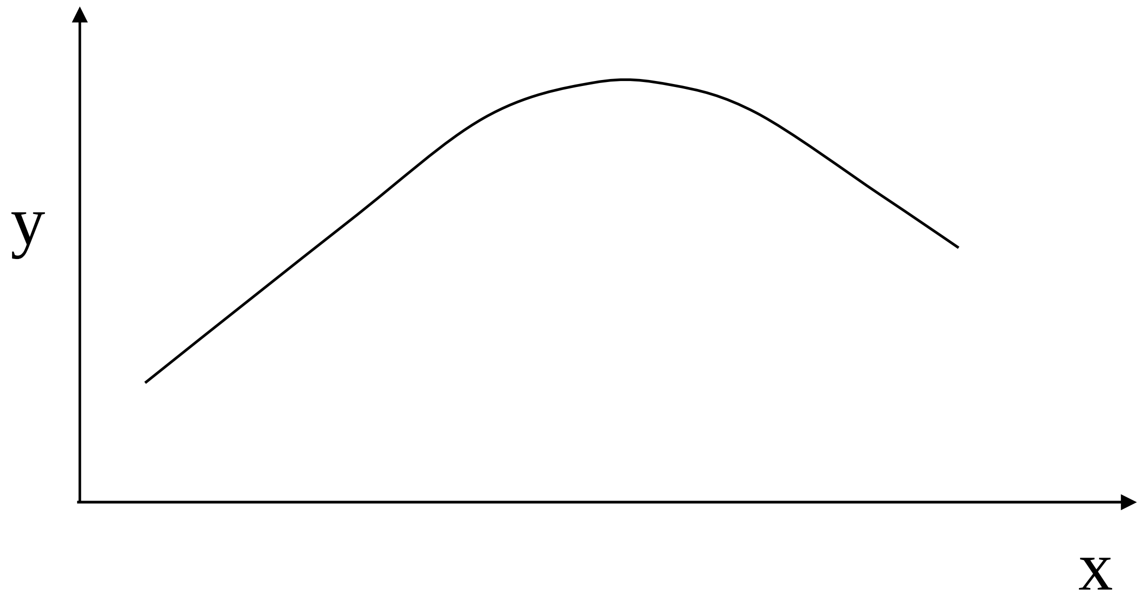
Basic Optimization Problem



3D Problem



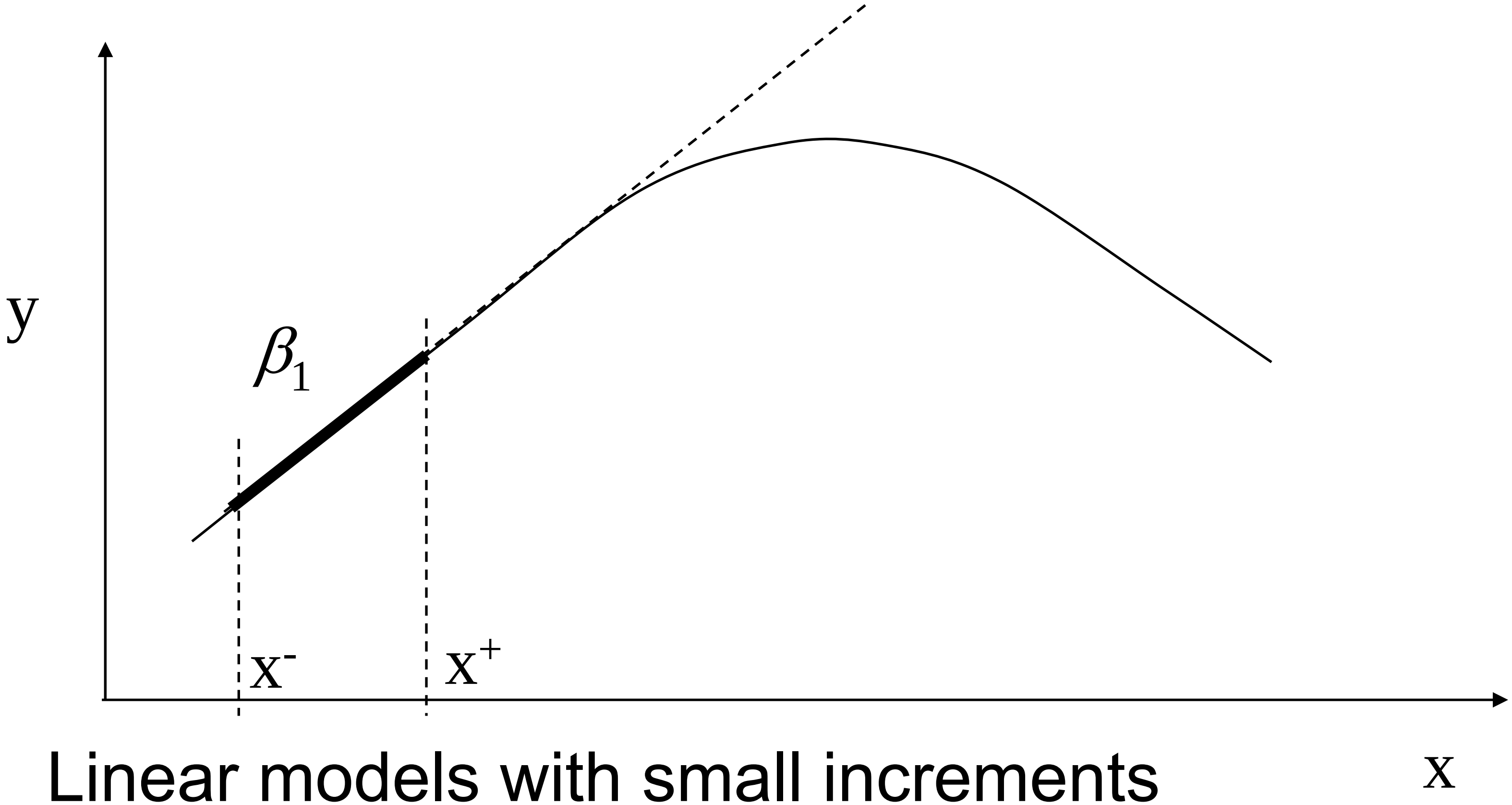
Analytical



$$\frac{\partial y(x)}{\partial x} = 0$$

