Lecture 6

Basics of Magnetostatics

Today's topics

1. New topic – magnetostatics
2. Present a parallel development as we did for electrostatics
3. Derive basic relations from a fundamental force law
4. Derive the analogs of $\mathbf{E}$ and ϕ
5. Derive the analog of Gauss' theorem
6. Derive the integral formulation of magnetostatics
7. Derive the differential form of magnetostatics
8. A few simple problems

General comments

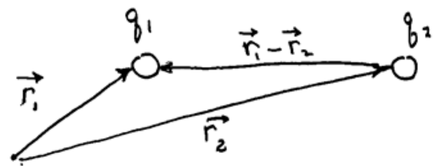
1. What is magnetostatics? The study of DC magnetic fields that arise from the motion of charged particles.
2. Since charged particles are present, does this mean we also must have an electric field? No!! Electrons can flow through ions in such a way that current flows but there is still no net charge, and hence no electric field. This is the situation in magnetostatics

3. Magnetostatics is more complicated than electrostatics.
4. Single magnetic charges do not exist in nature. The simplest basic element in magnetostatics is the magnetic dipole, which is equivalent to two, closely paired equal and opposite charges in electrostatics. In magnetostatics the dipole forms from a small circular loop of current.
5. Magnetic geometries are also more complicated than in electrostatics. The vector nature of the magnetic field is less intuitive.

The basic force of magnetostatics

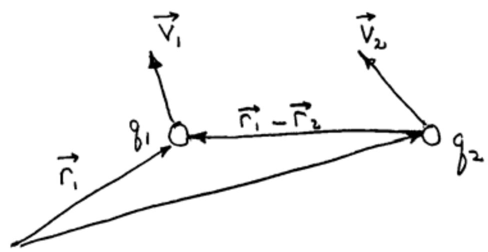
1. Recall electrostatics.

$$\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\mathbf{r}_1 - \mathbf{r}_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3}$$



2. In magnetostatics there is an equivalent empirical law which we take as a postulate

$$\mathbf{F}_{12} = \frac{q_1 q_2 \mu_0}{4\pi} \frac{\mathbf{v}_1 \times \left[\mathbf{v}_2 \times \left(\mathbf{r}_1 - \mathbf{r}_2\right)\right]}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3}$$



3. The force is proportional to $1/r^2$.
4. The force is perpendicular to $\mathbf{v}_1$: $\mathbf{F}_{12} \cdot \mathbf{v}_1 = 0$.
5. The force depends on the particle velocities $\mathbf{v}_1, \mathbf{v}_2$
6. The constant of proportionality is equal to $\mu_0 = 4\pi \times 10^{-7}$ $Henry's/meter$.

The magnetic field

1. We define the magnetic field in terms of the force as follows

$$\text{Electrostatics} \qquad \mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\mathbf{r}_1 - \mathbf{r}_2}{\left|\mathbf{r}_1 - \mathbf{r}_1\right|^3} \equiv q_1 \mathbf{E}$$

$$\mathbf{E} = \frac{q_2}{4\pi\varepsilon_0} \frac{\mathbf{r}_1 - \mathbf{r}_2}{\left|\mathbf{r}_1 - \mathbf{r}_1\right|^3}$$

$$\text{Magnetostatics} \qquad \mathbf{F}_{12} = \frac{\mu_0 q_1 q_2}{4\pi} \frac{\mathbf{v}_1 \times \left[\mathbf{v}_2 \times \left(\mathbf{r}_1 - \mathbf{r}_2\right)\right]}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3} \equiv q_1 \mathbf{v}_1 \times \mathbf{B}$$

$$\mathbf{B} = \frac{\mu_0 q_2}{4\pi} \frac{\mathbf{v}_2 \times \left(\mathbf{r}_1 - \mathbf{r}_2\right)}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3}$$

