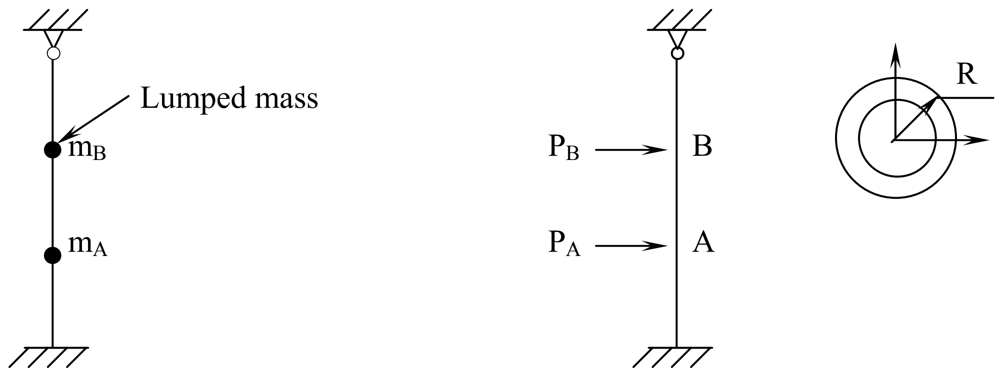
2.314/1.56/2.084/13.14 Fall 2006

Problem Set IX Solution

Solution:
The pipe can be modeled as in Figure 1.



Pipe | Beam with the same support

Geometry and properties:
L=3m, R=0.105m and t=0.007m
E=200GPa, ρ=8500kg/m^3 and ρ_{water}=750kg/m^3

$$m_A = m_B = \frac{\pi(R^2 - R_i^2)L\rho + (\pi R_i^2 + 1.1\pi R^2)L\rho_{water}}{2} = 133.724Kg$$

The governing motion eqution for dynamic response of this pipe can be expressed as:
$[M]\{\ddot{u}\} + [D]\{\dot{u}\} + [K]\{u\} = \{F\}$
where:

- [M]=mass matrix, $M = \begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix}$
- [D]=damping matrix
- [K]=stiffness matrix
- [F]=vector of loads, earthquake load in our case, $\{F\} = [M] \cdot \{S_a\}$
- {u}=vector of nodal displacements

First of all, we should calculate the stiffnes matrix of the pipe using beam theory:
Suppose the displacements of point A (L/3) and B (2L/3) are u_A and u_B, respectively.
Assuming that there is only a force P_A acting on point A (at L/3), we can calculate the corresponding displacements of points A and B, satisfying the boundary conditions:

$$u(0) = 0$$

$$u(L) = u(3) = 0$$

$$\theta_y(0) = \left.\frac{du(z)}{dz}\right|_{z=0} = 0$$

For $L>z>L/3$:

$$M_y = V_x(3-z) = -\frac{dM_y}{dz}(3-z)$$

Solving this equation with the boundary condition $M_y(L)=0$, we get

$$M_y = C_1(3-z)$$

For $z<L/3$:

$$M_y = 2P_A + V_x(3-z) = 2P_A - \frac{dM_y}{dz}(3-z)$$

Solving this equation:

$$M_y = C_2(3-z) + 2P_A$$

Because of continuity at $z=L/3$, we have

$$C_1 = C_2 + P_A$$

Meanwhile,

$$M_y = -\int_A \sigma_z x dA = -\int_A E\varepsilon_z x dA = EK_y \int_A x^2 dA = EK_y I, where\ K_y = \frac{d\theta}{dz} = \frac{d^2u}{dz^2}$$

$$where\ I = \int_A x^2 dA = 2\int_0^{\pi} \cos^2\theta d\theta \int_{0.098}^{0.105} r^3 dr = 2.3023\times 10^{-5}$$

