22.101 Applied Nuclear Physics (Fall 2006)

Lecture 10 (10/18/06)

Nuclear Shell Model

References:

W. E. Meyerhof, *Elements of Nuclear Physics* (McGraw-Hill, New York, 1967), Chap.2.
P. Marmier and E. Sheldon, *Physics of Nuclei and Particles* (Academic Press, New York, 1969), vol. II, Chap.15.2.
Bernard L. Cohen, *Concepts of Nuclear Physics* (McGraw-Hill, New York, 1971).

There are similarities between the electronic structure of atoms and nuclear structure. Atomic electrons are arranged in orbits (energy states) subject to the laws of quantum mechanics. The distribution of electrons in these states follows the Pauli exclusion principle. Atomic electrons can be excited up to normally unoccupied states, or they can be removed completely from the atom. From such phenomena one can deduce the structure of atoms. In nuclei there are two groups of like particles, protons and neutrons. Each group is separately distributed over certain energy states subject also to the Pauli exclusion principle. Nuclei have excited states, and nucleons can be added to or removed from a nucleus.

Electrons and nucleons have intrinsic angular momenta called intrinsic spins. The total angular momentum of a system of interacting particles reflects the details of the forces between particles. For example, from the coupling of electron angular momentum in atoms we infer an interaction between the spin and the orbital motion of an electron in the field of the nucleus (the spin-orbit coupling). In nuclei there is also a coupling between the orbital motion of a nucleon and its intrinsic spin (but of different origin). In addition, nuclear forces between two nucleons depend strongly on the relative orientation of their spins.

The structure of nuclei is more complex than that of atoms. In an atom the nucleus provides a common center of attraction for all the electrons and inter-electronic forces generally play a small role. The predominant force (Coulomb) is well understood.

Nuclei, on the other hand, have no center of attraction; the nucleons are held together by their mutual interactions which are much more complicated than Coulomb interactions.

All atomic electrons are alike, whereas there are two kinds of nucleons. This allows a richer variety of structures. Notice that there are ~ 100 types of atoms, but more than 1000 different nuclides. Neither atomic nor nuclear structures can be understood without quantum mechanics.

Experimental Basis

There exists considerable experimental evidence pointing to the shell-like structure of nuclei, each nucleus being an assembly of nucleons. Each shell can be filled with a given number of nucleons of each kind. These numbers are called magic numbers; they are **2, 8, 20, 28, 50, 82, and 126**. (For the as yet undiscovered superheavy nuclei the magic numbers are expected to be N = 184, 196, (272), 318, and Z = 114, (126), 164 [Marmier and Sheldon, p. 1262].) Nuclei with magic number of neutrons or protons, or both, are found to be particularly stable, as can be seen from the following data.

(i) Fig. 9.1 shows the abundance of stable isotones (same N) is particularly large for nuclei with magic neutron numbers.

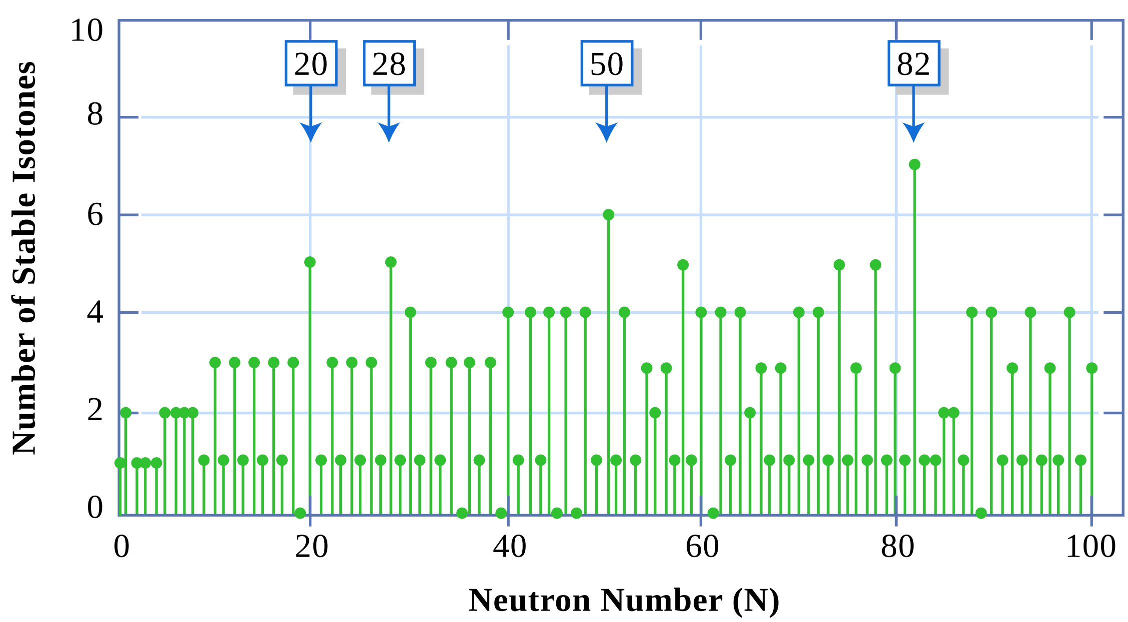

Figure by MIT OCW. Adapted from Meyerhof.

Fig. 9.1. Histogram of stable isotones showing nuclides with neutron numbers 20, 28, 50, and 82 are more abundant by 5 to 7 times than those with non-magic neutron numbers [from Meyerhof].

(ii) Fig. 9.2 shows that the neutron separation energy S_n is particularly low for nuclei with one more neutron than the magic numbers, where

$$S_n = [M(A-1,Z) + M_n - M(A,Z)]c^2 \quad (9.1)$$

This means that nuclei with magic neutron numbers are more tightly bound.

