Quantum Information Science : 2/23/2006
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Homework Corrections : Problem Set 2
- Problem 2 : $S = \langle ZZI, IZZ \rangle$
  (e) $B = \{ U_1 = ZYX,\ U_2 = YZI \}$

TODAY: Toric Codes

* 9-qubit quantum code

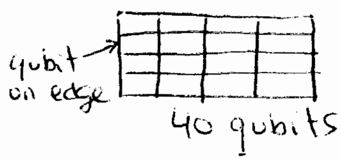
$|0_L\rangle = 1/\sqrt{8}\,(|000\rangle + |111\rangle)^{\otimes 3}$

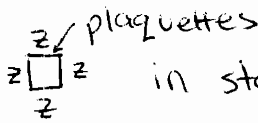
$|1_L\rangle = 1/\sqrt{8}\,(|000\rangle - |111\rangle)^{\otimes 3}$

Suppose error $Z^{(1)}$ then state is $1/\sqrt{8}\,(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)^{\otimes 2}$
$Z^{(2)}$ then state is " "

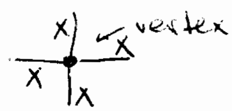
=> this code is degenerate

* Toric Code : special kind of CSS code
Consider grid with qubits on each edge





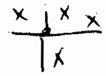
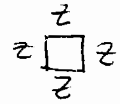
in stabilizer group



commute => TORIC (CSS) CODE

What is dimension? 16 group generators (constraints) for plaquettes
=> 16 + 24 = 40
24 group generators for vertices

=> This code encodes 0-dim subspace.

- TORIC CODE : identify the boundaries like a torus.
  There are now 15 Z-constraints and 15 X-constraints
  and 32 edges => encode 2 qubits
- What Pauli products commute with  and 
  and are not generated by these?

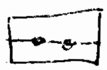
Put Z's on edges so commute with  $\forall$ vertices

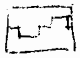
Euler's theorem => can look at simple cycles of Z's.

For example:

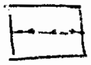
 is this generated by $z \overset{z}{\underset{z}{\square}} z$? yes.

yes, because we can decompose the simple cycle into squares However, this is not the case for cycles across the torus

For example: * or * ()

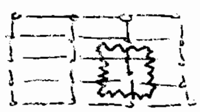
and  can be decompsed in terms of *.

Logical Qubit operations

 $|\psi_1 \psi_2\rangle_L = |\psi_1\rangle Z_L |\psi_2\rangle$

 $= Z_L^{(1)}$

How do we represent X operations?



The following operators commutes with X's:  and 

$\overset{\shortparallel}{X_L^{(2)}}$ $\overset{\shortparallel}{X_L^{(1)}}$

* Toric Code

$2K^2$ qubits

Smallest code distance K encodes two qubits

=> this is a $[[2K^2, 2, K]]$ quantum code.

• How do we correct errors?

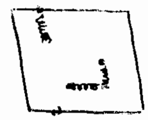
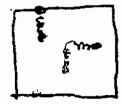
Measure the ~~Z~~ X errors:  points do not satisfy $\frac{x|x}{x|x}$

we want to make the points satisfy $\frac{x|x}{x|x}$.

Consider $\frac{x|x}{x|x}|\psi\rangle = -|\psi\rangle$

then multiply with Z_i on one of four edges:

$\frac{x|x}{x|x} \,\pm^{z}\, |\psi\rangle = - \pm^{z} \frac{x|x}{x|x} = \pm^{z} |\psi\rangle$

Smallest correction is set of paths joining all X error vertices.  or 

degenerate

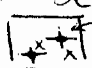
- Hamiltonian

Energy: $\sum_v \frac{1}{2}(1 - X_{v\uparrow} X_{v\downarrow} X_{v\rightarrow} X_{v\leftarrow}) + \sum_{\square} \frac{1}{2}(1 - Z_{top} Z_{bot} Z_{left} Z_{right})$

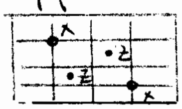
Energy gives the number of violated constraints:

Lowest Energy 0 : codeword

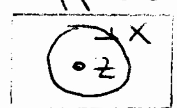
Second Energy 2 : because violations must be even.

 quasi particles with energy 1.

Suppose we had:

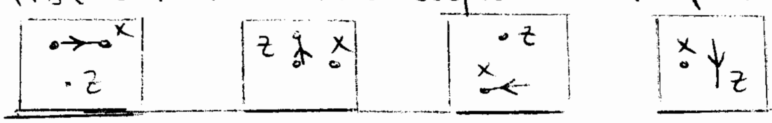


what happens when:



(move X particle around Z)

First consider the sequence of operations



That is, $X_1 \cdots X_{v_1} X_{v_J} \cdots Z_1 \cdots Z_{J} Z_{v_J} \cdots X_{v_i} X_{v_1} \cdots Z_{v_i} Z_2 Z_1 |\psi\rangle$

V_i V_I intersect $1 \cdots k$ in exactly one spot

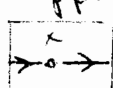
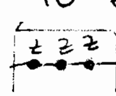
$\Rightarrow X_s Z_s X_s Z_s |\psi\rangle = -|\psi\rangle$

This is equivalent to the X particle moving around Z particle. Then $|\psi\rangle \rightarrow -|\psi\rangle$

People call X and Z magnetic and electric charges or anyons

- what would happen if we moved X around a torus

This would apply a logical $\sigma_Z^{(i)}$ to encoded state.

Because  is equivalent to 

* Quantum Codes on Qutrits

$\{|0\rangle, |1\rangle, |2\rangle\}$

What are analogs of X, Y, Z?

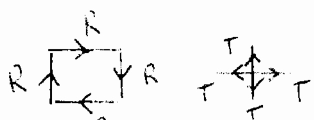
$$T = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \quad R = \begin{pmatrix} 1 & & \\ & \omega^2 & \\ & & \omega^2 \end{pmatrix} \quad \omega = -1/2 + \sqrt{3}/2\, i = 1^{1/3}$$

$$RT = \begin{pmatrix} & \omega & \\ & & \omega^2 \\ 1 & & \end{pmatrix} = \omega TR$$

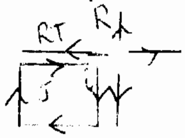
- Qutrit codes: Instead of X, Y, Z, find tensor products of $R^a T^b$ which all commute.

Find quantum subspace so $g_i |\psi\rangle = |\psi\rangle \ \forall i$

- What are the generators for a toric code?



; $\uparrow T = \downarrow T^{-1}$



$= R_i R_J^2 T_i T_J = T_i T_J R_i R_J^2 \omega^3 \Rightarrow$ commute

$m_\omega \quad m_{\omega^2}$

$e_\omega \quad e_{\omega^2}$

get ω or ω^2 depending on clock- or counter-clockwise.

Anyons because you can get any phase.
These are Abelian anyons.

Non-abelian anyons are created from non-abelian groups. In this case, ⟳ applies a unitary operation to the encoded subspace (instead of only a phase for abelian anyons).
For sufficiently complicated non-abelian anyons give universal quantum computation.

- What are the elementary operations that anyons can create out of vacuum?

(1) (b) ↙ • ↘ (a) particle/antiparticle creation

(2) move around each other (braiding)

(3) fuse two anyons
 See what type of particle you get

These operations maybe together with classically controlled operations give universal quantum computation.

Additionally, ~~The~~ anyons are naturally fault-tolerant if you keep anyons far apart.