So that

$$K_y = \frac{M_y}{EI} = \begin{cases} \frac{C_1}{EI}(3-z), 1<z<3 \\ \frac{C_2}{EI}(3-z) + \frac{2P_A}{EI}, 0<z<1 \end{cases}$$

$$\theta_y = \int K_y dz = \begin{cases} -\frac{C_1}{EI}\frac{(3-z)^2}{2} + \frac{9C_2}{2EI} + \frac{4P_A}{EI} \\ -\frac{C_2}{EI}\frac{(3-z)^2}{2} + \frac{2P_A}{EI}z + \frac{9C_2}{2EI} \end{cases} \quad \text{at } z=0,\ \theta_y=0$$

$$u(z) = \int \theta_y dz = \begin{cases} \frac{C_1}{EI}\frac{(3-z)^3}{6} + \left[\frac{9C_2}{2EI} + \frac{4P_A}{EI}\right]z - \frac{13P_A}{3EI} - \frac{9C_2}{2EI} \\ \frac{C_2}{EI}\frac{(3-z)^3}{6} + \frac{P_A}{EI}z^2 + \frac{9C_2}{2EI}z - \frac{9C_2}{2EI} \end{cases}$$

At last, we have another boundary condtion, $u(L)=0$

So that

$$3\left[\frac{9C_2}{2EI}+\frac{4P_A}{EI}\right]-\frac{13P_A}{3EI}-\frac{9C_2}{2EI}=0$$

$$C_2=-\frac{23}{27}P_A$$

$$C_1=C_2+P_A=\frac{4}{27}P_A$$

Then, we obtain

$$\begin{bmatrix}u_A\\u_B\end{bmatrix}=\begin{bmatrix}\frac{C_2}{EI}\frac{8}{6}+\frac{P_A}{EI}+\frac{9C_2}{2EI}-\frac{9C_2}{2EI}\\\frac{C_1}{EI}\frac{1}{6}+\frac{9C_2}{EI}+\frac{8P_A}{EI}-\frac{13P_A}{3EI}-\frac{9C_2}{2EI}\end{bmatrix}=\begin{bmatrix}-0.1358\frac{P_A}{EI}\\-0.142\frac{P_A}{EI}\end{bmatrix}$$

Similarly, assuming that there is only a force P_B acting on point B (at *2L/3*), we can calculate the corresponding displacements of points A and B, satisfying the boundary conditions:

For *L>z>2L/3*:

$$M_y=V_x(3-z)=-\frac{dM_y}{dz}(3-z)$$

Solving this equation with the boundary condition $M_y(L)=0$, we get

$$M_y=C_1(3-z)$$

For *z<2L/3*:

$$M_y=P_B+V_x(3-z)=P_B-\frac{dM_y}{dz}(3-z)$$

Solving this equation:

$$M_y=C_2(3-z)+P_B$$

Because of continuity at *z=2L/3*, we have

$$C_1=C_2+P_B$$

Meanwhile,

$$M_y=-\int_A\sigma_z x\,dA=-\int_A E\varepsilon_z x\,dA=EK_y\int_A x^2dA=EK_yI,\ where\ K_y=\frac{d\theta}{dz}=\frac{d^2u}{dz^2}$$

$$where\ I=\int_A x^2dA=2\int_0^{\pi}\cos^2\theta\,d\theta\int_{0.098}^{0.105}r^3dr=2.3023\times10^{-5}$$

So that

$$K_y=\frac{M_y}{EI}=\begin{cases}\frac{C_1}{EI}(3-z),2<z<3\\\frac{C_2}{EI}(3-z)+\frac{P_B}{EI},z<2\end{cases}$$

$$\theta_y=\int K_y dz=\begin{cases}-\frac{C_1}{EI}\frac{(3-z)^2}{2}+\frac{9C_2}{2EI}+\frac{5P_B}{2EI}\\-\frac{C_2}{EI}\frac{(3-z)^2}{2}+\frac{P_B}{EI}z+\frac{9C_2}{2EI}\end{cases}\text{ at z=0, }\theta_y=0$$

$$u(z)=\int \theta_y dz=\begin{cases}\frac{C_1}{EI}\frac{(3-z)^3}{6}+\left[\frac{9C_2}{2EI}+\frac{5P_B}{2EI}\right]z-\frac{9C_2}{2EI}-\frac{19P_B}{6EI}\\ \frac{C_2}{EI}\frac{(3-z)^3}{6}+\frac{P_B}{2EI}z^2+\frac{9C_2}{2EI}z-\frac{9C_2}{2EI}\end{cases}$$