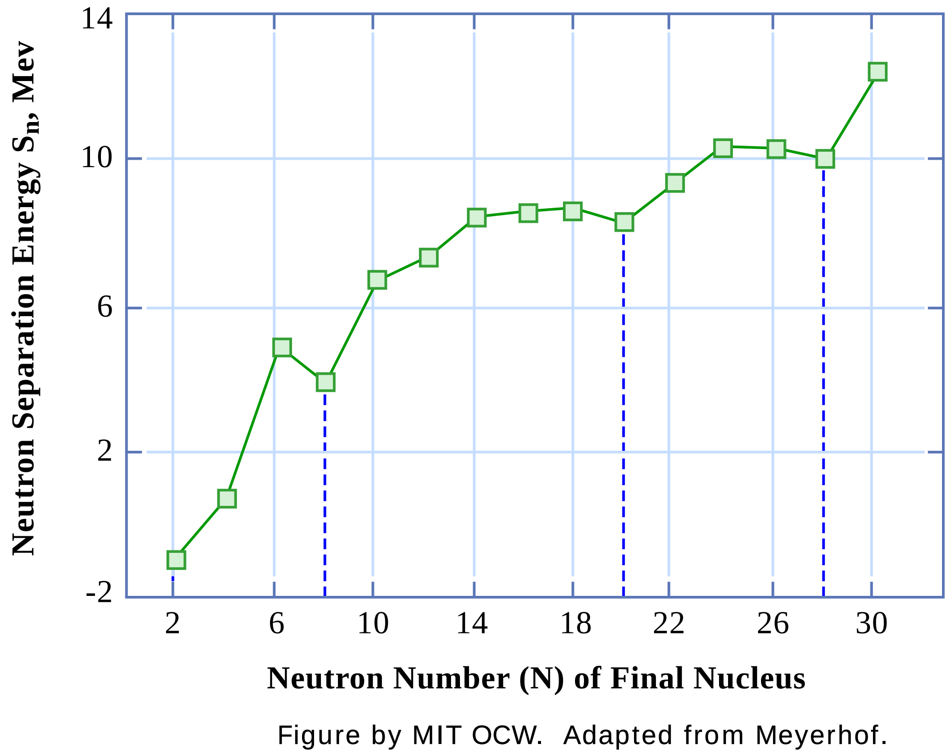

Fig. 9.2. Variation of neutron separation energy with neutron number of the final nucleus M(A,Z) [from Meyerhof].

(iii) The first excited states of even-even nuclei have higher than usual energies at the magic numbers, indicating that the magic nuclei are more tightly bound (see Fig. 9.3).

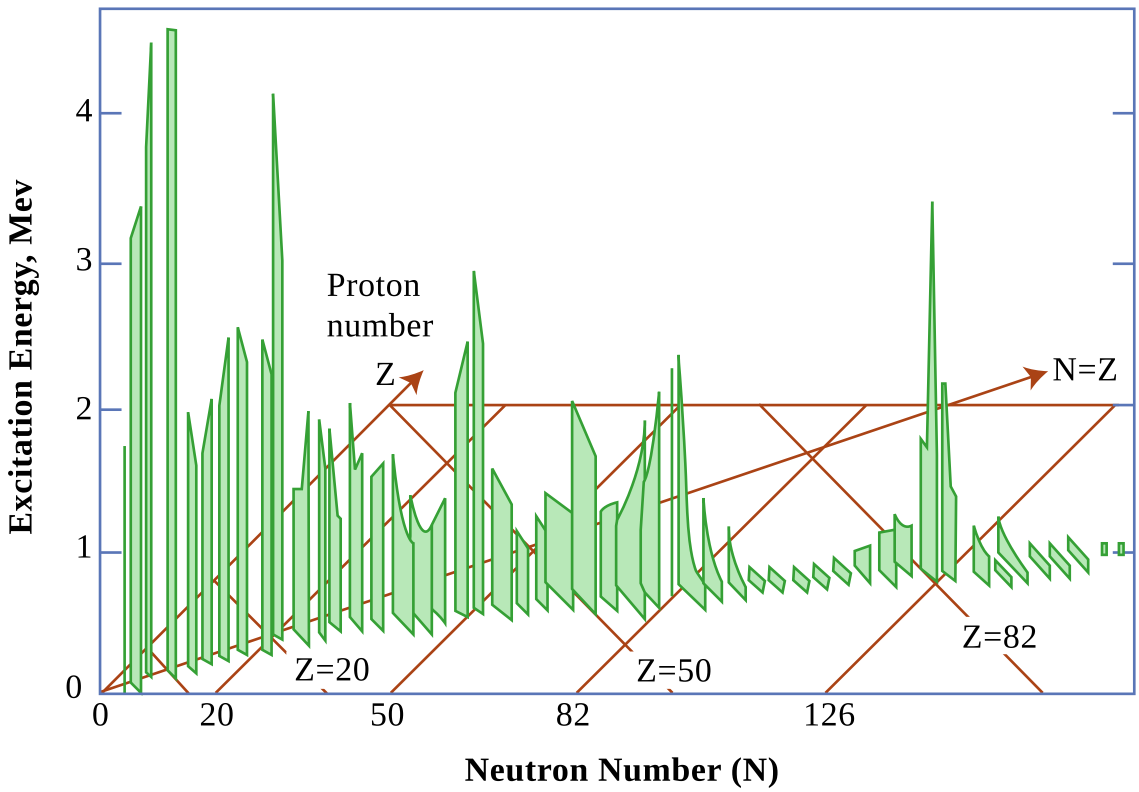

Figure by MIT OCW. Adapted from Meyerhof.

Fig. 9.3. First excited state energies of even-even nuclei [from Meyerhof].

(iv) The neutron capture cross sections for magic nuclei are small, indicating a wider spacing of the energy levels just beyond a closed shell, as shown in Fig. 9.4.

