22.01 Fall 2016, Problem Set 1 Solutions

September 20, 2016

Complete all the assigned problems, and do make sure to show your intermediate work.

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1 (50 points) Retracing Chadwick's Discovery of the Neutron

In thess questions, you will recreate some of James Chadwick's logic as he hypothesized and proved the existence of the neutron. Read the papers provided, "Possible Existence of a Neutron" and "The Existence of a Neutron," and answer the following questions.

1. What made James first hypothesize an uncharged particle with the mass of a proton?
   James noticed that the radiation emanating from the bombardment of beryllium by alpha particles produced over 30,000 ion pairs, while a gamma ray of the expected energy would have only produced about 10,000 ion pairs. Therefore, if a gamma ray was responsible, it was too high in energy, which would have violated energy conservation.

2. What was the competing hypothesis to explain the observed results?
   The competing hypothesis was that a more massive, but still uncharged, particle must be responsible for the ion pairs.

3. Write the nuclear reaction of alpha particles (helium nuclei) bombarding beryllium. You may want to look up the stable isotope of Be here: http://atom.kaeri.re.kr/
   The reaction of alpha particles bombarding beryllium is as follows:

   $$^4_2He +^9_4 Be \rightarrow^1_0 n +^{12}_6 C \tag{1}$$

   Of course now we know that this is the correct reaction. The competing (and incorrect) hypothesis would have been the creation of carbon-13:

   $$^4_2He +^9_4 Be \rightarrow^{13}_6 C + \gamma \tag{2}$$

4. Why would a neutron have greater "penetrating power" (range) through matter compared to charged particles? What does a neutron not interact with?
   The neutron should have a greater penetrating power (range) because it is uncharged. It therefore does not strongly interact with the electrons in the nucleus.

5. On p. 694 of the second paper, Chadwick states that "The source of polonium was prepared from a solution of radium by deposition on a disc of silver." How could polonium be produced directly from radium?
   Looking at the KAERI Table of Nuclides, one can see that the longest-lived isotope of Radium is ^{226}Ra***, which has a half life of 1600 years. It decays by alpha decay to*** ^{222}Rn***, which itself decays to*** ^{218}Po***. Polonium has quite a short half life, which means that almost as soon as it's made, it emits its characteristic alpha particle.***

6. On p. 698 of the second paper, Chadwick states that "the mass of the neutron is equal to that of the proton..." Is this true? What are the masses of the proton, neutron, and electron? Is the mass of Rutherford's "neutron," consisting of a proton and an electron, equal to the neutron's mass? Why or why not (where does the energy discrepancy come from)? Why couldn't Chadwick discern between the masses of these two particles?
   The masses of the proton and the neutron are not equal, though they are very close, so it's conceivable that back in the 1930's, measuring the differences in mass would have been too difficult. The masses are as follows, as sourced from NIST

   Neutron mass: $1.674927471 \cdot 10^{-27}$ kg
   Proton mass: $1.672621898 \cdot 10^{-27}$ kg
   Electron mass:$9.10938356 \cdot 10^{-31}$ kg
   This leaves a mass discrepancy of $6.80381056 \cdot 10^{-31}$ kg***, almost 3/4 of the mass of an electron. This extra energy comes from the conversion of some of the mass of the neutron to kinetic energy of the proton and electron leaving the reaction.***

7. On pp. 701-702, why is the kinetic energy of ^{11}B not accounted for, and what does it mean for kinetic energies to be given in "mass units?" Convert these "mass unit" energies to energies in electron volts (eV). What is the approximate kinetic energy of ^{11}B in eV at room temperature?
   The kinetic energy of ^{11}B ***is ignored, because it is so very, very small compared to the MeV energies involved in nuclear reactions. The value of this kinetic energy can be found by multiplying Boltzmann's constant*** $\left(8.6173324 \cdot 10^{-5} \frac{eV}{K}\right)$ ***by room temperature*** $(298\,K)$***, we get 0.025 eV. Giving energies in mass units means that we are equating mass and energy by Einstein's relation:***
$$E = mc^2 \tag{3}$$

2 (50 points) Getting Used to Nuclear Quantities

In these questions, you will calculate a number of quantities related to nuclear reactions and power generation. You will have to look up certain reactions and values from *primary sources* in the literature (books, papers, databases). Make sure to state which values you look up or assume, and *cite your sources* using proper citation methods.

These calculations are useful, especially when arguing the benefits and costs of nuclear power. If you can derive them quickly and by yourselves, you don't have to rely on as many other sources of information to make your point.