2. Let's extend the definition to include many charges

$$\text{Electrostatics} \qquad \mathbf{E}(\mathbf{r}) = \sum_j \frac{q_j}{4\pi\varepsilon_0} \frac{\mathbf{r} - \mathbf{r}_j}{\left|\mathbf{r} - \mathbf{r}_j\right|^3}$$

$$\text{Magnetostatics} \qquad \mathbf{B}(\mathbf{r}) = \sum_j \frac{\mu_0 q_j}{4\pi} \frac{\mathbf{v}_j \times \left(\mathbf{r} - \mathbf{r}_j\right)}{\left|\mathbf{r} - \mathbf{r}_j\right|^3}$$

3. Let's extend the analysis to a continuum

$$\text{Electrostatics} \qquad \sum_j q_j \quad \rightarrow \quad \int \rho \, d\mathbf{r}'$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \rho \frac{\mathbf{r} - \mathbf{r}'}{\left|\mathbf{r} - \mathbf{r}'\right|^3} d\mathbf{r}'$$

$$\text{Magnetostatics} \qquad \sum_j q_j \mathbf{v}_j \quad \rightarrow \quad \int \rho \mathbf{v} \, d\mathbf{r}' = \int \mathbf{J} \, d\mathbf{r}'$$

$$\mathbf{J} = \rho \mathbf{v} = \text{ current density in Amps/m}^2$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}\left(\mathbf{r}'\right) \times \left(\mathbf{r} - \mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|^3} d\mathbf{r}'$$

4. Now, recall that

$$\frac{\mathbf{r} - \mathbf{r}'}{\left|\mathbf{r} - \mathbf{r}'\right|^3} = -\nabla \frac{1}{\left|\mathbf{r} - \mathbf{r}'\right|} = +\nabla' \frac{1}{\left|\mathbf{r} - \mathbf{r}'\right|}$$

5. This allows us to introduce the vector potential $\mathbf{A}$

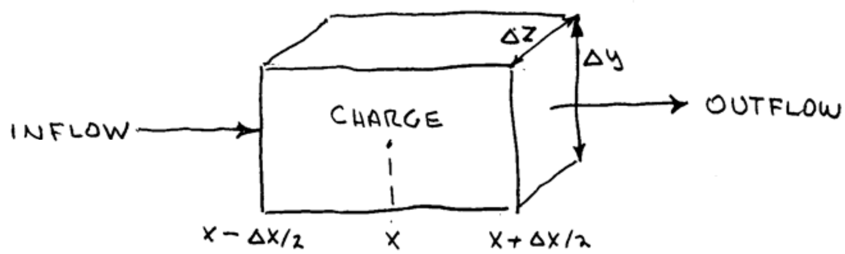
$$
\begin{aligned}
\text{Electrostatics}\qquad \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\int \rho \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\,d\mathbf{r}' \\
&= -\frac{1}{4\pi\varepsilon_0}\int \rho \nabla \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\,d\mathbf{r}' \\
&= -\nabla \frac{1}{4\pi\varepsilon_0}\int \rho \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\,d\mathbf{r}' \\
&= -\nabla\phi \\
\phi &= \frac{1}{4\pi\varepsilon_0}\int \rho \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\,d\mathbf{r}' \\
\text{Magnetostatics}\qquad \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\,d\mathbf{r}' \\
&= -\frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\times\nabla \frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\,d\mathbf{r}' \\
&= \frac{\mu_0}{4\pi}\int \nabla\times \frac{\mathbf{J}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}\,d\mathbf{r}' \\
&= \nabla\times\frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}\,d\mathbf{r}' \\
&= \nabla\times\mathbf{A} \\
\mathbf{A} &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}\,d\mathbf{r}'
\end{aligned}
$$

6. $\mathbf{A}$ is the vector potential and the integral expression for $\mathbf{A}$ is known as the Biot-Savart law.
7. A basic property of magnetostatics

$$
\begin{array}{llll}
\text{Electrostatics} & \mathbf{E} = -\nabla\phi & \rightarrow & \nabla\times\mathbf{E} = 0 \\
\text{Magnetostatics} & \mathbf{B} = \nabla\times\mathbf{A} & \rightarrow & \nabla\cdot\mathbf{B} = 0
\end{array}
$$

A new relation – conservation of charge

1. A basic physical property is that charge is conserved. It cannot be created or destroyed.
2. This implies a relation between ρ and $\mathbf{J}$.
3. Let's derive this relation for a simple 1-D case



