# Session 2

## Basic Probabilistic Concepts

# BINARY EVENTS AND PROBABILITIES

A Binary Event Has Only Two Possible Outcomes:

- Success
- Failure

For a Series of N Identical Trials of a Binary Random Event:

$$p \equiv \text{Probability of Success} = \lim_{N\to\infty}\left(\frac{\text{\# of Successes}}{\text{\# of Trials, N}}\right)$$

$$q \equiv \text{Probability of Failure} = \lim_{N\to\infty}\left(\frac{\text{\# of Failures}}{\text{\# of Trials, N}}\right)$$

} Frequentist definition of probability

For a Single Trial, the Probability of Some Outcome Occurring Equals Unity, or $1 = p + q$.

# THINGS TREATED AS RANDOM EVENT OR PHENOMENA

RANDOM EVENTS

- Radioactive Decay
- Quantum State Transitions

STATISTICS OF LARGE POPULATIONS OF SIMILAR OBJECTS
(many statistics obey a normal distribution)

- Human Fates and Attributes
- Pump Fates
- Car Wrecks

SENSITIVE DETERMINISTIC EVENTS

- Flipped Coin Fates
- Card Hands
- Weather

RARE EVENTS

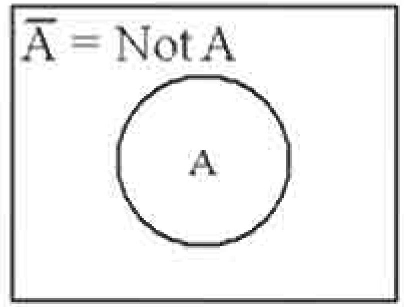
- Aircraft Crashes
- Infrequent Accidents

POORLY UNDERSTOOD EVENTS

- Your Teenager's Mood
- Short Term Stock Price Changes
- EDG Wearout

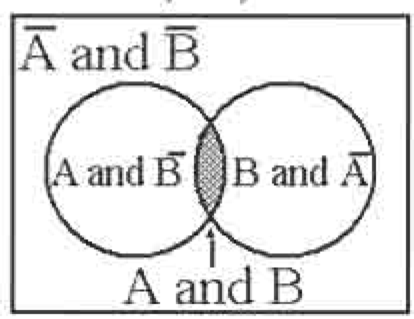
# VENN DIAGRAM

Events A and $\bar{A}$




← Universe of Possible Events

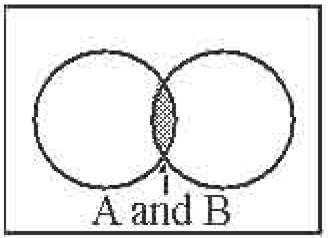
Events A, $\bar{A}$; B and $\bar{B}$



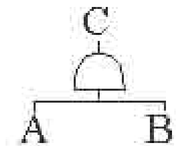

$$P(A) + P(\bar{A}) = 1$$

$$P(B) + P(\bar{B}) = 1$$

# PROBABILITIES OF COMBINED EVENTS

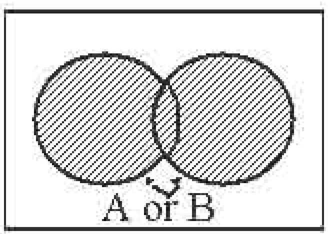



Event (A and B) $\equiv$ $A \cdot B$ Boolian operator

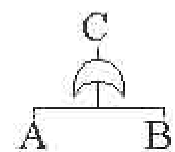



Event (B, given A) $\equiv$ $B/A$

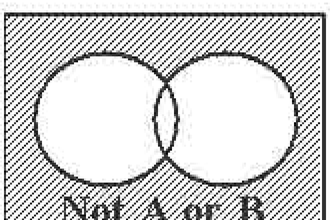
$$P(A \cdot B) = \begin{cases} P(A) \cdot P(B/A) \\ P(B) \cdot P(A/B) \\ P(A) \cdot P(B) \text{ only if A and B are Independent Events} \end{cases}$$




Event (A or B) $\equiv$ $A + B$ Boolian operator




$$P(A + B) = \begin{cases} P(A) + P(B) - P(A \cdot B) \\ P(A) + P(B) - P(A) * P(B) \text{ only if A and B are Independent Events} \end{cases}$$