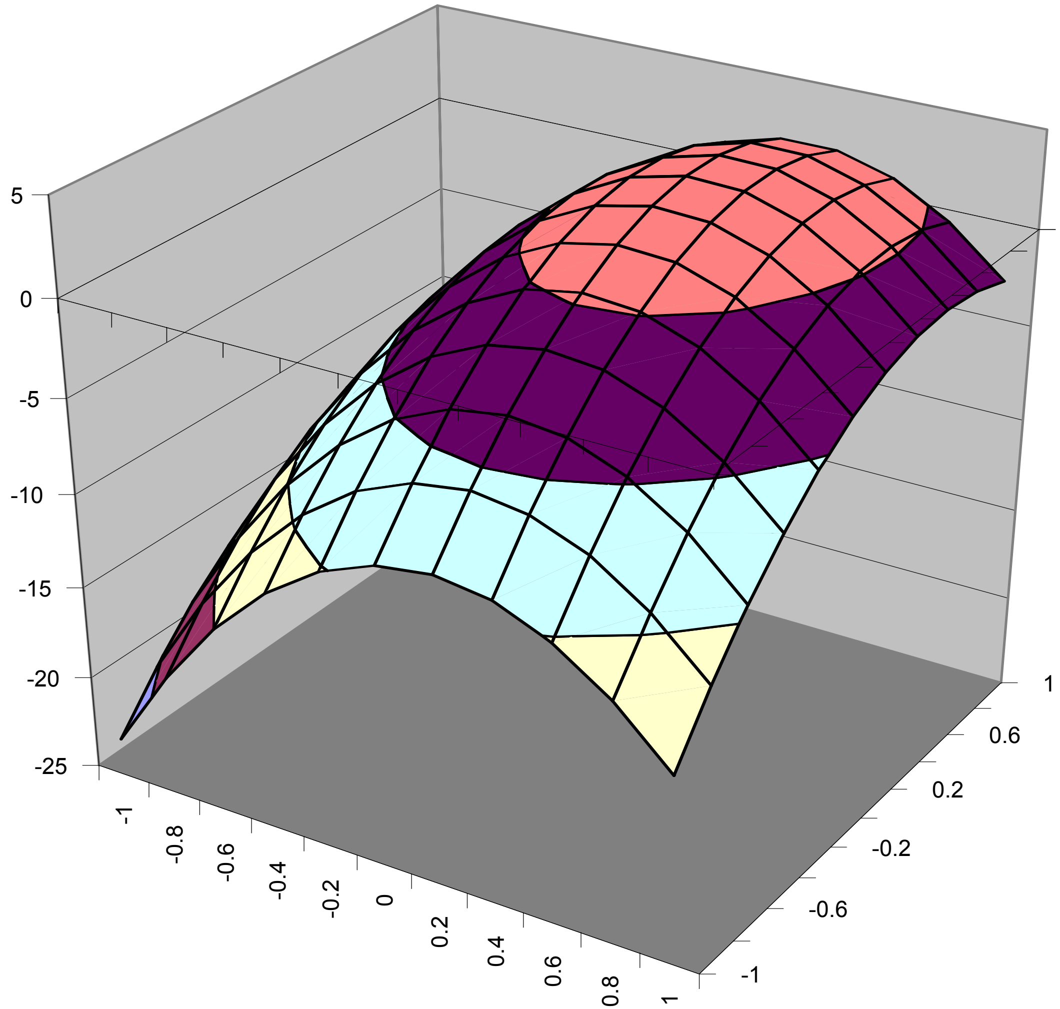
- Need Accurate *y(x)*
  - Analytical Model
  - Dense x increments in Experiment
- Difficult with Sparse Experiments
  - Easy to missing optimum