At last, we have another boundary condtion, *u(L)=0*

So that

$$3\left[\frac{9C_2}{2EI}+\frac{5P_B}{2EI}\right]-\frac{9C_2}{2EI}-\frac{19P_B}{6EI}=0$$

$$C_2=-\frac{13}{27}P_B$$

$$C_1=C_2+P_B=\frac{14}{27}P_B$$

Then, we obtain

$$\begin{bmatrix}u_A\\u_B\end{bmatrix}=\begin{bmatrix}\frac{C_2}{EI}\frac{8}{6}+\frac{P_B}{2EI}+\frac{9C_2}{2EI}-\frac{9C_2}{2EI}\\ \frac{C_1}{EI}\frac{1}{6}+\frac{9C_2}{EI}+\frac{5P_B}{EI}-\frac{9C_2}{2EI}-\frac{19P_B}{6EI}\end{bmatrix}=\begin{bmatrix}-0.142\frac{P_B}{EI}\\-0.247\frac{P_B}{EI}\end{bmatrix}$$

So that, according to superposition when P_A and P_B act on the system simultaneously,

$$\begin{bmatrix}u_A\\u_B\end{bmatrix}=\frac{1}{EI}\begin{bmatrix}-0.1358 & -0.142\\-0.142 & -0.247\end{bmatrix}\begin{bmatrix}P_A\\P_B\end{bmatrix}$$

In the meantime,

$$\begin{bmatrix}P_A\\P_B\end{bmatrix}=K\begin{bmatrix}u_A\\u_B\end{bmatrix}$$

We can get

$$K=EI\begin{bmatrix}0.1358 & 0.142\\0.142 & 0.247\end{bmatrix}^{-1}=EI\begin{bmatrix}18.46232 & -10.614\\-10.614 & 10.15054\end{bmatrix}=4.6046\times10^6\begin{bmatrix}18.46232 & -10.614\\-10.614 & 10.15054\end{bmatrix}$$

Secondly, we should compute the damping matrix:

The damping matrix has only two non-zero elements, located on the diagonal. These elements are equal and give two percent critical damping for vibration at the system fundmental frequency.

Thus, we should calculate the undamped fundmental frequency first.

$$[M]\{\ddot{u}\}+[K]\{u\}=0$$

$$\begin{bmatrix}133.724 & 0\\0 & 133.724\end{bmatrix}\begin{bmatrix}\ddot{u}_A\\\ddot{u}_B\end{bmatrix}+EI\begin{bmatrix}18.46232 & -10.614\\-10.614 & 10.15054\end{bmatrix}\begin{bmatrix}u_A\\u_B\end{bmatrix}=0$$

Solving this equation by assuming $u_A=Ae^{\omega t}$ *and* $u_B=Be^{\omega t}$, we get

$$\begin{bmatrix} 133.724\omega^2 + 18.46232EI & -10.614EI \\ -10.614EI & 133.724\omega^2 + 10.15054EI \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

$$(133.724\omega^2 + 18.46232EI)(133.724\omega^2 + 10.15054EI) - 112.657(EI)^2 = 0$$

$$17882.1\omega^4 + 3826.226EI\omega^2 + 74.7455(EI)^2 = 0$$

$$\omega^2 = -1.00126 \times 10^5 or -8.851 \times 10^5$$

Then, the fundamental undamped frequencys are
$\omega_1 = 940.8, \omega_2 = 316.43$ in radius/s; note that $\omega = 2\pi f$
and corresponding eigenvectors are

$$u_1 = \begin{bmatrix} 0.826 \\ -0.56364 \end{bmatrix}, u_2 = \begin{bmatrix} 0.56364 \\ 0.826 \end{bmatrix}$$