2.1 Relative Power Densities

Calculate the energy in ***Joules*** released from burning 1kg of coal, natural gas, uranium, and deuterium. Use the CRC Handbook of Chemistry and Physics, available through the MIT Libraries site (libraries.mit.edu), to find chemical binding energies (otherwise known as enthalpies of formation, or ΔH_0^f) data for your answers.

Now repeat this calculation for the nuclear fission of uranium into ^{90}Sr and ^{145}Xe (two typical fission products), and the nuclear fusion of ^{2}H with ^{3}H. Use the KAERI Table of Nuclides to find the nuclear binding energies for your answers. Neglect electrons entirely for simplicity.
The equations for the five reactions asked for are as follows:

$$C + O_2 \rightarrow CO_2 + E_1 \tag{4}$$

$$CH_4 + 2O_2 \rightarrow CO_2 + 2H_2O + E_2 \tag{5}$$

$$U + O_2 \rightarrow UO_2 + E_3 \tag{6}$$

$$2D_2 + O_2 \rightarrow 2D_2O + E_4 \tag{7}$$

Species	Binding Energy	Unit
CO_2 (g)	393.5	$\frac{kJ}{mol}$
CH_4 (g)	74.6	$\frac{kJ}{mol}$
H_2O (g)	285.8	$\frac{kJ}{mol}$
UO_2 (s)	1,085	$\frac{kJ}{mol}$
^{235}U	1,784	MeV
^{90}Sr	782.6	MeV
^{145}Xe	1,180	MeV
^{2}H	2.225	MeV
^{3}H	8.482	MeV
^{4}He	28.294	MeV

Table 1: Binding energies for Problem 2.1

Energy	***Energy***	***Unit***	***eV/atom***	***Molar Mass*** $\left(\frac{g}{mol}\right)$	***Efficiency***	***J/kg***
E_1	***393.5***	$\frac{kJ}{mol}$	***4.1***	***12.011***	***0.75***	$2.458 \cdot 10^7$
E_2	***802.5***	$\frac{kJ}{mol}$	***8.3***	***16.043***	***1***	$4.970 \cdot 10^7$
E_3	***1,085***	$\frac{kJ}{mol}$	***11.3***	***235***	***1***	$4.616 \cdot 10^6$
E_4	***483.6***	$\frac{kJ}{mol}$	***5.04***	***4***	***1***	$1.209 \cdot 10^8$
E_5	***179.3***	***MeV***	$1.793 \cdot 10^8$	***235***	***1***	$7.324 \cdot 10^{13}$
E_6	***17.59***	***MeV***	$1.759 \cdot 10^7$	***5***	***1***	$3.377 \cdot 10^{15}$

Table 2: Final energies in Problem 2.1

$$^{235}_{92}U +^1_0 n \rightarrow FP_1 + FP_2 + E_5 \tag{8}$$

$$^2_1H +^3_1 H \rightarrow^4_2 He +^1_0 n + E_6 \tag{9}$$

For Equation 4, we may first assume that coal is not 100% carbon, according to the Energy Information Administration (EIA) ranges from 60% for lignite to 80% from anthracite[1]. Let's assume that it's 75% carbon. We can also assume that methane and uranium don't incur efficiency factors for "burning," and by that we mean in the chemical sense.

First, we can use the CRC Handbook from the MIT libraries site to look up chemical binding energies in $\frac{kJ}{mol}$ ***for equations 4-7. For equations 8-9, we can use the KAERI table of nuclides to look up the binding energies of the nuclei involved, a direct analogue to the chemical binding energies. Remember that the chemical binding energies of pure elements, as well as the nuclear binding energies of lone nucleons, are zero. Table 1 shows the binding energies from the two sources. Using these values, we can calculate the missing energies in Equations 4-9, shown in Table 2.***

These energies must then be all converted to the same value for direct comparison, let's use $\frac{eV}{atom}$ ***as the unit. The MeV energies are easy, just multiply by 1,000,000. For*** $\frac{kJ}{mol}$***, we use the following equation:***

$$E\left[\frac{\cancel{kJ}}{\cancel{mol}}\right] * \left[\frac{1\,\cancel{mol}}{6 \cdot 10^{23}\,atoms}\right] * \left[\frac{1\,eV}{1.6 \cdot 10^{-22}\,\cancel{kJ}}\right] = E\left[\frac{eV}{atom}\right] \tag{10}$$