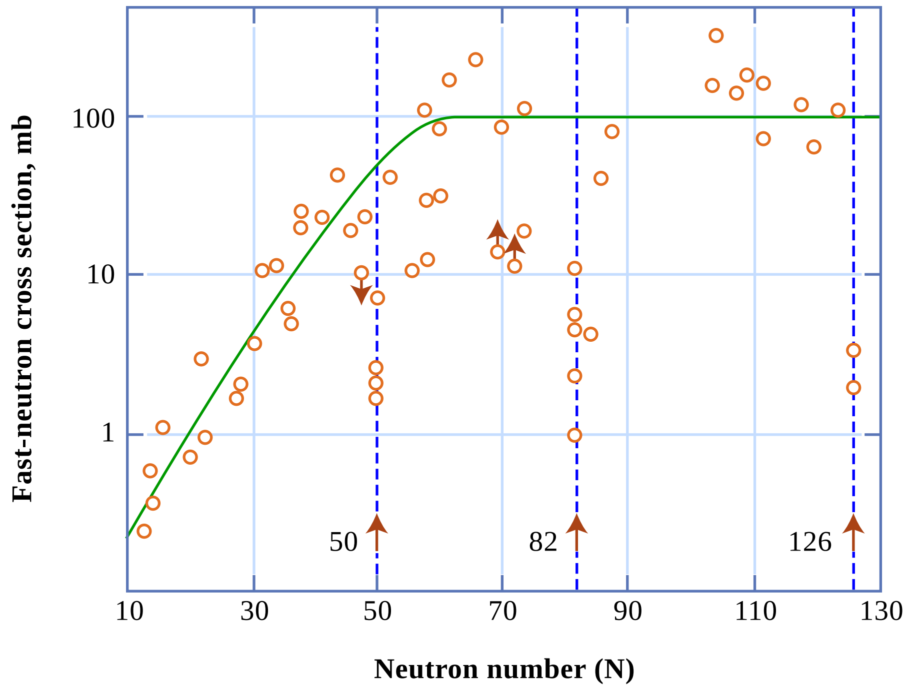

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Fig. 9.4. Cross sections for capture at 1 Mev [from Meyerhof].

Simple Shell Model

The basic assumption of the shell model is that the effects of internuclear interactions can be represented by a single-particle potential. One might think that with very high density and strong forces, the nucleons would be colliding all the time and therefore cannot maintain a single-particle orbit. But, because of Pauli exclusion the nucleons are restricted to only a limited number of allowed orbits. A typical shell-model potential is

$$V(r) = -\frac{V_o}{1 + \exp[(r - R)/a]} \tag{9.1}$$

where typical values for the parameters are $V_o \sim 57$ Mev, $R \sim 1.25A^{1/3}$ F, $a \sim 0.65$ F. In addition one can consider corrections to the well depth arising from (i) symmetry energy from an unequal number of neutrons and protons, with a neutron being able to interact with a proton in more ways than n-n or p-p (therefore n-p force is stronger than n-n and p-p), and (ii) Coulomb repulsion. For a given spherically symmetric potential V(r), one can examine the bound-state energy levels that can be calculated from radial wave equation for a particular orbital angular momentum ℓ,

$$-\frac{\hbar}{2m}\frac{d^2u_\ell}{dr^2}+\left[\frac{\ell(\ell+1)\hbar^2}{2mr^2}+V(r)\right]u_\ell(r)=Eu_\ell(r) \qquad (9.2)$$

Fig. 9.5 shows the energy levels of the nucleons for an infinite spherical well and a harmonic oscillator potential, $V(r)=m\omega^2r^2/2$. While no simple formulas can be given for the former, for the latter one has the expression

$$E_\nu=\hbar\omega(\nu+3/2)=\hbar\omega(n_x+n_y+n_z+3/2) \qquad (9.3)$$

where ν = 0, 1, 2, …, and n_x, n_y, n_z = 0, 1, 2, … are quantum numbers. One should notice the degeneracy in the oscillator energy levels. The quantum number ν can be divided into *radial* quantum number n (1, 2, …) and *orbital* quantum numbers ℓ (0, 1, …) as shown in Fig. 9.5. One can see from these results that a central force potential is able to account for the first three magic numbers, 2, 8, 20, but not the remaining four, 28, 50, 82, 126. This situation does not change when more rounded potential forms are used. The implication is that something very fundamental about the single-particle interaction picture is missing in the description.

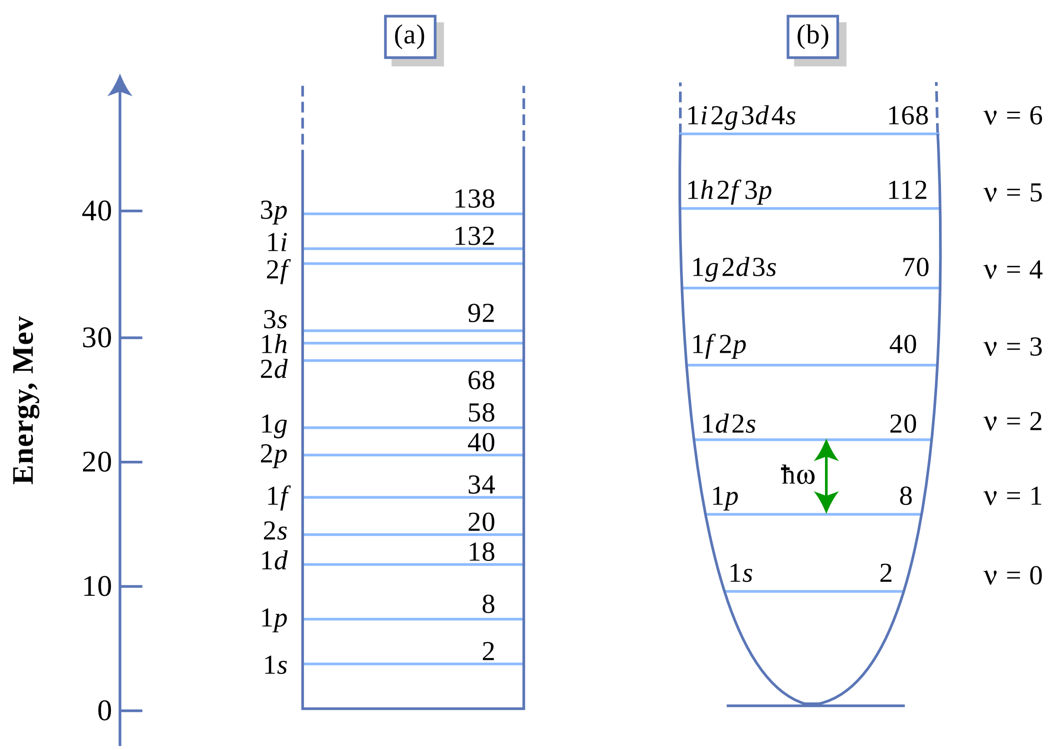


Figure by MIT OCW. Adapted from Meyerhof.

