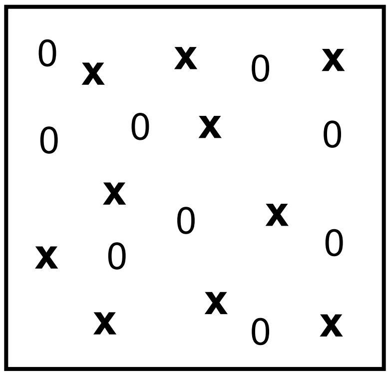
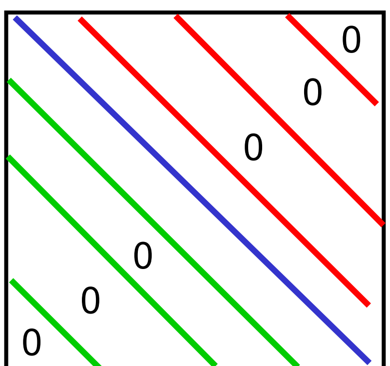
Introduction to Numerical Analysis for Engineers

- Systems of Linear Equations — Mathews
  - Cramer's Rule
  - Gaussian Elimination — 3.3-3.5
    - Numerical implementation 3.3-3.4
    - Numerical stability
      - Partial Pivoting
      - Equilibration
      - Full Pivoting
    - Multiple right hand sides
    - Computation count
    - LU factorization — 3.5
    - Error Analysis for Linear Systems — 3.4
      - Condition Number
    - Special Matrices
  - Iterative Methods — 3.6
    - Jacobi's method
    - Gauss-Seidel iteration
    - Convergence

Linear Systems of Equations
Iterative Methods

Sparse, Full-bandwidth Systems





Rewrite Equations

$$\overline{\overline{\mathbf{A}}}\,\overline{\mathbf{x}} = \overline{\mathbf{b}} \quad \Leftrightarrow \quad \sum_{j=1}^{n} a_{ij} x_j = b_i$$

$$a_{ii} \neq 0 \Rightarrow x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}} \;,\; i = 1, \ldots n$$

Iterative, Recursive Methods

Jacobi's Method

$$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}} \;,\; i = 1, \ldots n$$

Gauss-Seidels's Method

$$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}} \;,\; i = 1, \ldots n$$

Linear Systems of Equations
Iterative Methods

Convergence

$$\left\|\overline{\mathbf{x}}^{(k+1)} - \overline{\mathbf{x}}\right\| \rightarrow 0 \ \text{ for } \ k \rightarrow \infty$$

Iteration – Matrix form

$$\overline{\mathbf{x}}^{(k+1)} = \overline{\overline{\mathbf{B}}}\overline{\mathbf{x}}^{(k)} + \overline{\mathbf{c}} \ , \ k = 0, \ldots$$

Decompose Coefficient Matrix

$$\overline{\overline{\mathbf{A}}} = \overline{\overline{\mathbf{D}}}\left(\overline{\overline{\mathbf{L}}} + \overline{\overline{\mathbf{I}}} + \overline{\overline{\mathbf{U}}}\right)$$

with

$$\overline{\overline{\mathbf{D}}} = \operatorname{diag} a_{ii}$$

$$\overline{\overline{\mathbf{L}}} = \begin{cases} a_{ij}/a_{ii} \ , & i > j \\ 0, & i \leq j \end{cases}$$

$$\overline{\overline{\mathbf{U}}} = \begin{cases} a_{ij}/a_{ii} \ , & i < j \\ 0, & i \geq j \end{cases}$$

Jacobi's Method

$$\overline{\mathbf{x}}^{(k+1)} = -\left(\overline{\overline{\mathbf{L}}} + \overline{\overline{\mathbf{U}}}\right)\overline{\mathbf{x}}^{(k)} + \overline{\overline{\mathbf{D}}}^{-1}\overline{\mathbf{b}}$$