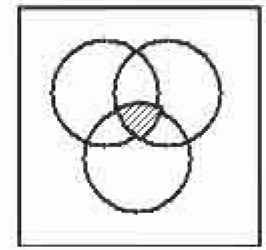



$\equiv$ $\overline{A} \cdot \overline{B}$

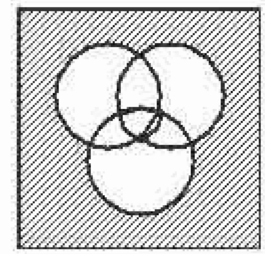
$$P(\overline{A} \cdot \overline{B}) = \begin{cases} 1 - P(A + B) \\ 1 - [P(A) + P(B) - P(A) * P(B)] \text{ only if A and B are Independent Events} \\ P(\overline{A}) \cdot P(\overline{B}) \text{ only if A and B are Independent Events} \end{cases}$$

# PROBABILITIES OF DIFFERENT OUTCOMES

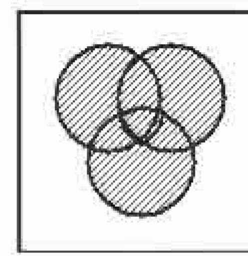
Three Race Horses, a, b, c, Where Each Horse Runs in a Different Race:



P(All Horses Win) = $p_a \, p_b \, p_c$ = (0.1) (0.5) (0.7) = 0.035



P(No Horse Wins) = $q_a \, q_b \, q_c$ = (0.9) (0.5) (0.3) = 0.135



P(At Least One Horse Wins) = 1 - P(No Horse Wins)
= 1 - 0.135 = 0.85

Best Bet for a Winner: (Paul Revere), p = 0.7

# PROBABILITY OF COMBINED EVENTS

In Obtaining the Probability of a Combination of Independent Multiple Events We Must Consider

- The Number of Permutations (Reflecting the Order of the Individual Events) Within the Combination (Reflecting the Respective Numbers of Events of Each Type Within a Permutation).

- The Probability, P, of a Single Permutation

Prob.[Combination] = [No. of Permutations] $\supseteq$ [Prob.(One Permutation)]

Example: White and Black Balls in Different Positions

| | | | | | |
|---|---|---|---|---|---|
| Permutation | Color: | W | B | W | B |
| | Place: | 1 | 2 | 3 | 4 |
| Combination | (2W, 2B) | | | | |

# FOR THE EXAMPLE OF THREE SUCCESSIVE INDEPENDENT TRIALS

Let P(i, j) = Probability (i Successes, j Failures)

| Combination | Number of Permutations | Single Prob.(Permutation) | Prob.(Combination) |
|---|---|---|---|
| (3 Successes, 0 Failures) | (S, S, S); $\Rightarrow 1$ | $p \cdot p \cdot p = p^3$ | $p^3$ |
| (2 Successes, 1 Failure) | $\left.\begin{pmatrix} S, & S, & F \\ S, & F, & S \\ F, & S, & S \end{pmatrix}\right\} \Rightarrow 3$ | $p \cdot p \cdot q = p^2q$ | $3p^2q$ |
| (1 Success, 2 Failures) | $\left.\begin{pmatrix} S, & F, & F \\ F, & S, & F \\ F, & F, & S \end{pmatrix}\right\} \Rightarrow 3$ | $p \cdot q \cdot q = pq^2$ | $3pq^2$ |
| (0 Successes, 3 Failures) | (F, F, F); $\Rightarrow 1$ | $q \cdot q \cdot q = q^3$ | $q^3$ |

Probability of Some Outcome Occurring in Three Trials = P(3, 0) + P(2, 1) + P(1, 2) + P(0, 3) = 1

**Remember: p + q = 1**

# EXPECTED OUTCOMES

$$\langle f(x)\rangle = E(f(x)) = \sum_i f(x_i)P_i$$

Let $x_i$ Be Distributed According to Probability Mass Function, $P(x_i)$

Example of Three Trials (N = 3):

Expected number of successes, $\langle S\rangle = \begin{bmatrix} \text{(3 successes)} \cdot P(3, 0) \\ + \text{(2 successes)} \cdot P(2, 1) \\ + \text{(1 success)} \cdot P(1, 2) \\ + \text{(0 successes)} \cdot P(0, 3) \end{bmatrix} \quad \begin{bmatrix} 3 \cdot p^3 \\ + 2 \cdot 3p^2q \\ + 1 \cdot 3pq^2 \\ + 0 \cdot q^3 \end{bmatrix} = 3p$

$$\langle S\rangle = \sum_{i=0}^{N} S_i P(i, j), \quad i + j = N$$

Expected number of failures, $\langle F\rangle = \begin{bmatrix} \text{(0 failures)} \cdot P(3, 0) \\ + \text{(1 failure)} \cdot P(2, 1) \\ + \text{(2 failures)} \cdot P(1, 2) \\ + \text{(3 failures)} \cdot P(0, 3) \end{bmatrix} = \begin{bmatrix} 0 \cdot p^3 \\ + 1 \cdot 3p^2q \\ + 2 \cdot 3pq^2 \\ + 3 \cdot q^3 \end{bmatrix} = 3q$