Fig. 9.5. Energy levels of nucleons in (a) infinite spherical well (range R = 8F) and (b) a parabolic potential well. In the spectroscopic notation (n, ℓ), n refers to the number of times the orbital angular momentum state ℓ has appeared. Also shown at certain levels are the cumulative number of nucleons that can be put into all the levels up to the indicated level [from Meyerhof].

Shell Model with Spin-Orbit Coupling

It remains for M. G. Mayer and independently Haxel, Jensen, and Suess to show (1949) that an essential missing piece is an attractive interaction between the orbital angular momentum and the intrinsic spin angular momentum of the nucleon. To take into account this interaction we add a term to the Hamiltonian H,

$$H = \frac{p^2}{2m} + V(r) + V_{so}(r)\underline{s} \cdot \underline{L} \qquad (9.4)$$

where V_{so} is another central potential (known to be attractive). This modification means that the interaction is no longer spherically symmetric; the Hamiltonian now depends on the relative orientation of the spin and orbital angular momenta. It is beyond the scope of this class to go into the bound-state calculations for this Hamiltonian. In order to understand the meaning of the results of such calculations (eigenvalues and eigenfunctions) we need to digress somewhat to discuss the addition of two angular momentum operators.

The presence of the spin-orbit coupling term in (9.4) means that we will have a different set of eigenfunctions and eigenvalues for the new description. What are these new quantities relative to the eigenfunctions and eigenvalues we had for the problem without the spin-orbit coupling interaction? We first observe that in labeling the energy levels in Fig. 9.5 we had already taken into account the fact that the nucleon has an orbital angular momentum (it is in a state with a specified ℓ), and that it has an intrinsic spin of ½ (in unit of $\hbar$). For this reason the number of nucleons that we can put into each level has been counted correctly. For example, in the 1s ground state one can put two nucleons, for zero orbital angular momentum and two spin orientations (up and down).

The student can verify that for a state of given ℓ, the number of nucleons that can go into that state is $2(2\ell+1)$. This comes about because the eigenfunctions we are using to describe the system is a representation that *diagonalizes* the square of the orbital angular momentum operator L^2, its z-component, L_z, the square of the intrinsic spin angular momentum operator S^2, and its z-component S_z. Let us use the following notation to label these eigenfunctions (or representation),

$$\left|\ell, m_\ell, s, m_s\right\rangle \equiv Y_\ell^{m_\ell}\chi_s^{m_s} \tag{9.5}$$

where $Y_\ell^{m_\ell}$ is the spherical harmonic we first encountered in Lec4, and we know it is the eigenfunction of the square of the orbital angular momentum operator L^2 (it is also the eigenfunction of L_z). The function $\chi_s^{m_s}$ is the spin eigenfunction with the expected properties,

$$S^2\chi_s^{m_s} = s(s+1)\hbar^2\chi_s^{m_s}, \qquad s=1/2 \tag{9.6}$$

$$S_z\chi_s^{m_s} = m_s\hbar\chi_s^{m_s}, \qquad -s \le m_s \le s \tag{9.7}$$

The properties of $\chi_s^{m_s}$ with respect to operations by S^2 and S_z completely mirror the properties of $Y_\ell^{m_\ell}$ with respect to L^2 and L_z. Going back to our representation (9.5) we see that the eigenfunction is a "ket" with indices which are the good quantum numbers for the problem, namely, the orbital angular momentum and its projection (sometimes called the magnetic quantum number m, but here we use a subscript to denote that it goes with the orbital angular momentum), the spin (which has the fixed value of ½) and its projection (which can be +1/2 or -1/2).

The representation given in (9.5) is no longer a good representation when the spin-orbit coupling term is added to the Hamiltonian. It turns out that the good representation is just a linear combination of the old representation. It is sufficient for our purpose to just know this, without going into the details of how to construct the linear combination. To understand the properties of the new representation we now discuss angular momentum addition.

The two angular momenta we want to add are obviously the orbital angular momentum operator $\underline{L}$ and the intrinsic spin angular momentum operator $\underline{S}$, since they are the only angular momentum operators in our problem. Why do we want to add them? The reason lies in (9.4). Notice that if we define the total angular momentum as

$$\underline{j} = \underline{S} + \underline{L} \tag{9.8}$$

we can then write

$$\underline{S} \cdot \underline{L} = (j^2 - S^2 - L^2)/2 \tag{9.9}$$

so the problem of diagonalizing (9.4) is the same as diagonalizing j^2, S^2, and L^2. This is then the basis for choosing our new representation. In analogy to (9.5) we will denote the new eigenfunctions by $\left| jm_j \ell s \right\rangle$, which has the properties

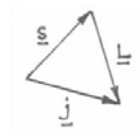
$$j^2 \left| jm_j \ell s \right\rangle = j(j+1)\hbar^2 \left| jm_j \ell s \right\rangle, \qquad |\ell - s| \le j \le \ell + s \tag{9.10}$$

$$j_z \left| jm_j \ell s \right\rangle = m_j \hbar \left| jm_j \ell s \right\rangle, \qquad -j \le m_j \le j \tag{9.11}$$

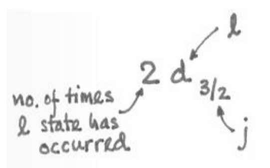
$$L^2 \left| jm_j \ell s \right\rangle = \ell(\ell+1)\hbar^2 \left| jm_j \ell s \right\rangle, \qquad \ell = 0, 1, 2, \ldots \tag{9.12}$$

$$S^2 \left| jm_j \ell s \right\rangle = s(s+1)\hbar^2 \left| jm_j \ell s \right\rangle, \qquad s = 1/2 \tag{9.13}$$

In (9.10) we indicate the values that j can take for given ℓ and s (=1/2 in our discussion), the lower (upper) limit corresponds to when $\underline{S}$ and $\underline{L}$ are antiparallel (parallel) as shown in the sketch.



