22.38 - PS#6 Solutions

4-5) in 1984, population will follow a log-normal with median $p_{50_{84}} = +(.1\, p_{50_{74}}) + p_{50_{74}}$ and $COV = .05 = \xi$

$p_{50_{74}} = e^{\lambda_7}$, where $\lambda_7 = \ln \mu_7 - \frac{1}{2}\xi^2 = 9.2$

$p_{50_{84}} = 2(p_{50_{74}}) = 2(e^{9.2}) = e^{\lambda_8} = 19{,}975 \Rightarrow \lambda_8 = 9.962$

μ_{84}: $\lambda_8 = \ln \mu_8 - \frac{1}{2}\xi^2 \Rightarrow \underline{\mu_{84} = 20{,}000}$

distribution: $f_{P_{84}}(p) = \frac{1}{\sqrt{2\pi}\,\xi p} \exp\left[-\frac{1}{2}\left(\frac{\ln p - \lambda}{\xi}\right)^2\right]$ where $\xi = .05$, $\lambda = 9.962$
of population

then consumption: $f_{X_{84}}(X) =$ normal w/ $\mu = 104.2$, $\sigma = .975$

b) $\underline{\mu_D:}$ $X = 19.5 \ln\left(\frac{\mu_{84}}{40}\right) - 17 \Rightarrow \bar{X} = 104.18$ gp/person

$\mu_D = 104.18$ (20,000 people) $= \underline{2.08 \times 10^6 \text{ gpd consumption}}$

$\underline{\text{Var:}}$ $Var(D) \simeq Var(P)\left(\frac{dg}{dP}\right)^2$ where $g(x) = P\left(19.5 \ln\left(\frac{P}{40}\right) - 17\right)$

- $\frac{dg}{dP} = 19.5 \ln\left(\frac{P}{40}\right) - 17 + 19.5(1) = 19.5 \ln\left(\frac{P}{40}\right) + 19.5$

$Var(P) = \mu_8^2 (e^{\xi^2} - 1) = 1.0 \times 10^6$

$\left(\frac{dg}{dp}\right)^2 = \left[19.5 \ln\left(\frac{\mu_8}{40}\right) + 19.5\right]^2 = 1.528 \times 10^4$

$Var(D) = 1 \times 10^6 \cdot 1.528 \times 10^4 = \underline{1.53 \times 10^{10} = Var(D)}$

4.9)

$.1 < C < .35$ where $C = R_H - R_L$ * assume they are statistically independent distribution

$R_L: \mu_L = 20cm$, $COV = .01$ $\Rightarrow \sigma^2 = .04$

$R_H: \mu_h = 20.2$, $COV = .02$ $\Leftrightarrow \sigma^2 = .163$

$\mu_c = \mu_h - \mu_L = .2$; $\sigma^2 = \sigma^2_h + \sigma^2_L = .203 \Rightarrow \sigma = .451$

$P(.1 \leq C \leq .35) = \phi(S_{.35}) - \phi(S_{.1})$

$S_{.1} = \frac{.1-.2}{.451} = -.22$; $S_{.35} = \frac{.35-.2}{.451} = .333$

$P(.1 \leq C \leq .35) = .6293 - .413 = \underline{.216}$

4.19)

a) they are all dependent on R, therefore they are dependent.

b) T is a normal distribution.

$E(T) = \mu_A + \mu_B + \mu_c$ where $\begin{cases} \mu_A = .2(15) + .3 \\ \mu_B = .15(15) + .4 \\ \mu_c = .03(15) \end{cases}$

$\underline{E(T) = 6.4 \text{ mg}}$

$Var(T) = (.2 + .15 + .03)^2 Var(x) = .38^2 \cdot 2^2 = \underline{.5776}$

$\hookrightarrow T = .38R + .7$

c) S is a normal distribution, $N(31.4, .92)$

$S = T - I - M - E + 30$; $E(S) = \mu_T - \mu_I - \mu_m - \mu_E + 30 = \underline{31.4 = \mu_S}$