4. Statement of charge conservation

$$\begin{array}{ccccc} \text{Rate of increase of charge} & = & \text{flow in of charge} & - & \text{flow out of charge} \\ (1) & = & (2) & - & (3) \end{array}$$

5. The separate terms are as follows

$$\begin{aligned} (1) &= \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} \\ (2) &= \text{ flux of charge } \times \text{ area } = \rho v \Delta y \Delta z\big|_{x-\Delta x/2} \\ (3) &= \text{ flux of charge } \times \text{ area } = \rho v \Delta y \Delta z\big|_{x+\Delta x/2} \end{aligned}$$

6. Recall that $\rho \mathbf{v} = \mathbf{J} =$ current density. The conservation law becomes

$$\begin{aligned} \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} &= \Delta y \Delta z \left(\rho v\big|_{x-\Delta x/2} - \rho v\big|_{x+\Delta x/2} \right) \\ &= \Delta y \Delta z \left(J_x\big|_{x-\Delta x/2} - J_x\big|_{x+\Delta x/2} \right) \\ &\approx \Delta y \Delta z \left(J_x(x) - \frac{\partial J_x}{\partial x}\frac{\Delta x}{2} - J_x(x) - \frac{\partial J_x}{\partial x}\frac{\Delta x}{2} \right) \\ &= -\Delta x \Delta y \Delta z \frac{\partial J_x}{\partial x} \end{aligned}$$

7. Therefore, in 1-D

$$\frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} = 0$$

8. In 3-D

$$\frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

9. Note, in electrostatics conservation of charge implies $\partial \rho / \partial t = 0$. This is a trivial result since we are looking at static situations with no time variation
10. However, in magnetostatics conservation of charge implies that $\nabla \cdot \mathbf{J} = 0$. The current is divergence free for static problems.
11. This implies something about $\nabla \cdot \mathbf{A}$ as follows

$$\begin{aligned}
\mathbf{A} &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \\
\nabla \cdot \mathbf{A} &= \frac{\mu_0}{4\pi} \int \nabla \cdot \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \\
&= \frac{\mu_0}{4\pi} \int \left[-\nabla' \cdot \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{J}(\mathbf{r}') \right] d\mathbf{r}' \\
&= -\frac{\mu_0}{4\pi} \int \nabla' \cdot \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \\
&= -\frac{\mu_0}{4\pi} \int \frac{\mathbf{n}' \cdot \mathbf{J}(S')}{|\mathbf{r} - \mathbf{r}'|} dS' \\
&= 0 \text{ for a localized current distribution: } \mathbf{J}(S') = 0 \text{ as } S' \to \infty \\
\nabla \cdot \mathbf{A} &= 0
\end{aligned}$$

The analog of Poisson's equation

1. In electrostatics we have

$$\mathbf{E} = -\nabla \phi$$

$$\nabla^2 \phi = -\rho / \varepsilon_0$$

2. In magnetostatics

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'$$

$$\nabla^2 \mathbf{A} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'$$

$$= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi \delta(\mathbf{r} - \mathbf{r}') d\mathbf{r}'$$

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

3. This is the equivalent to Poisson's equation.
4. Also, we find that

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A})$$

$$= \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$= -\nabla^2 \mathbf{A}$$

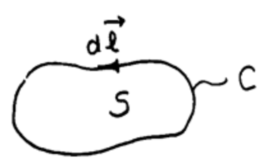
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

5. This is Ampere's law for magnetostatics.
6. We can now easily obtain the integral formulation of magnetostatics. First equation.

$$\int_V \nabla \cdot \mathbf{B}\, d\mathbf{r} = 0 \qquad \rightarrow \qquad \int_S \mathbf{n} \cdot \mathbf{B}\, dS = 0$$

7. Second equation.

$$\int_S (\nabla \times \mathbf{B} - \mu_0 \mathbf{J}) \cdot \mathbf{n}\, dS = 0 \qquad \rightarrow \qquad \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n}\, dS$$