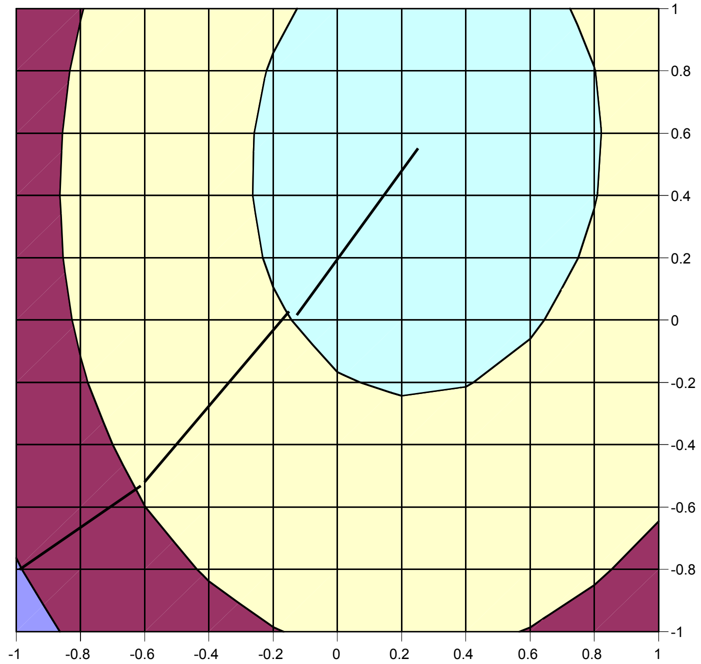
Sparse Data Procedure



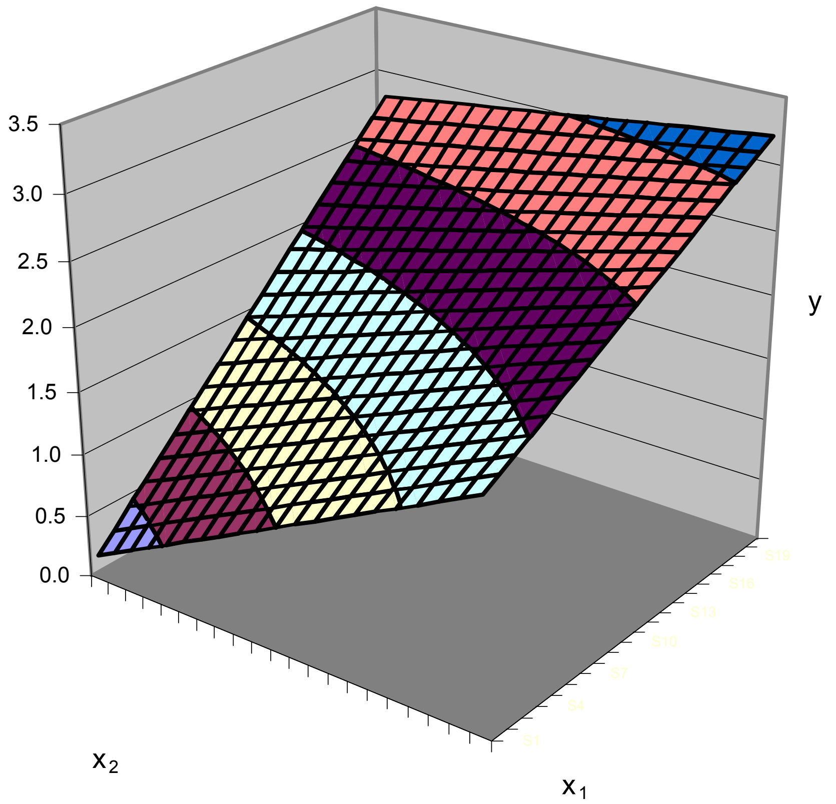
- Linear models with small increments
- Move along desired gradient
- Near zero slope change to quadratic model