Then, we should also calculate the critical damping matrix,
Suppose that the critical damping of lumped mass is considered seperately:

$$D_{cr} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

$$[M]\{\ddot{u}\} + [D]\{\dot{u}\} + [K]\{u\} = 0$$

and the solution is $u(t) = \sum_{n=1}^{N} A_n \cos \omega_n t u_n$

Multiply the above equation with A_N^T leftly, and also replace {u} with A_N {u}, where

$$A_N = [u_1 \quad u_2] = \begin{bmatrix} 0.826 & 0.56364 \\ -0.56364 & 0.826 \end{bmatrix}$$

we get

$$\begin{bmatrix} 133.724\omega^2 + C_1'\omega + 25.7EI & 0 \\ 0 & 133.724\omega^2 + C_2'\omega + 2.9EI \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

If there are nontrivial solutions for A_1 and A_2, we obtain

$$133.724\omega^2 + C_1'\omega + 25.7EI = 0$$

$$133.724\omega^2 + C_2'\omega + 2.9EI = 0$$

So that critical damping matrix:

$$D_{cr}' = \begin{bmatrix} C_1' & 0 \\ 0 & C_2' \end{bmatrix} = \begin{bmatrix} 251592.2 & 0 \\ 0 & 84514.2 \end{bmatrix}$$

$$D_{cr} = A_N D_{cr}' A_N^T = \begin{bmatrix} 198513 & -77788 \\ -77788 & 137593.5 \end{bmatrix}$$

Damping matrix in this case

$$D' = 0.02D_{cr}' = \begin{bmatrix} 5031.844 & 0 \\ 0 & 1690.284 \end{bmatrix}$$

At last, the force due to earthquake

$$\{F\} = -[M]\cdot\{\ddot{u}_g\}$$

Natural frequency

$$f_1 = \frac{\omega_1}{2\pi} = 50.36 \; and \; f_2 = \frac{\omega_2}{2\pi} = 149.7$$

According to the response spectrum of Fig 8 in note M-32, extrapolating to the calculated frequencies, we obtain the maximum displacements corresponding to S_a=0.33g are

$$S_{d1} \approx \frac{S_a}{\omega_1^{\,2}} = \frac{0.33\cdot 9.8}{940.8^2} = 3.6538\times 10^{-6}\, m$$

$$S_{d2} \approx \frac{S_a}{\omega_2^{\,2}} = \frac{0.33\cdot 9.8}{316.43^2} = 3.23\times 10^{-5}\, m$$

So

$$\{F\} = -\begin{bmatrix} 133.724 & 0 \\ 0 & 133.724 \end{bmatrix}\ddot{u}_g$$

Finally, we obtain

$$\begin{bmatrix} 133.724 & 0 \\ 0 & 133.724 \end{bmatrix}\begin{bmatrix} \ddot{u}_A \\ \ddot{u}_B \end{bmatrix} + \begin{bmatrix} 5031.844 & 0 \\ 0 & 1690.284 \end{bmatrix}\begin{bmatrix} \dot{u}_A \\ \dot{u}_B \end{bmatrix} + \begin{bmatrix} 25.7EI & 0 \\ 0 & 2.9EI \end{bmatrix}\begin{bmatrix} u_A \\ u_B \end{bmatrix} = A_N^{\;T}\begin{bmatrix} -133.724 \\ -133.724 \end{bmatrix}\ddot{u}_g$$

$$\begin{bmatrix} \ddot{u}_A \\ \ddot{u}_B \end{bmatrix} + \begin{bmatrix} 37.63 & 0 \\ 0 & 12.64 \end{bmatrix}\begin{bmatrix} \dot{u}_A \\ \dot{u}_B \end{bmatrix} + \frac{1}{133.724}\begin{bmatrix} 25.7EI & 0 \\ 0 & 2.9EI \end{bmatrix}\begin{bmatrix} u_A \\ u_B \end{bmatrix} = -\begin{bmatrix} 0.2624 \\ 1.39 \end{bmatrix}\ddot{u}_g$$