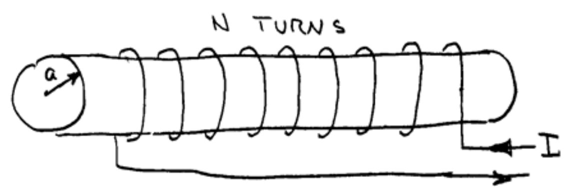


Summary and comparison

	Magnetostatics	Electrostatics
Integral relation	$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\lvert\mathbf{r}-\mathbf{r}'\rvert} d\mathbf{r}'$	$\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{\lvert\mathbf{r}-\mathbf{r}'\rvert} d\mathbf{r}'$
Differential formulation	$\mathbf{B} = \nabla\times\mathbf{A}$	$\mathbf{E} = -\nabla\phi$
	$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$	$\nabla^2\phi = -\rho/\varepsilon_0$
Integral formulation	$\int \mathbf{n}\cdot\mathbf{B}\, dS = 0$	$\oint \mathbf{E}\cdot d\mathbf{l} = 0$
	$\oint \mathbf{B}\cdot d\mathbf{l} = \int \mathbf{J}\cdot\mathbf{n}\, dS$	$\int \mathbf{n}\cdot\mathbf{E}\, dS = \int \rho\, d\mathbf{r}$
Force density	$\mathbf{F} = \frac{q\mathbf{E}}{\Delta\mathbf{r}} = \rho\mathbf{E}$	$\mathbf{F} = \frac{q\mathbf{v}\times\mathbf{B}}{\Delta\mathbf{r}} = \mathbf{J}\times\mathbf{B}$

Example 1

1. Find the magnetic field in a long solenoid.



2. By symmetry there is only a z component of field: $\mathbf{B} = B_z\mathbf{e}_z$
3. With no sources at infinity B_z must vanish outside the coil: $B_z = 0$ for $r > a$.
4. By symmetry $B_z(r,\theta,z) = B_z(r)$
5. Inside the solenoid

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}$$

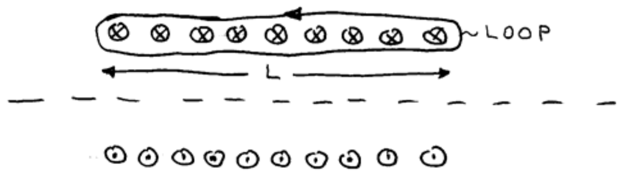
$$\frac{\partial B_z}{\partial r} = -\mu_0 J_\theta = 0$$

6. The solution is

$$B_z = B_0 = const.$$

7. Express B_0 in terms the current flowing in the wire I by using Amperes law.

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot dS$$
$$B_0 L = \mu_0 N I$$

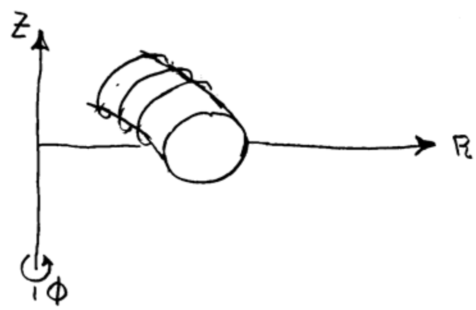


8. Therefore, inside a long solenoid the field is uniform with a value

$$B_0 = \frac{\mu_0 N I}{L}$$

Example 2

1. Find the field inside a torus.



2. By symmetry we see that $\partial / \partial\phi = 0$
3. The only non-zero field component is $\mathbf{B} = B_\phi \mathbf{e}_\phi$.
4. As in the straight solenoid $B_\phi = 0$ outside the torus.
5. Inside the torus we have

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\frac{1}{R}\frac{\partial R B_\phi}{\partial R} = 0$$

6. The solution is

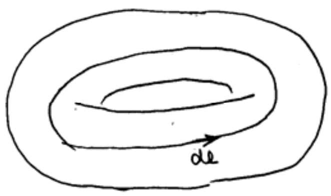
$$B_\phi = \frac{K}{R}$$

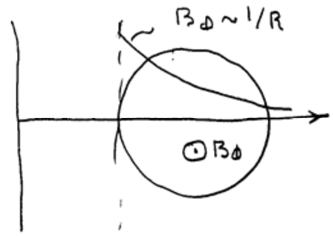
7. Express K in terms of the coil current by using the integral form of Ampere's law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot \mathbf{n}\, dS$$
$$2\pi R B_\phi = \mu_0 N I$$