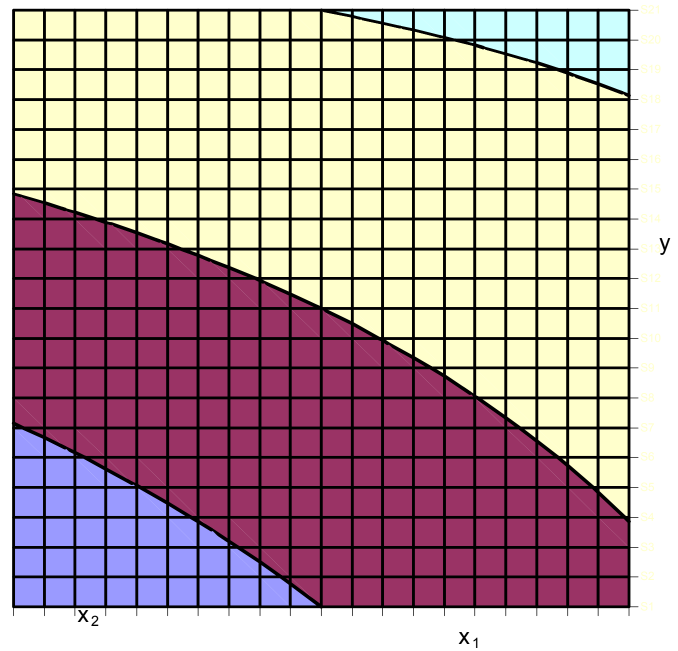
Extension to 3D





Linear Model Gradient Following





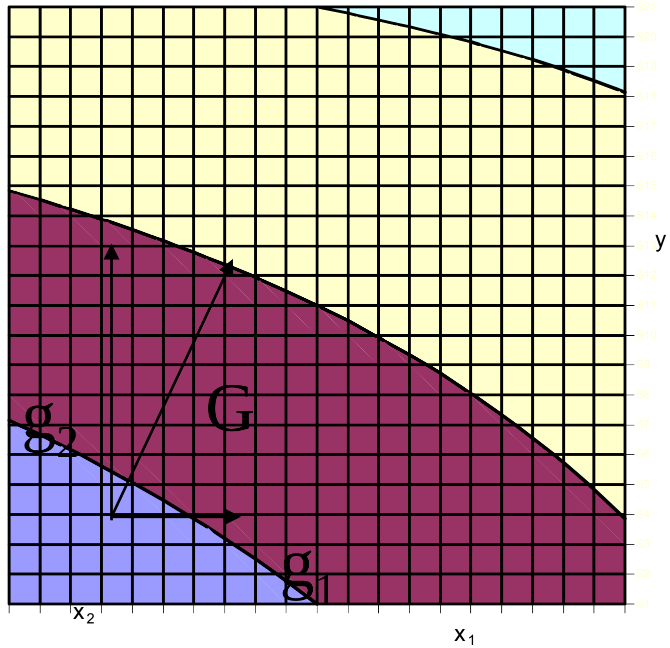
$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