Iteration Matrix form

$$\begin{aligned} \overline{\overline{\mathbf{B}}} &= -\left(\overline{\overline{\mathbf{L}}} + \overline{\overline{\mathbf{U}}}\right) \\ \overline{\mathbf{c}} &= \overline{\overline{\mathbf{D}}}^{-1}\overline{\mathbf{b}} \end{aligned}$$

Convergence Analysis

$$\begin{aligned} \overline{\mathbf{x}}^{(k+1)} &= \overline{\overline{\mathbf{B}}}\overline{\mathbf{x}}^{(k)} + \overline{\mathbf{c}} \\ \overline{\mathbf{x}} &= \overline{\overline{\mathbf{B}}}\overline{\mathbf{x}} + \overline{\mathbf{c}} \\ \hline \overline{\mathbf{x}}^{(k+1)} - \overline{\mathbf{x}} &= \overline{\overline{\mathbf{B}}}\left(\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right) \\ &= \overline{\overline{\mathbf{B}}} \cdot \overline{\overline{\mathbf{B}}}\left(\overline{\mathbf{x}}^{(k-1)} - \overline{\mathbf{x}}\right) \\ &\cdot \\ &= \overline{\overline{\mathbf{B}}}^{k+1}\left(\overline{\mathbf{x}}^{(0)} - \overline{\mathbf{x}}\right) \end{aligned}$$

$$\left\|\overline{\mathbf{x}}^{(k+1)} - \overline{\mathbf{x}}\right\| \leq \left\|\overline{\overline{\mathbf{B}}}^{k+1}\right\| \left\|\overline{\mathbf{x}}^{(0)} - \overline{\mathbf{x}}\right\| \leq \left\|\overline{\overline{\mathbf{B}}}\right\|^{k+1} \left\|\overline{\mathbf{x}}^{(0)} - \overline{\mathbf{x}}\right\|$$

Sufficient Convergence Condition

$$\boxed{\left\|\overline{\overline{\mathbf{B}}}\right\| < 1}$$

Linear Systems of Equations
Iterative Methods

Sufficient Convergence Condition

$$\left\|\overline{\overline{\mathbf{B}}}\right\| < 1$$

Jacobi's Method

$$b_{ij} = -\frac{a_{ij}}{a_{ii}} \ , \ i \neq j$$

$$\left\|\overline{\overline{\mathbf{B}}}\right\|_{\infty} = \max_{i} \sum_{j=1, j\neq i}^{n} \frac{|a_{ij}|}{|a_{ii}|}$$

Sufficient Convergence Condition

$$\sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}|$$

Diagonal Dominance

Stop Criterion for Iteration

$$\begin{aligned} \overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}} &= \overline{\overline{\mathbf{B}}}\left(\overline{\mathbf{x}}^{(k-1)} - \overline{\mathbf{x}}\right) \qquad \overline{\overline{\mathbf{B}}}\,\overline{\mathbf{x}}^{(k)} \\ &= -\overline{\overline{\mathbf{B}}}\left(\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}^{(k-1)}\right) + \overline{\overline{\mathbf{B}}}\left(\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right) \end{aligned}$$

$$\left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right\| \leq \left\|\overline{\overline{\mathbf{B}}}\right\| \left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}^{(k-1)}\right\| + \left\|\overline{\overline{\mathbf{B}}}\right\| \left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right\|$$

$$\left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right\| \leq \frac{\left\|\overline{\overline{\mathbf{B}}}\right\|}{1 - \left\|\overline{\overline{\mathbf{B}}}\right\|} \left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}^{(k-1)}\right\|$$

$$\boxed{\left\|\overline{\overline{\mathbf{B}}}\right\| < 1/2 \Rightarrow \left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}\right\| \leq \left\|\overline{\mathbf{x}}^{(k)} - \overline{\mathbf{x}}^{(k-1)}\right\|}$$

vib_string.m

```
n=99;
L=1.0;
h=L/(n+1);
k=2*pi;
kh=k*h
x=[h:h:L-h]';
a=zeros(n,n);
f=zeros(n,1);
o=1
```