Finally, we convert these energies in $\frac{eV}{atom}$ ***to*** $\frac{J}{kg}$ ***as follows:***

$$E\left[\frac{\cancel{eV}}{\cancel{atom}}\right] * \left[\frac{6 \cdot 10^{23}\,\cancel{atoms}}{1\,\cancel{mol}}\right] * \left[\frac{1.6 \cdot 10^{-19}\,J}{1\,\cancel{eV}}\right] * \left[\frac{1\,\cancel{mol}}{< Mol\,Mass > \cancel{g}}\right]$$

[1] ***http://www.eia.gov/coal/production/quarterly/co2_article/co2.html***

$$*\,[Efficiency]*\left[\frac{1000\,\cancel{g}}{1\,kg}\right]=E\,\left[\frac{J}{kg}\right] \tag{11}$$

One last note: We have assumed that all chemical fuels except for uranium started as gases, so we took the enthalpy of formation for the gas phase ($\Delta H_0^f(g)$) for each compound. If you took the value for the liquid phase and your answer is different, that's OK.

2.2 Accelerator Energetics

A common tool to provide data on nuclear reactions and to perform irradiations is the electrostatic accelerator. These work by accelerating charged particles through a large, static electric field. We consider here an accelerator that provides a 1.7MV potential drop over 2m for doubly charged iron ions $\left(Fe^{+2}\right)$, which enter the accelerator at ˜zero kinetic energy into the accelerator from an ion source.

2.2.1 What will be the kinetic energy of a nickel ion in eV, as it exits the accelerator?

The energy imparted by an accelerator is equal to the charge of the particle being accelerated times the voltage, so this equals

$$E_{kinetic}=2\left(1.6\cdot 10^{-19}\,C\right)*\left(1.7\cdot 10^{6}\,V\right)=5.44\cdot 10^{-13}\,J=3.4\,MeV \tag{12}$$

2.2.2 What will be its *total mass* (not its rest mass) as it exits the accelerator?

The accelerator transferred 3.4 MeV of energy to the Fe ion, imparting kinetic energy. First, we find the rest mass energy of the Fe ion. Let's choose a specific isotope, Fe-56, for this calculation, which has a mass of 55.935 amu:

$$E_{rest\,mass}=m_0c^2=52,103\,MeV \tag{13}$$

Checking our math for sanity, this is just about 59 times the rest mass of the proton (938.27 MeV). Then, we know the equation for the total energy is:

$$E_{total}=\gamma m_0c^2=E_{rest\,mass}+E_{kinetic} \tag{14}$$

Using this equation, we find the value of γ to be 1.000065, which is *waaay* not relativistic. Its total mass is just as follows:

$$m_{total}=m_0\gamma=9.289\cdot 10^{-26}\,kg \tag{15}$$

which is just 1.000065 times its rest mass.

2.2.3 If the ion source injects 2mA of current, what is the total number of particles leaving the accelerator per second?

2mA of current represents the motion of 0.002 Coulombs per second. Each ion is *doubly charged*, so each has a charge of $3.2\cdot 10^{-19}$ C. To find the number of particles, we convert as follows:

$$\frac{\#\,ions}{sec}=0.002\,\frac{C}{sec}*\frac{1\,ion}{3.2\cdot 10^{-19}\,C}=6.25\cdot 10^{15}\,\frac{ions}{sec} \tag{16}$$

2.2.4 What is the total power, in Watts, associated with pulling 2mA of current through 2MV of electrostatic potential? Where does this power go?

The power in a beam is the same as through any circuit:

$$P=IV=(0.002\,A)\left(1.7\cdot 10^{6}\,V\right)=3.4\,kW \tag{17}$$

This energy must be dissipated in the target that it hits, usually as heat, which must be removed.

2.3 Mass-Energy Equivalence

From special relativity, the total mass (m) of a moving particle can be expressed as follows:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_0 \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{18}$$

where m_0 is its rest mass, v is its velocity, and c is the speed of light.

2.3.1 What is the particle's mass at the following speeds: 1 $\frac{m}{s}$, 1 $\frac{km}{s}$, 1 $\frac{Mm}{s}$, 0.9c, 0.99c, c?