$$\langle F\rangle = \sum_{i=0}^{N} F_i P(i, j), \quad i + j = N$$

In general,

$$\langle S\rangle = Np, \quad \langle F\rangle = Nq = N(1 - p)$$

# BINOMIAL DISTRIBUTION

- An experiment has only two outcomes: "success" with probability p and "failure" with probability (1-p);
- Consider performing N such independent trials;
- X is the number of successful outcomes out of the total N trials;
- X has a Binomial distribution Binomial (k, N; p)

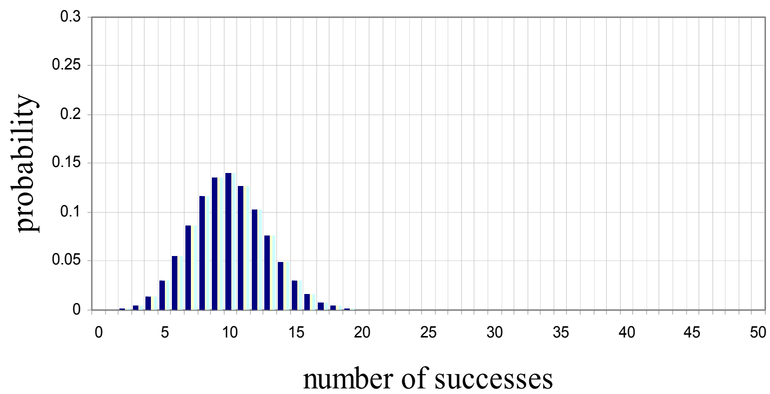
$$P(X = k) = \binom{N}{k}(1-p)^{N-k} p^k, k = 0,1,\ldots, N$$

$$\binom{N}{k} = \frac{N!}{k!(N-k)!} \quad , \quad N! = N(N-1)\cdots 1$$

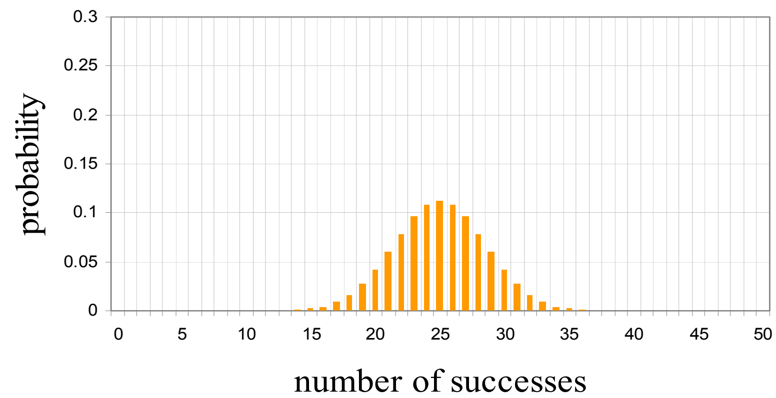
- For a given value of p any value of k $[0 \le k \le N]$ is possible
- The most likely value of k is pN