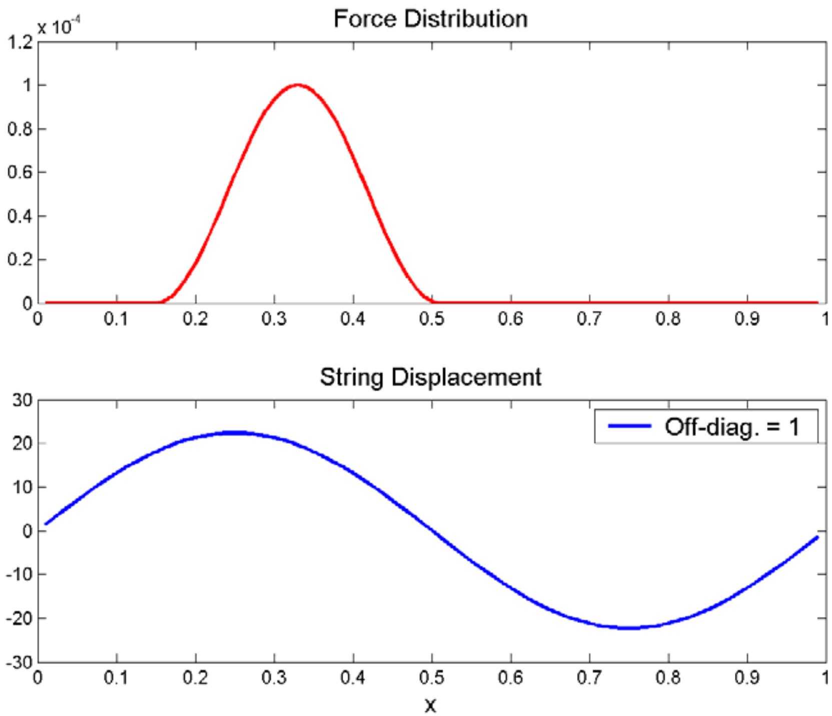
Off-diagonal values

```
a(1,1) =kh^2 - 2;
a(1,2)=o;

for i=2:n-1
    a(i,i)=a(1,1);
    a(i,i-1) = o;
    a(i,i+1) = o;
end
a(n,n)=a(1,1);
a(n,n-1)=o;

nf=round((n+1)/3);
nw=round((n+1)/6);
nw=min(min(nw,nf-1),n-nf);
figure(1)
hold off

nw1=nf-nw;
nw2=nf+nw;
f(nw1:nw2) = h^2*hanning(nw2-nw1+1);
subplot(2,1,1); plot(x,f,'r');
% exact solution
y=inv(a)*f;
subplot(2,1,2); plot(x,y,'b');
```