Using Equation 18, and expressing the mass in terms of m_0, we have the following masses:

$$v = 1\,\frac{m}{s} \Rightarrow m = m_0\,(to\,floating\,point\,precision) \tag{19}$$

$$v = 1\,\frac{km}{s} \Rightarrow m = 1.00000000001 m_0 \tag{20}$$

$$v = 1\,\frac{Mm}{s} \Rightarrow m = 1.0000056 m_0 \tag{21}$$

$$v = 0.9c \Rightarrow m = 2.294 m_0 \tag{22}$$

$$v = 0.99c \Rightarrow m = 7.089 m_0 \tag{23}$$

$$v = c \Rightarrow m = \infty \tag{24}$$

2.3.2 Derive an expression for the particle's kinetic energy (T) in terms of its total and rest masses.

The total energy of a particle is given in terms of its rest mass energy and kinetic energy:

$$E_{total} = T + E_{rest\,mass} = \gamma m_0 c^2 \tag{25}$$

and we know that the rest mass energy is given by $E_{rest\,mass} = m_0 c^2$, therefore:

$$T = (\gamma - 1)\, m_0 c^2 \tag{26}$$

2.3.3 Show that the particle's momentum (p) can be described in terms of its kinetic energy and rest mass as follows:

$$p = \frac{1}{c}\sqrt{T^2 + 2T m_0 c^2} \tag{27}$$

We can start with the expression for total relativistic energy of a particle:

$$E = \sqrt{p^2c^2 + E_{rest\,mass}^2} = \sqrt{p^2c^2 + m_0^2c^4} \tag{28}$$

We can then square each side, and isolate the term containing the momentum:

$$p^2c^2 = m^2c^4 - m_0^2c^4 \tag{29}$$

Now we recognize that the total mass (m) is related to the rest mass (m_0) by the factor γ:

$$p^2c^2 = m_0^2\gamma^2c^4 - m_0^2c^4 \tag{30}$$

Then we can use the relation in Equation 26 for the kinetic energy:

$$\frac{T}{m_0c^2} = (\gamma - 1) \Rightarrow \gamma = \frac{T}{m_0c^2} + 1 \tag{31}$$

We then plug this into Equation 30:

$$p^2c^2 = m_0^2c^4\left[\left[\frac{T}{m_0c^2}+1\right]^2 - 1\right] \tag{32}$$

Factoring this out, we get:

$$p^2c^2 = m_0^2c^4\left[\frac{T^2}{m_0^2c^4}+\frac{2T}{m_0c^2}+1\right] - 1 = T^2 + 2Tm_0c^2 \tag{33}$$

Now we just take the square root of each side and divide by the speed of light:

$$pc = \sqrt{T^2+2Tm_0c^2} \Rightarrow p = \frac{1}{c}\sqrt{T^2+2Tm_0c^2} \tag{34}$$

2.3.4 Radioactive decay typically proceeds with the emission of ˜1 MeV particles. For the case of an alpha particle, a beta particle, a neutrino, and a neutron of kinetic enregy 1 MeV, which ones must be treated in a relativistic manner? You will have to look up the rest masses of each particle in your answer. Note that the neutrino was only proven to have mass last year!

To answer this question, we can tabulate values of m_0 for each particle, and use the given kinetic energy of 1 MeV for every particle to determine γ, which gives us a feel for how relativistic each particle is. Remember that if $\gamma = 1$, then the particle is at rest, and is completely non-relativistic. In reality any particle in motion gains mass by definition from Equation 26, reproduced here:

$$T = (\gamma - 1)\, m_0c^2 \tag{35}$$

However, it is convenient to define a *cutoff* value for γ, above which we consider particles to be relativistic, and therefore require more complex treatments for things like stopping power and cross sections (which we will see in a few weeks). Let's say if the particle's motion increases its total mass by $\geq 1\%$, we will consider it significant, and therefore to be treated relativistically. The table below summarizes the rest masses for each particle, which are taken from Appendix A (p. 551) of the Turner book, available online through MIT Libraries. In addition, just for fun, we also calculate the kinetic energy for each particle that would result in it becoming relativistic by our definition ($\gamma = 1.01$):

Particle	m_0(kg)	m_0 (MeV)	T (MeV)	m_{total}(MeV)	γ	***Relativistic?***	$T_{\gamma=1.01}$ (MeV)
α	$6.6447 \cdot 10^{-27}$	3,727.4	1	3,728.4	1.0003	***No***	37.274
β^-	$9.1094 \cdot 10^{-31}$	0.51100	1	1.51100	2.9569	***Yes***	0.00511
ν_{e^-}	$1.7827 \cdot 10^{-37}$	$\sim 10^{-7}$[1]	1	1.0000001	10,000,001	***Yes***	$\sim 10^{-9}$
${}_0^1$n	$1.6749 \cdot 10^{-27}$	939.57	1	940.57	1.0011	***No***	9.3957

[1] A. Gando et al. (KamLAND-Zen Collaboration). "Search for Majorana Neutrinos Near the Inverted Mass Hierarchy Region with KamLAND-Zen." *Phys. Rev. Lett.*, 117:082503 (2016).