Steepest Descent

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

$$g_{x_1} = \frac{\partial y}{\partial x_1} = \beta_1 + \beta_{12} x_2$$

$$g_{x_2} = \frac{\partial y}{\partial x_2} = \beta_2 + \beta_{12} x_1$$



Make changes in x_1 and x_2 along G

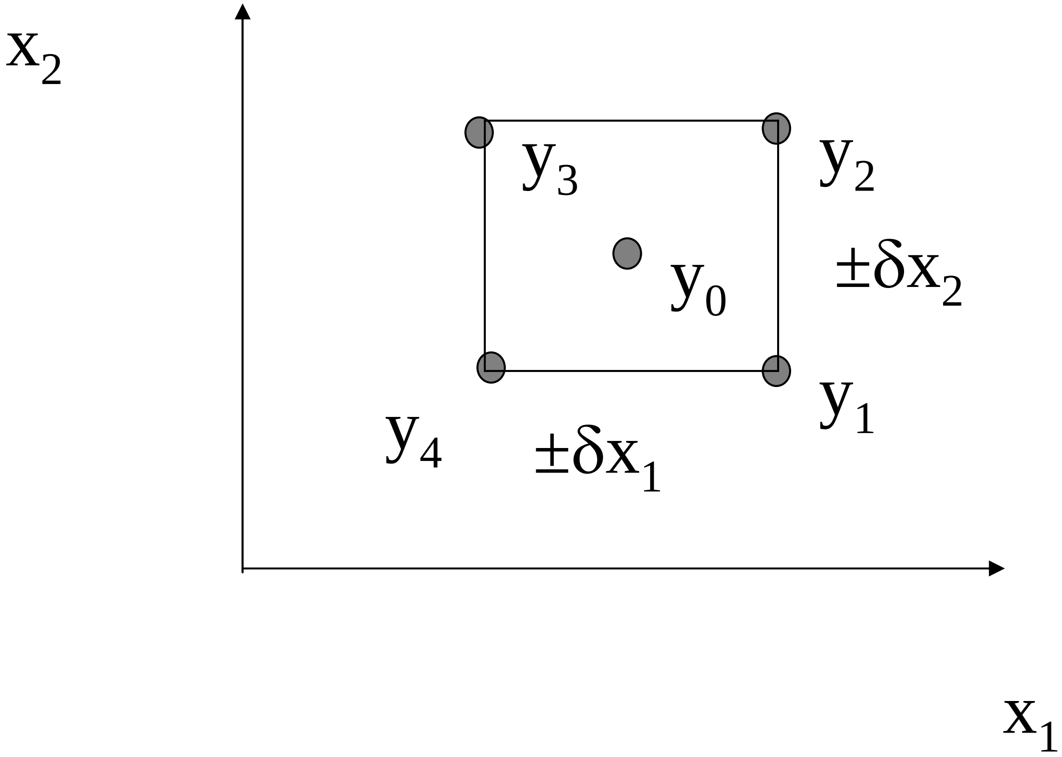
$$\Delta x_2 = \frac{g_{x_1}}{g_{x_2}} \Delta x_1$$

Experimental Optimization

- WHY NOT JUST PICK BEST POINT?
- Why not optimize on-line?
  - Skip the Modeling Step!
- Adaptive Methods
  - Learn how best to model as you go.
    - e.g. Adaptive OFACT

