BASIC RELATIONSHIPS IN ELASTICITY THEORY

I. Nomenclature

a) ε_i = strain in the i- direction
   i = x,y,z (Cartesian) or r, θ, z (cylindrical)
   γ_{ij} = shear strain in the i – j plane

b) u, v, w = displacements in the three directions (m)

c) σ_i = normal stress component (Pa)
   τ_{ij} = shear stress component (Pa)

d) E =modulus of elasticity in tension and compression (Pa)
   G = modulus of elasticity in shear (Pa) = $E/2(1+N)$
   ν = Poisson's ratio

e) $\bar{X}.\bar{Y}.\bar{Z}$ = body force components (N/m^3)
   $\bar{R},\bar{\Theta},\bar{Z}$ =body force components (N/m^3)

II. **Strain-Displacement Relationships**

a) Cartesian Coordinates

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

b) Cylindrical Coordinates

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \gamma_{\theta z} = \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

III. **Strain-Stress Relationships**

(Hooke's law for an isotropic medium)

a) Cartesian Coordinates

$$\varepsilon_x = \frac{1}{E}[\sigma_x - v(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - v(\sigma_x + \sigma_z)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - v(\sigma_x + \sigma_y)]$$

$$\tau_{xy} = G\gamma_{xy},\ \tau_{yz} = G\gamma_{yz},\ \tau_{xz} = G\gamma_{xz}$$

b) Cylindrical Coordinates

$$\varepsilon_r = \frac{1}{E}[\sigma_r - v(\sigma_\theta + \sigma_z)]$$

$$\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - v(\sigma_r + \sigma_z)]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - v(\sigma_r + \sigma_\theta)]$$

$$\tau_{r\theta} = G\gamma_{r\theta}, \tau_{rz} = G\gamma_{rz}, \tau_{\theta z} = G\gamma_{\theta z}$$

IV. Stress-Strain Relationships

a) Cartesian Coordinates

$$\sigma_x = \frac{E}{(v+1)(2v-1)}[(v-1)\varepsilon_x - v(\varepsilon_y + \varepsilon_z)]$$

$$\sigma_y = \frac{E}{(v+1)(2v-1)}[(v-1)\varepsilon_y - v(\varepsilon_x + \varepsilon_z)]$$

$$\sigma_z = \frac{E}{(v+1)(2v-1)}[(v-1)\varepsilon_z - v(\varepsilon_x + \varepsilon_y)]$$

b) Cylindrical Coordinates

$$\sigma_r = \frac{E}{(v+1)(2v-1)}[(v-1)\varepsilon_r - v(\varepsilon_\theta + \varepsilon_z)]$$

$$\sigma_\theta = \frac{E}{(v+1)(2v-1)}[(v-1)\varepsilon_\theta - v(\varepsilon_r + \varepsilon_z)]$$

$$\sigma_z = \frac{E}{(v+1)(2v-1)}[(v-1)\varepsilon_z - v(\varepsilon_r + \varepsilon_\theta)]$$

V. Equilibrium Equations

a) Cartesian Coordinates

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \overline{X} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + \overline{Y} = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \overline{Z} = 0$$

b) Cylidrical Coordinates

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + \frac{\partial \tau_{z\theta}}{\partial z} + \overline{\Theta} = 0$$

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_{rz}}{\partial z} + \overline{R} = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} + \overline{Z} = 0$$

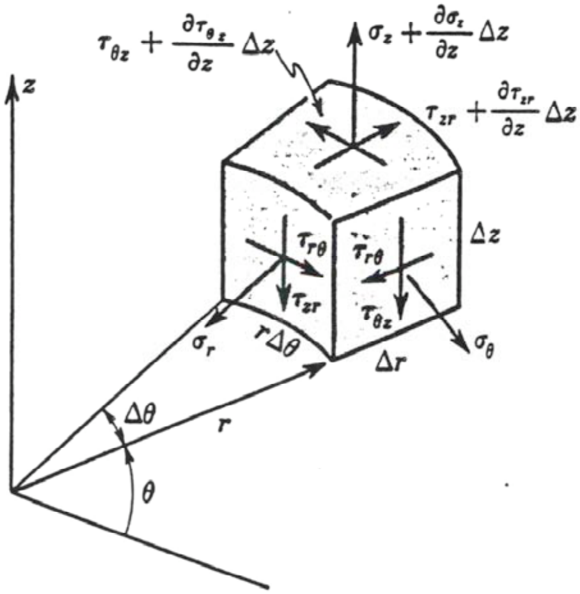
VI. Equation and Unknown Count

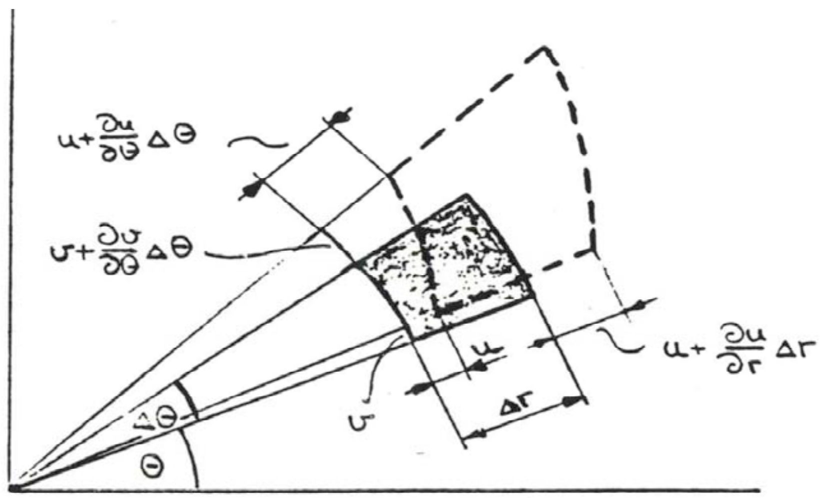
a) Equations

6 strain – displacement relationships
6 strain – stress relationships
3 Equilibrium equations
15 equations

b) Unknowns

6 strains
3 displacements
6 stresses
15 unknowns





$$\varepsilon_\theta = \lim_{\Delta\theta \to 0} \frac{[(r+u)\Delta\theta + v + \frac{\partial v}{\partial \theta}\Delta\theta] - [r\Delta\theta + v]}{r\Delta\theta}$$

$$\varepsilon_\theta = \lim_{\Delta\theta \to 0} [\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}] = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\varepsilon_r = \lim_{\Delta r \to 0} \frac{u + \frac{\partial u}{\partial r}\Delta r - u}{\Delta r} = \frac{\partial u}{\partial r}$$