Returning now to the energy levels of the nucleons in the shell model with spin-orbit coupling we can understand the conventional spectroscopic notation where the value of j is shown as a subscript.



This is then the notation in which the shell-model energy levels are displayed in Fig. 9.6.

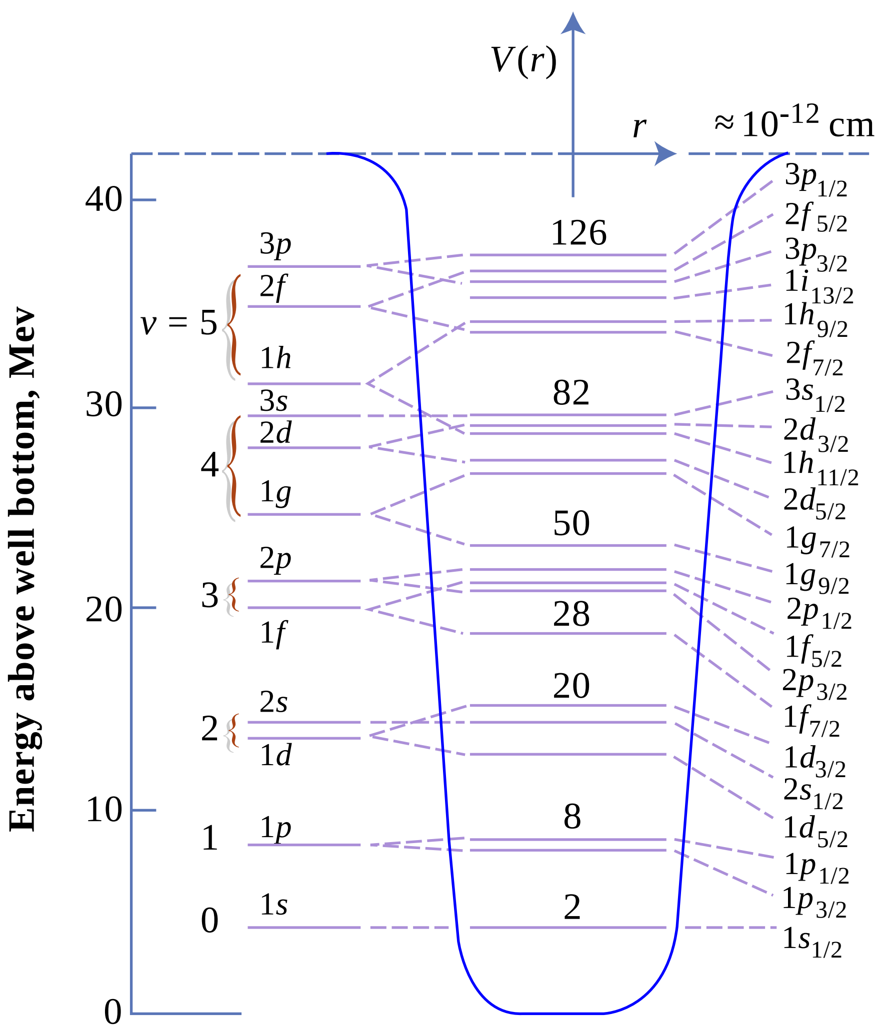


Figure by MIT OCW. Adapted from Meyerhof.

Fig. 9.6. Energy levels of nucleons in a smoothly varying potential well with a strong spin-orbit coupling term [from Meyerhof].

For a given (n, ℓ, j) level, the nucleon occupation number is 2j+1. It would appear that having 2j+1 identical nucleons occupying the same level would violate the Pauli exclusion principle. But this is not the case since each nucleon would have a distinct value of m_j (this is why there are 2j+1 values of m_j for a given j).

We see in Fig. 9.6 the shell model with spin-orbit coupling gives a set of energy levels having breaks at the seven magic numbers. This is considered a major triumph of the model, for which Mayer and Jensen were awarded the Noble prize in physics. For our purpose we will use the results of the shell model to predict the ground-state spin and parity of nuclei. Before going into this discussion we leave the student with the following comments.

1. The shell model is most useful when applied to closed-shell or near closed-shell nuclei.
2. Away from closed-shell nuclei collective models taking into account the rotation and vibration of the nucleus are more appropriate.
3. Simple versions of the shell model do not take into account pairing forces, the effects of which are to make two like-nucleons combine to give zero orbital angula momentum.
4. Shell model does not treat distortion effects (deformed nuclei) due to the attraction between one or more outer nucleons and the closed-shell core. When the nuclear core is not spherical, it can exhibit "rotational" spectrum.

Prediction of Ground-State Spin and Parity

There are three general rules for using the shell model to predict the total angular momentum (spin) and parity of a nucleus in the ground state. These do not always work, especially away from the major shell breaks.

1. Angular momentum of odd-A nuclei is determined by the angular momentum of the last nucleon in the species (neutron or proton) that is odd.
2. Even-even nuclei have zero ground-state spin, because the net angular momentum associated with even N and even Z is zero, and even parity.
3. In odd-odd nuclei the last neutron couples to the last proton with their intrinsic spins in parallel orientation.