$Var(S) = Var(T) + Var(I) + Var(M) + Var(E) = \underline{.84 = Var(S)}$

d) $P(S \geq 30) = 1 - \phi(S_{30})$

$S = \frac{30 - 31.4}{.92} = -1.53$ $\Rightarrow P(S \geq 30) = \phi(1.53)$

$\underline{P(S \geq 30) = .937}$

4-24) a) $F(x) = 1 - e^{-\lambda x} = P(X \le x)$;

weather
good: $P = .75$, $\bar{T} = 1$, $\lambda = 1$
bad: $P = .25$, $\bar{T} = 2$; $\lambda = .5$

$$P_T(x \ge 1.5) = .75(e^{-1.5}) + .25(e^{-1.5/2}) = \underline{.2854}$$

b) $$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|a|} f_{X,Y}\left(\frac{z - by}{a}\right) f_Y(y)\,dy \quad \text{where } Z = aX + bY$$

$$f_Z(z) = \int_0^{\infty} f_x(z-y) f_y(y)\,dy = \int_0^{z} \lambda_x e^{-\lambda_x(z-y)} \cdot \lambda_y e^{-\lambda_y y}\,dy \quad (*)$$

$$= -\frac{\lambda_x \lambda_y}{(\lambda_y - \lambda_x)} e^{-\lambda_x z}\left[e^{-(\lambda_y - \lambda_x)z} - 1\right]$$

$$P(Z > 2 \mid \text{good weather}) = \int_2^{\infty} \frac{-\lambda_x \lambda_y}{(\lambda_y - \lambda_x)} e^{-\lambda_x z}\left[e^{-(\lambda_y - \lambda_x)z} - 1\right]dz \; ; \quad \lambda_x = 1, \; \lambda_y = 2$$

$$= \int_2^{\infty} -2\left(e^{-2z} - e^{-z}\right)dz = 2\left(e^{-2} - \tfrac{1}{2}e^{-4}\right) = \underline{0.252}$$

note: in $(*)$ you integrate $\int_0^z$, not $\int_0^\infty$ because you want only to capture the values that satisfy $Z = X + Y$

4-31) $T = 4\sqrt{N_A}$ $\Rightarrow$ T is log-normally distributed,

so the distribution of N_A is as follows:

$\ln T = \ln 4 + \frac{\ln N_A}{2}$ $\Rightarrow$ $\ln T$ is normal $\therefore$ $\ln N_A$ is also normal

$\therefore$ N_A is lognormally distributed w/ parameters:

$\mu_{N_A} = \Sigma N_i = 25$; $Var(N_a) = \Sigma \sigma_i^2 = 15$

$\Rightarrow \lambda = \ln \mu - \frac{1}{2}\zeta^2$; $\zeta^2 = \ln(1 + (\frac{\sigma}{\mu})^2)$ $\Rightarrow \lambda = 3.207$; $\zeta = 0.154$

$P(T > 25) = 1 - P(T \leq 25) = 1 - P(N_a \leq (\frac{25}{4})^2) = 1 - \Phi(S_{N_a})$

$S_{N_a} = \frac{\ln((\frac{25}{4})^2) - 3.207}{0.154} = 2.975$

$P(T > 25) = 1 - \phi(2.975) = \underline{0.00146}$

4.32) $I_c = \frac{I_A}{100} + \frac{I_B}{400}$

a) then $E(I_c) = \frac{\mu_A}{100} + \frac{\mu_B}{400} = \underline{15 = E(I_c)}$

$Var(I_c) = (\frac{1}{100})^2 Var A + (\frac{1}{400})^2 Var B = \underline{1.25}$

b) assume I_c is lognormally distributed,

$\Rightarrow 2I \Rightarrow \mu_{2I} = 30$, $\sigma^2 = 1.5$

$\Rightarrow D = 40 \ln 2I$

$\Rightarrow E(D) = 40(\ln(30)) = \underline{136}$