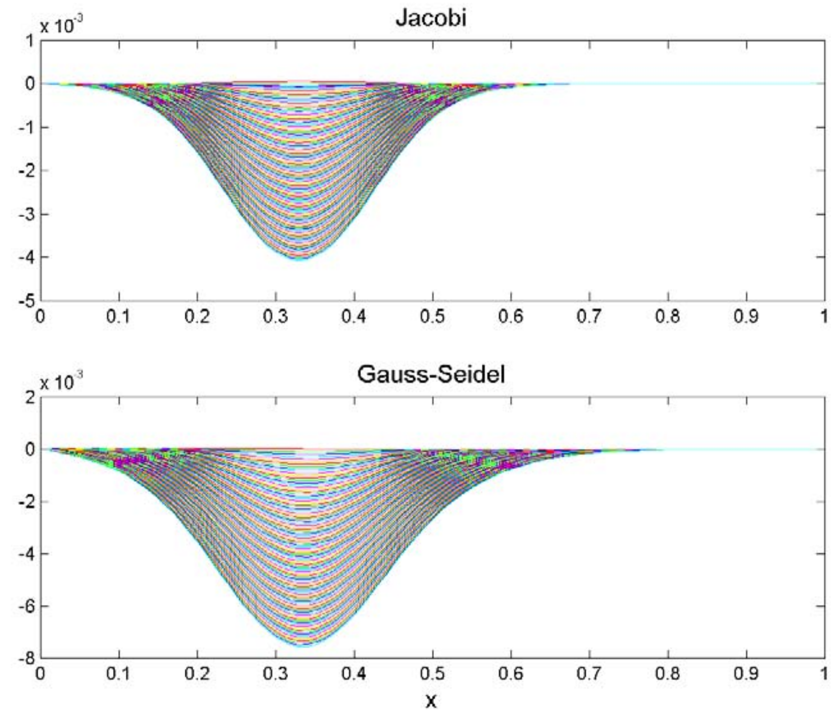
```
% Iterative solution using Jacobi and Gauss-Seidel
b=-a;
c=zeros(n,1);
for i=1:n
    b(i,i)=0;
    for j=1:n
        b(i,j)=b(i,j)/a(i,i);
        c(i)=f(i)/a(i,i);
    end
    end

nj=100;
xj=f;
xgs=f;

figure(2)
nc=6
col=['r' 'g' 'b' 'c' 'm' 'y']
hold off
for j=1:nj
    xj=b*xj+c;
    xgs(1)=b(1,2:n)*xgs(2:n) + c(1);
    for i=2:n-1
        xgs(i)=b(i,1:i-1)*xgs(1:i-1) + b(i,i+1:n)*xgs(i+1:n) +c(i);
    end
    xgs(n)= b(n,1:n-1)*xgs(1:n-1) +c(n);
    cc=col(mod(j-1,nc)+1);
    subplot(2,1,1); plot(x,xj,cc); hold on;
    subplot(2,1,2); plot(x,xgs,cc); hold on;
    hold on
end
```

vib_string.m
o=1.0

Exact Solution

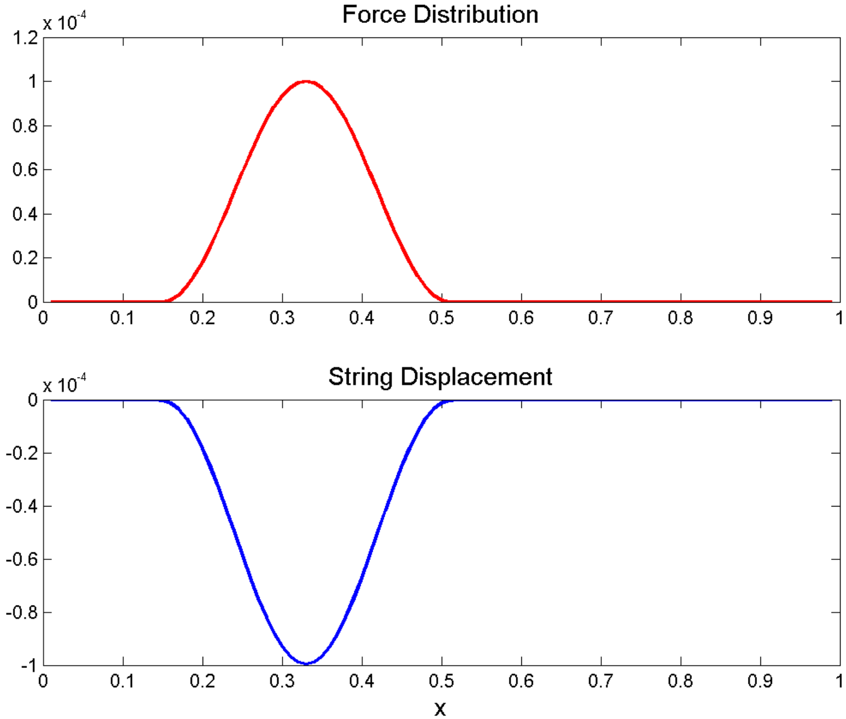


Iterative Solutions



vib_string.m
o = 0.5

Exact Solution



Iterative Solutions