To illustrate how these rules work, we consider an example for each case. Consider the odd-A nuclide Be^9 which has 4 protons and 5 neutrons. With the last nucleon being the fifth neutron, we see in Fig. 9.6 that this nucleon goes into the state $1p_{3/2}$ (ℓ=1, j=3/2). Thus we would predict the spin and parity of this nuclide to be $3/2^-$. For an even-even nuclide we can take A^{36}, with 18 protons and neutrons, or Ca^{40}, with 20 protons and neutrons. For both cases we would predict spin and parity of 0^+. For an odd-odd nuclide we take Cl^{38}, which has 17 protons and 21 neutrons. In Fig. 9.6 we see that the 17^{th} proton goes into the state $1d_{3/2}$ (ℓ=2, j=3/2), while the 21^{st} neutron goes into the state $1f_{7/2}$ (ℓ=3, j=7/2). From the ℓ and j values we know that for the last proton the orbital and spin angular momenta are pointing in opposite direction (because j is equal to ℓ-1/2). For the last neutron the two momenta are pointing in the same direction (j = ℓ +1/2). Now the rule tells us that the two spin momenta are parallel, therefore the orbital angular momentum of the odd proton is pointing in the opposite direction from the orbital angular momentum of the odd neutron, with the latter in the same direction as the two spins. Adding up the four angular momenta, we have +3+1/2+1/2-2 = 2. Thus the total angular momentum (nuclear spin) is 2. What about the parity? The parity of the nuclide is the product of the two parities, one for the last proton and the other for the last neutron. Recall that the parity of a state is determined by the orbital angular momentum quantum number ℓ, $\pi = (-1)^\ell$. So with the proton in a state with ℓ = 2, its parity is even, while the neutron in a state with ℓ = 3 has odd parity. The parity of the nucleus is therefore odd. Our prediction for Cl^{38} is then 2^-. The student can verify, using for example the Nuclide Chart, the foregoing predictions are in agreement with experiment.

Potential Wells for Neutrons and Protons

We summarize the qualitative features of the potential wells for neutrons and protons. If we exclude the Coulomb interaction for the moment, then the well for a proton is known to be deeper than that for a neutron. The reason is that in a given nucleus usually there are more neutrons than protons, especially for the heavy nuclei, and the n-p interactions can occur in more ways than either the n-n or p-p interactions on account of the Pauli exclusion principle. The difference in well depth ΔV_s is called the symmetry energy; it is approximately given by

$$\Delta V_s = \pm 27 \frac{(N-Z)}{A} \text{ Mev} \tag{9.14}$$

where the (+) and (-) signs are for protons and neutrons respectively. If we now consider the Coulomb repulsion between protons, its effect is to raise the potential for a proton. In other words, the Coulomb effect is a positive contribution to the nuclear potential which is larger at the center than at the surface.

Combining the symmetry and the Coulomb effects we have a sketch of the potential for a neutron and a proton as indicated in Fig. 9.7. One can also estimate the

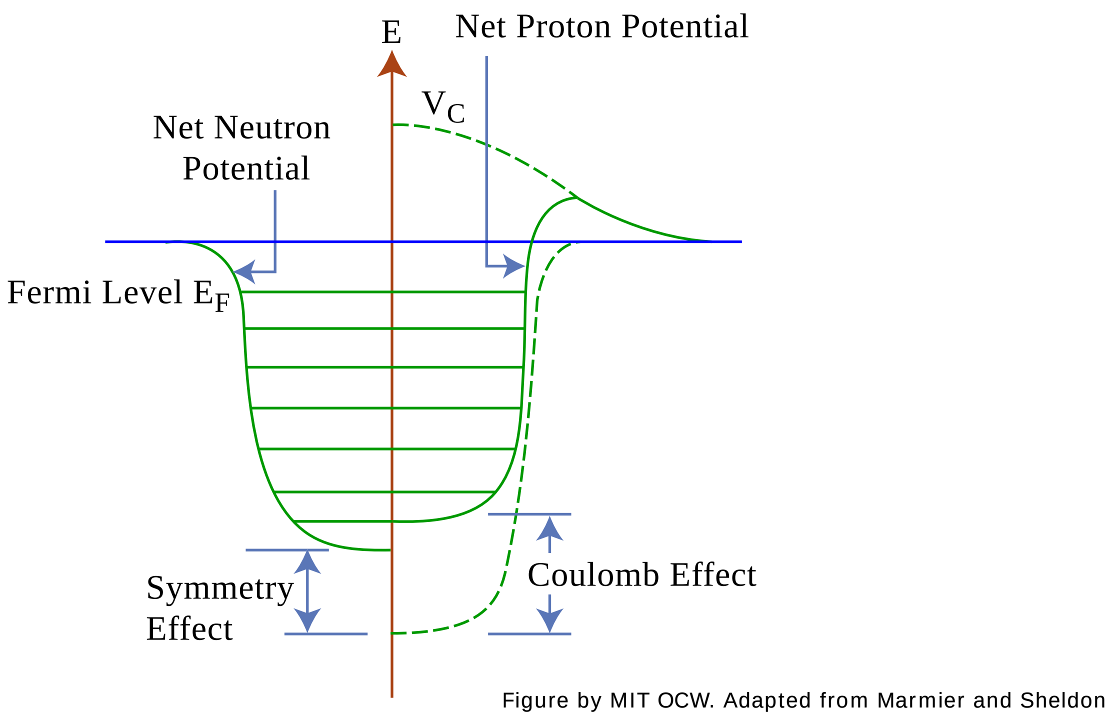


Fig. 9.7. Schematic showing the effects of symmetry and Coulomb interactions on the potential for a neutron and a proton [from Marmier and Sheldon].

well depth in each case using the Fermi Gas model. One assumes the nucleons of a fixed kind behave like a fully degenerate gas of fermions (degeneracy here means that the states are filled continuously starting from the lowest energy state and there are no unoccupied states below the occupied ones), so that the number of states occupied is equal to the number of nucleons in the particular nucleus. This calculation is carried out separately for neutrons and protons. The highest energy state that is occupied is called the *Fermi level*, and the magnitude of the difference between this state and the ground state is called the *Fermi energy* E_F. It turns out that E_F is proportional to $n^{2/3}$, where n is the number of nucleons of a given kind, therefore E_F (neutron) > E_F (proton). The sum of E_F and the separation energy of the last nucleon provides an estimate of the well depth. (The separation energy for a neutron or proton is about 8 Mev for many nuclei.) Based on these considerations one obtains the results shown in Fig. 9.8.