8. This gives

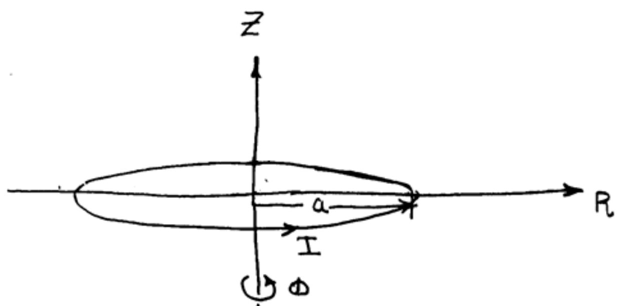
$$B_\phi(R) = \frac{\mu_0 N I}{2\pi R}$$





Example 3

1. Find the field due to the current flowing in a loop of wire



2. This problem is more complicated than you would think. Proceed as follows.
3. Since there are no conductors and the current is localized it makes sense to start with the Biot-Savart law

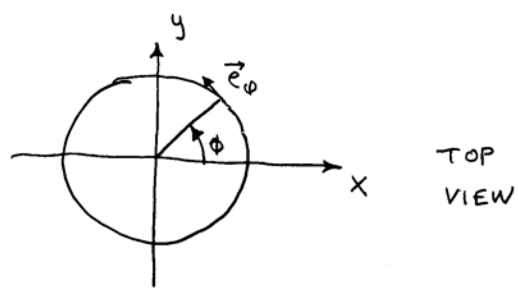
$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'$$

4. Note that the current density is given by

$$\mathbf{J}(\mathbf{r}') = I\delta(R'-a)\delta(Z')\mathbf{e}_\phi$$
$$\int \mathbf{J}\cdot\mathbf{n}\,dS' = \int I\delta(R'-a)\delta(Z')\,dR'dZ' = I$$

5. One tricky problem is that $\mathbf{e}_{\phi'} \neq \mathbf{e}_\phi$. As we integrate around the loop the direction of $\mathbf{e}_{\phi'}$ is constantly changing while at the fixed observation point $\mathbf{e}_\phi$ does not change.
6. This difficulty is avoided by expressing the unit vectors in rectangular coordinates which always stay fixed. Thus,

$$\mathbf{e}_\phi = -\sin\phi\mathbf{e}_x + \cos\phi\mathbf{e}_y$$
$$\mathbf{e}_{\phi'} = -\sin\phi'\mathbf{e}_x + \cos\phi'\mathbf{e}_y$$



7. The Biot-Savart law becomes

$$\begin{aligned}\mathbf{A} &= A_x\mathbf{e}_x + A_y\mathbf{e}_y \\ &= \frac{\mu_0}{4\pi}\int \frac{J_{\phi'}\mathbf{e}_{\phi'}}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' \\ &= \frac{\mu_0}{4\pi}\int \frac{J_{\phi'}(-\sin\phi'\mathbf{e}_x + \cos\phi'\mathbf{e}_y)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\end{aligned}$$

8. We see that

$$A_x = \frac{\mu_0 I}{4\pi}\int \frac{\delta(R'-a)\delta(Z')(-\sin\phi')}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$
$$A_y = \frac{\mu_0 I}{4\pi}\int \frac{\delta(R'-a)\delta(Z')(\cos\phi')}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

9. The next step is to combine these terms to evaluate $A_\phi(\mathbf{r}) = -A_x \sin\phi + A_y \cos\phi$ at the observation point.

$$
\begin{aligned}
A_\phi &= \frac{\mu_0 I}{4\pi} \int \frac{\delta\left(R'-a\right)\delta\left(Z'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}\left(\cos\phi\cos\phi' + \sin\phi\sin\phi'\right) d\mathbf{r}' \\
&= \frac{\mu_0 I}{4\pi} \int \frac{\delta\left(R'-a\right)\delta\left(Z'\right)\cos\left(\phi'-\phi\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} d\mathbf{r}'
\end{aligned}
$$