EVOP

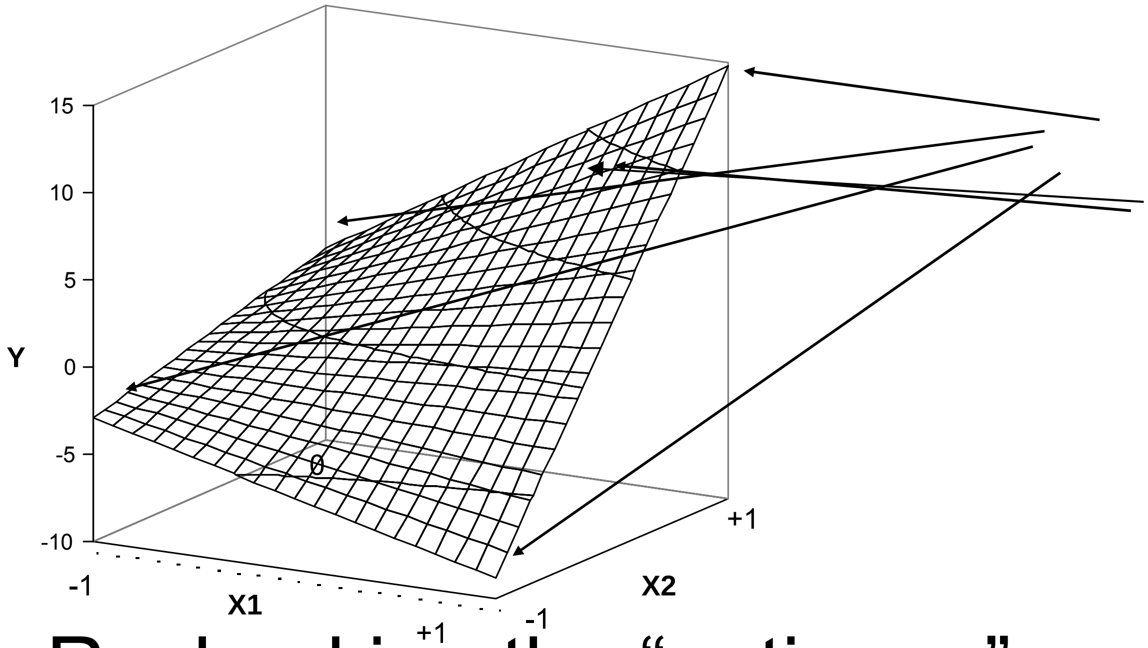
- Evolutionary Operation



- Pick “best” y_i
- Re-center process
- Do again.

Confirming Experiments

- Checking Intermediate points



- Data only at corners
- Test at interior point
- Evaluate error
- Consider Central Composite?

- Rechecking the “optimum”

A Procedure for DOE/Optimization

- Study Physics of Process
  - Define Important Inputs
  - Intuition about model
  - Limits on inputs
- Define Optimization Penalty Function
  - J=f(x)

$$\max_{\underline{x}} J \qquad \min_{\underline{x}} J$$

For us, $\underline{x} = \underline{u}$ or $\underline{\alpha}$

Procedure

- Identify model (linear, quadratic, terms to include)
- Define inputs and ranges
- Identify "noise" parameters to vary if possible ($\Delta\alpha$'s)
- Perform Experiment
  - Appropriate order
    - randomization
    - blocking against nuisance or confounding effects

Procedure

- Solve for $\underline{\beta}$'s
- Apply ANOVA
  - Data significant?
  - Terms significant?
  - Lack of Fit Significant?
- Drop Insignificant Terms
- Add Higher Order Terms as needed

Procedure

- Search for Optimum
  - Analytically
  - Piecewise
  - Continuously

Procedure

- Find Optimum value x*
- Perform Confirming experiment
  - Test Model at x*
  - Evaluate error with respect to model
  - Test hypothesis that $y(\underline{x}^*) = \hat{y}(\underline{x}^*)$

Procedure

- If hypothesis fails
  - Consider new ranges for inputs
  - Consider higher order model as needed
  - Boundary may be optimum!

Summary

- Fractional Factorial Designs
- Aliasing Patterns
- Implications for Model Construction
- Process Optimization using DOE