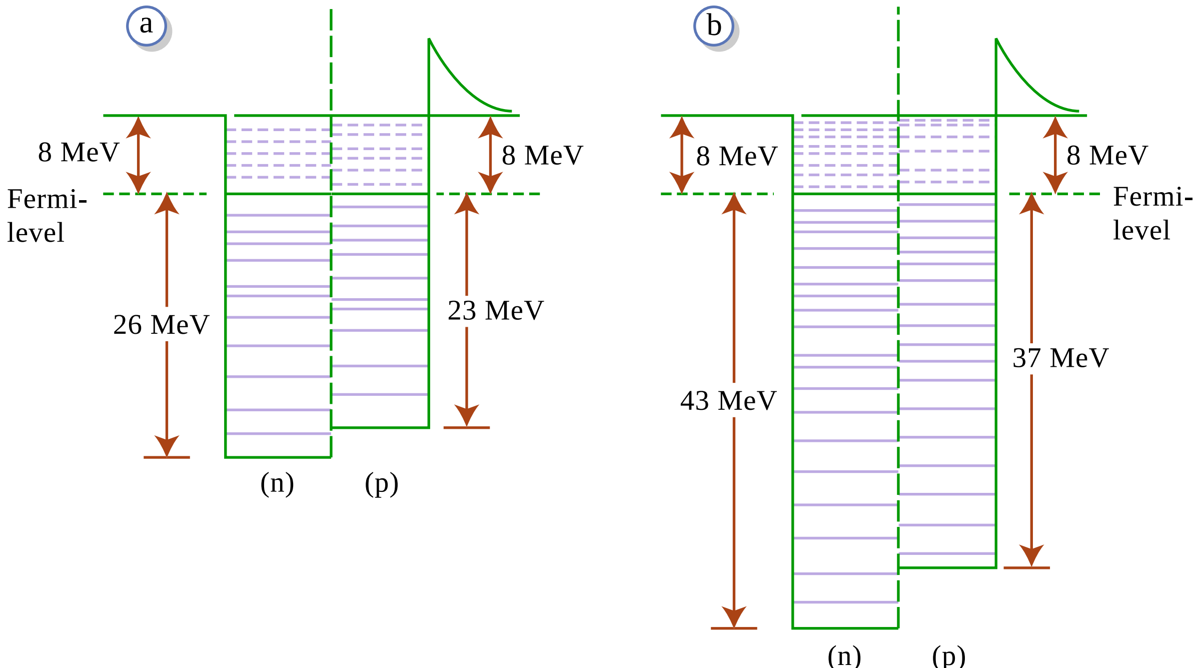


Figure by MIT OCW. Adapted from Marmier and Sheldon.

Fig. 9.8. Nuclear potential wells for neutrons and protons according to the Fermi-gas model, assuming the mean binding energy per nucleon to be 8 Mev, the mean relative nucleon admixture to be N/A ~ 1/1.8m Z/A ~ 1/2.2, and a range of 1.4 F (a) and 1.1 F (b) [from Marmier and Sheldon].

We have so far considered only a spherically symmetric nuclear potential well. We know there is in addition a centrifugal contribution of the form $\ell(\ell+1)\hbar^2/2mr^2$ and a spin-orbit contribution. As a result of the former the well becomes narrower and shallower for the higher orbital angular momentum states. Since the spin-orbit coupling is attractive, its effect depends on whether S is parallel or anti-parallel to L. The effects are illustrated in Figs. 9.9 and 9.10. Notice that for $\ell = 0$ both are absent.

We conclude this chapter with the remark that in addition to the bound states in the nuclear potential well there exist also virtual states (levels) which are positive energy states in which the wave function is large within the potential well. This can happen if the deBroglie wavelength is such that approximately standing waves are formed within the well. (Correspondingly, the reflection coefficient at the edge of the potential is large.)

A virtual level is therefore not a bound state; on the other hand, there is a non-negligible probability that inside the nucleus a nucleon can be found in such a state. See Fig. 9.11.