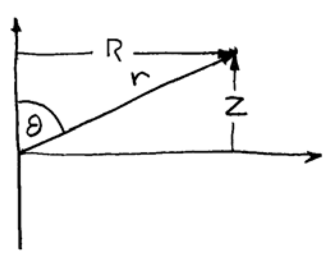
10. To evaluate this integral note that

$$
\begin{aligned}
\left|\mathbf{r}-\mathbf{r}'\right| &= \left[\left(x'-x\right)^2 + \left(y'-y\right)^2 + \left(z'-z\right)^2\right]^{1/2} \\
&= \left[\left(R'\cos\phi' - R\cos\phi\right)^2 + \left(R'\sin\phi' - R\sin\phi\right)^2 + \left(Z'-Z\right)^2\right]^{1/2} \\
&= \left[R^2 + R'^2 - 2RR'\cos\left(\phi'-\phi\right) + \left(Z'-Z\right)^2\right]^{1/2}
\end{aligned}
$$

11. We can now carry out the integration over R', Z'.

$$
\begin{aligned}
A_\phi &= \frac{\mu_0 I}{4\pi} \int \frac{\delta\left(R'-a\right)\delta\left(Z'\right)\cos\left(\phi'-\phi\right)}{\left|\mathbf{r}-\mathbf{r}'\right|} R'\,dR'\,dZ'\,d\phi' \\
&= \frac{\mu_0 I a}{4\pi} \int \frac{\cos\left(\phi'-\phi\right)}{\left[R^2 + a^2 - 2Ra\cos\left(\phi'-\phi\right) + Z^2\right]^{1/2}} d\phi'
\end{aligned}
$$

12. This integral can be evaluated in terms of the complete elliptic integrals $K(k), E(k)$. This does not provide too much insight.
13. A more useful approach is to evaluate the integrals far from the loop of wire.
14. As shown below we let $R = r\sin\theta \quad Z = r\cos\theta$ and assume $r \gg a$.



15. In this limit

$$\frac{1}{\left[R^2 + a^2 - 2Ra\cos\left(\phi' - \phi\right) + Z^2\right]^{1/2}} = \frac{1}{\left[r^2 + a^2 - 2ra\sin\theta\cos\left(\phi' - \phi\right)\right]^{1/2}}$$
$$\approx \frac{1}{r}\left[1 - \frac{a}{r}\sin\theta\cos\left(\phi' - \phi\right)\right]$$

16. The vector potential becomes

$$A_\phi = \frac{\mu_0 Ia}{4\pi}\int \frac{\cos\left(\phi' - \phi\right)}{r}\left[1 - \frac{a}{r}\sin\theta\cos\left(\phi' - \phi\right)\right]d\phi'$$
$$= \frac{\mu_0\left(\pi a^2 I\right)}{4\pi}\frac{\sin\theta}{r^2}$$

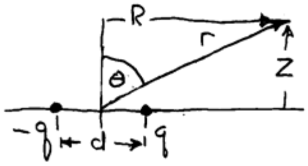
17. Comparison with electrostatics:

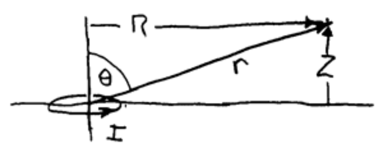
Electric dipole: $\phi = \dfrac{qd}{4\pi\varepsilon_0}\dfrac{\sin\theta}{r^2} = \dfrac{P}{4\pi\varepsilon_0}\dfrac{\sin\theta}{r^2}$

$P = qd =$ dipole moment

Magnetic dipole: $A_\phi = \dfrac{\mu_0\left(\pi a^2 I\right)}{4\pi}\dfrac{\sin\theta}{r^2} = \dfrac{\mu_0 M}{4\pi}\dfrac{\sin\theta}{r^2}$

$M = \pi a^2 I = AI =$ magnetic moment





18. A circular loop of wire produces a magnetic dipole field