# EFFECTS OF P ON BINOMIAL DISTRIBUTION — N=50

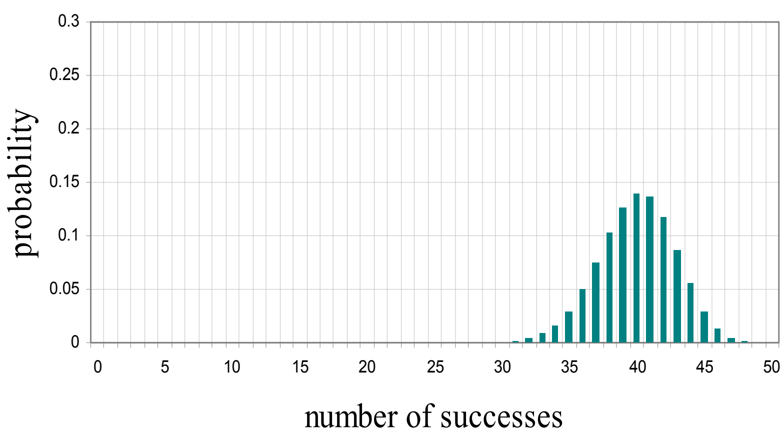
p=0.2, N=50



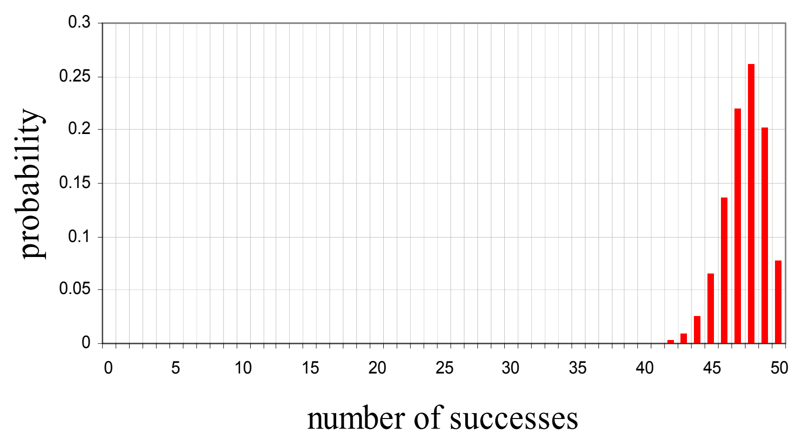

p=0.5, N=50



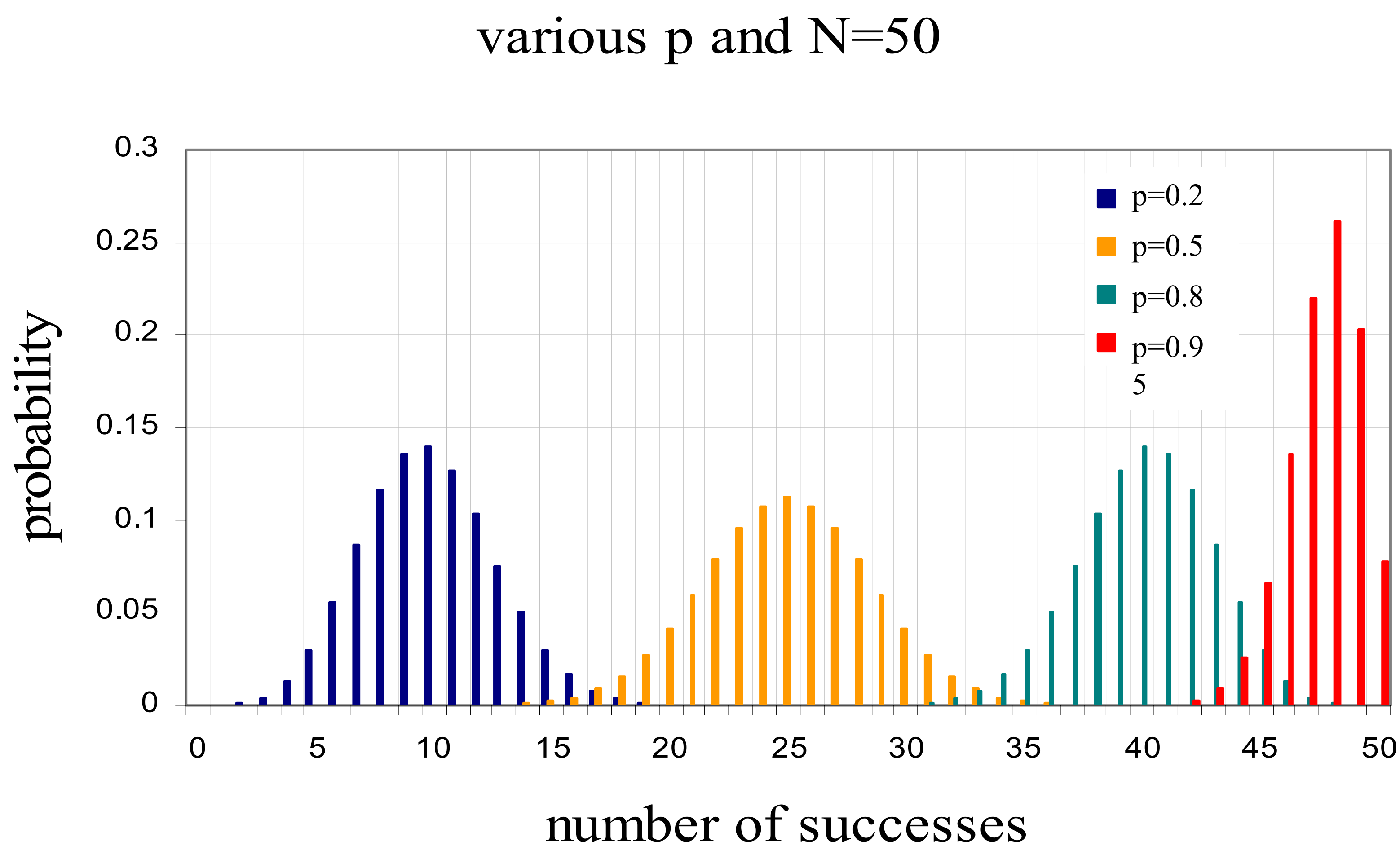

p=0.8, N=50




p=0.95, N=50

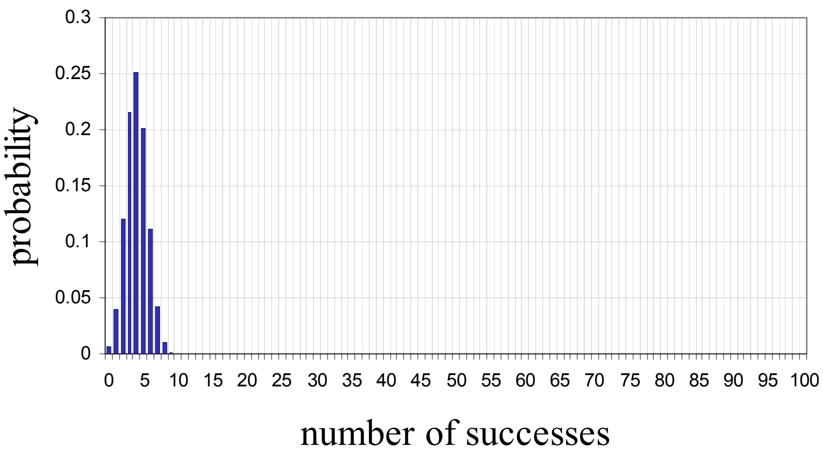