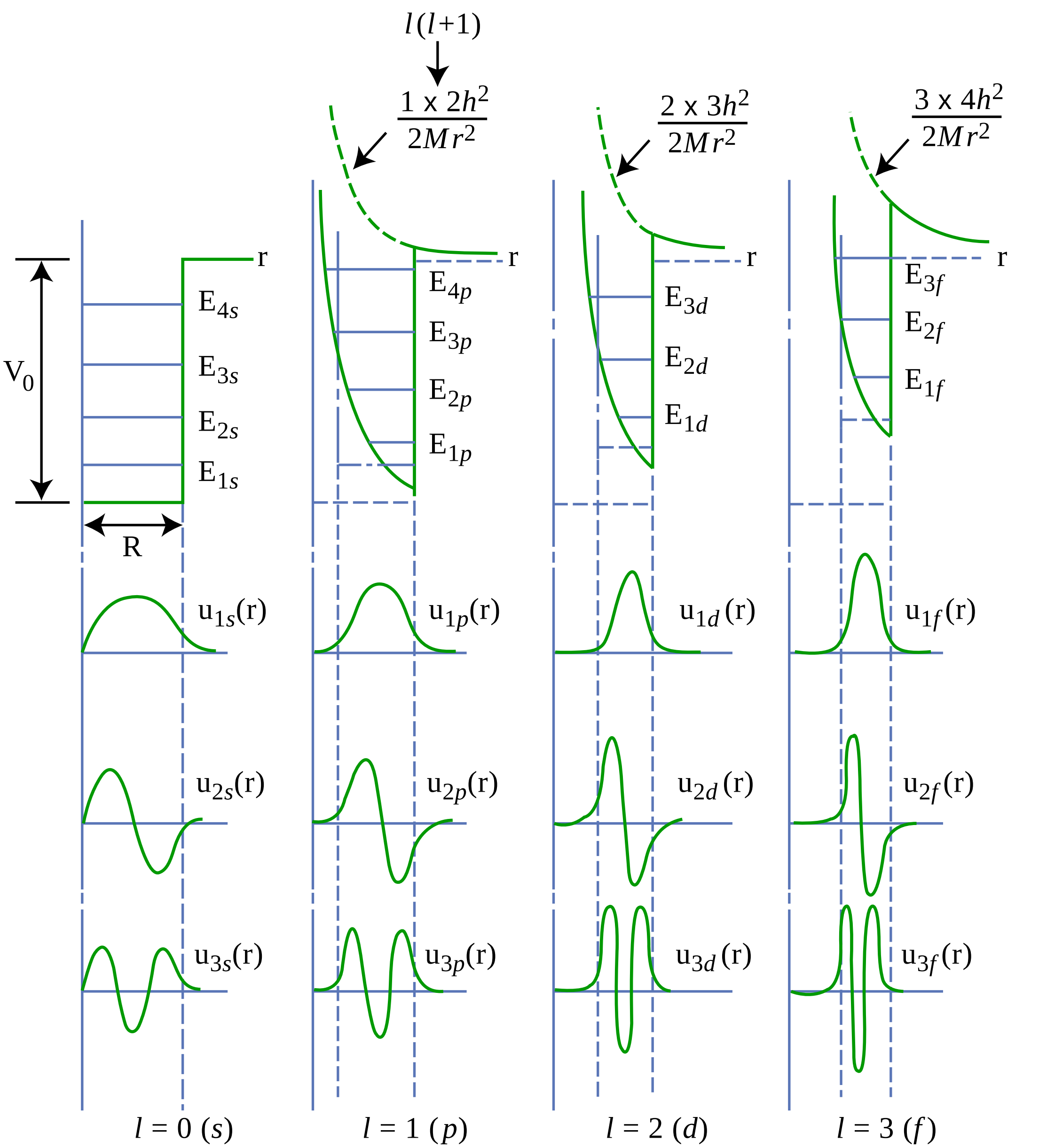


Figure by MIT OCW. Adapted from Cohen.

Fig. 9.9. Energy levels and wave functions for a square well for $\ell = 0, 1, 2,$ and 3 [from Cohen].

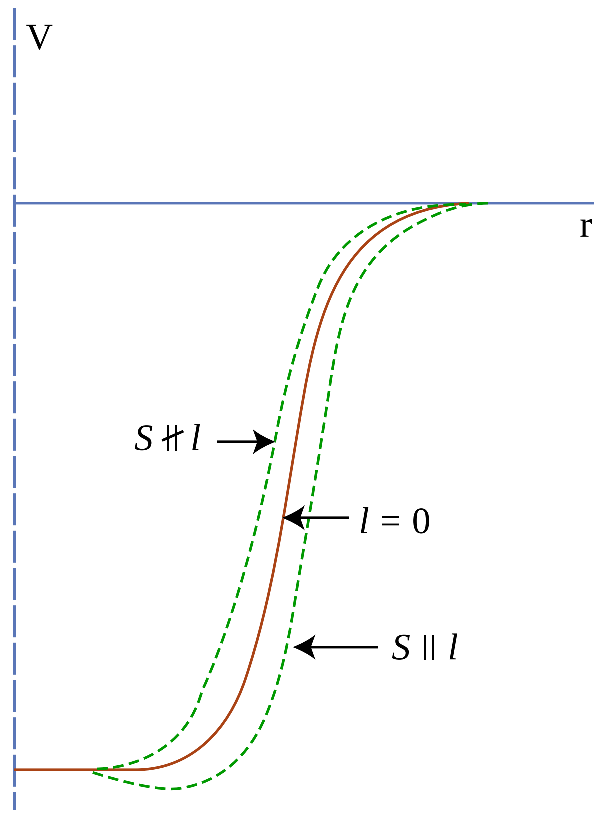


Figure by MIT OCW. Adapted from Cohen.

Fig. 9.10. The effect of spin-orbit interaction on the shell-model potential [from Cohen].

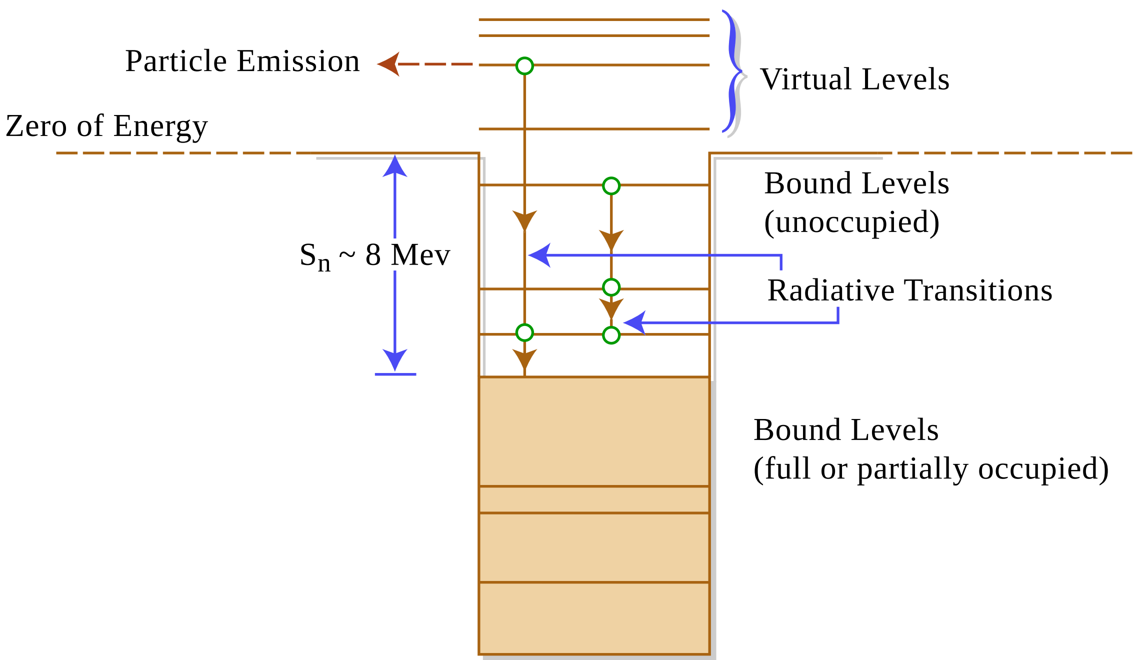


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Fig. 9.11. Schematic representation of nuclear levels [from Meyerhof].