Then, maximum displacement

$$u_{1,\max} = 0.2624\cdot S_{d1}\cdot u_1 = \begin{bmatrix} 7.92\times 10^{-7} \\ -5.4\times 10^{-7} \end{bmatrix}$$

$$u_{2,\max} = 1.39\cdot S_{d2}\cdot u_2 = \begin{bmatrix} 2.53\times 10^{-5} \\ 3.71\times 10^{-5} \end{bmatrix}$$

$$P_{1,\max} = \left[(C\omega u)^2 + (Ku)^2\right]^{0.5}$$

$$= \left[\left(0.02\cdot 940.8\begin{bmatrix} 198513 & -77788 \\ -77788 & 137593.5 \end{bmatrix}\begin{bmatrix} 7.92\times 10^{-7} \\ -5.4\times 10^{-7} \end{bmatrix}\right)^2 + \left(EI\begin{bmatrix} 18.46232 & -10.614 \\ -10.614 & 10.15054 \end{bmatrix}\begin{bmatrix} 7.92\times 10^{-7} \\ -5.4\times 10^{-7} \end{bmatrix}\right)^2\right]^{0.5}$$

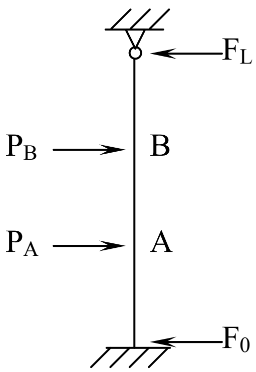
$$= \left[\begin{bmatrix} 3.75 \\ -2.56 \end{bmatrix}^2 + \begin{bmatrix} 93.72 \\ -63.95 \end{bmatrix}^2\right]^{0.5} = \begin{bmatrix} 93.795 \\ 64 \end{bmatrix}$$

$$P_{2,\max} = \left[(C\omega u)^2 + (Ku)^2\right]^{0.5}$$

$$= \left[\left(0.02 \cdot 316.43 \begin{bmatrix} 198513 & -77788 \\ -77788 & 137593.5 \end{bmatrix} \begin{bmatrix} 2.53\times10^{-5} \\ 3.71\times10^{-5} \end{bmatrix}\right)^2 + \left(EI \begin{bmatrix} 18.46232 & -10.614 \\ -10.614 & 10.15054 \end{bmatrix} \begin{bmatrix} 2.53\times10^{-5} \\ 3.71\times10^{-5} \end{bmatrix}\right)^2\right]^{0.5}$$

$$= \left[\begin{bmatrix} 13.52 \\ 19.85 \end{bmatrix}^2 + \begin{bmatrix} 337.6 \\ 497.4 \end{bmatrix}^2\right]^{0.5} = \begin{bmatrix} 337.87 \\ 497.8 \end{bmatrix}$$

Now, given the forces of P_A and P_B, we calculate the forces acted on supports:



Assuming the forces that act on the supports are F_0 and F_L, respectively, as in the above figure.

$$F_L = \left.\frac{dM_y}{dz}\right|_{z=L} = EI\left.\frac{dK_y}{dz}\right|_{z=L} = \frac{4}{27}P_A + \frac{14}{27}P_B$$

$$F_0 = P_A + P_B - F_L = \frac{23}{27}P_A + \frac{13}{27}P_B$$

For $P_{1,\max}$:

$$F_1 = \begin{bmatrix} F_0 \\ F_L \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 23 & 13 \\ 4 & 14 \end{bmatrix} P_{1,\max} = \begin{bmatrix} 110.71 \\ 47.08 \end{bmatrix}$$

Likewise, for $P_{2,\max}$:

$$F_2 = \begin{bmatrix} F_0 \\ F_L \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 23 & 13 \\ 4 & 14 \end{bmatrix} P_{2,\max} = \begin{bmatrix} 527.5 \\ 308.2 \end{bmatrix}$$

Again, $F = \left[F_1^2 + F_2^2\right]^{0.5} = \begin{bmatrix} 539 \\ 311.8 \end{bmatrix} Newton$

So, the peak forces that act on the support *z=0* and *z=L* are *539* and *311.8 Newton*, respectively.