# EFFECTS OF P ON BINOMIAL DISTRIBUTION — N=50 (continued)

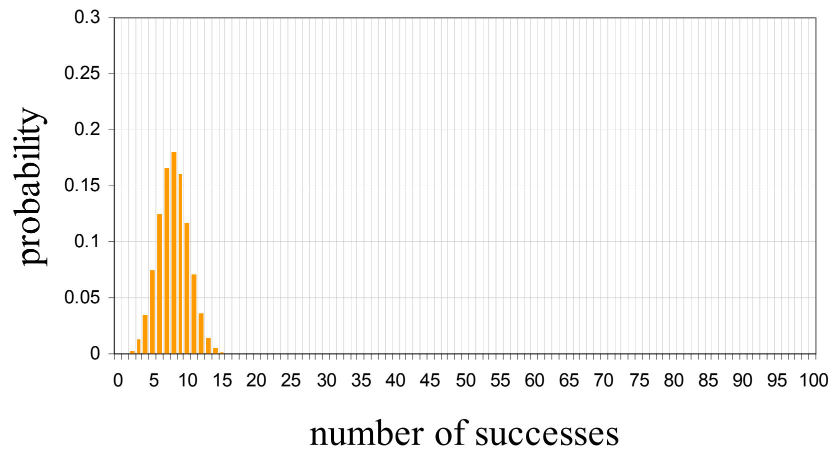


# EFFECTS OF N ON BINOMIAL DISTRIBUTION

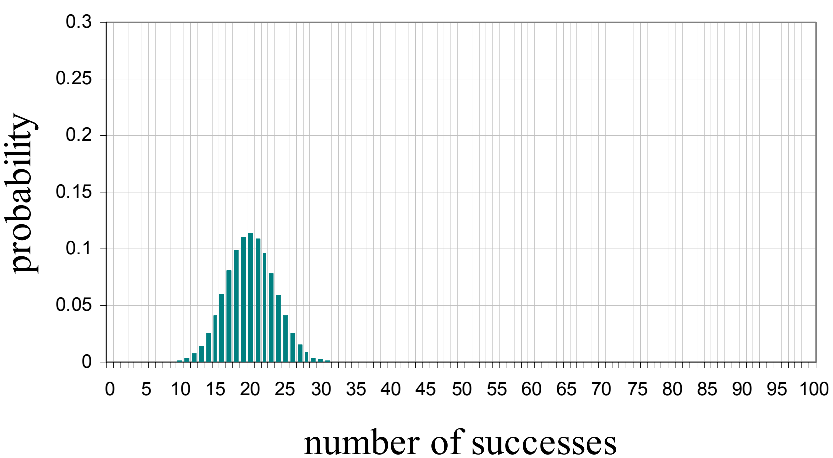
Binomial distribution with p=0.4, N=10




Binomial distribution with p=0.4, N=20




Binomial distribution with p=0.4, N=50




Binomial distribution with p=0.4, N=100