22.51 Quantum Theory of Radiation Interactions

Mid-Term Exam
Thursday Oct. 29, 2009

Problem 1 Simple two-level system *20 points*

The operator σ_x (σ_y) corresponds to the component of the spin of an electron along the x-axis (y-axis), in units of $\hbar/2$. In the basis $\{|0\rangle = |\sigma_z = +\hbar/2\rangle, |1\rangle = |\sigma_z = -\hbar/2\rangle\}$ the state $|\psi\rangle$ is represented by the vector

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

where α and β are complex numbers.

a) Calculate the probability that the spin x-component is positive.

b) Calculate the expectation value $\langle\sigma_y\rangle$

Solution

a) The eigenvectors of the σ_x operators are $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ with positive and negative eigenvalue respectively. We can express the z-basis eigenkets in terms of this new basis as $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$. Thus:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \frac{1}{\sqrt{2}}[\alpha(|+\rangle + |-\rangle) + \beta(|+\rangle - |-\rangle)] = \frac{1}{\sqrt{2}}(\alpha + \beta)|+\rangle + \frac{1}{\sqrt{2}}(\alpha - \beta)|-\rangle$$

Then $p(x > 0) = |\langle+|\psi\rangle|^2 = |\alpha + \beta|^2/2$

b) $\langle\sigma_y\rangle = \langle\psi|\sigma_y|\psi\rangle$ with $\sigma_y = i(|1\rangle\langle0| - |0\rangle\langle1|)$. Expanding the ket, we find

$$\langle\sigma_y\rangle = i(\alpha^*\langle0| + \beta^*\langle1|)(|1\rangle\langle0| - |0\rangle\langle1|)(\alpha|0\rangle + \beta|1\rangle) = -i(\alpha^*\beta - \beta^*\alpha) = 2Im(\alpha^*\beta)$$

Problem 2 Operators on composite systems *10 points*

We have seen in class how to calculate $e^{-i\sigma_z\phi}$ by expanding the exponential as a series:

$$e^{-i\sigma_z\phi} = \sum_k \frac{1}{k!}(-i\sigma_z\phi)^k = \sum_{k,even} \frac{1}{k!}(-i\phi)^k \mathbb{1} + \sum_{k,odd} \frac{1}{k!}(-i\phi)^k \sigma_z = \cos(\phi)\mathbb{1} - i\sin(\phi)\sigma_z$$

where we used the fact that $(\sigma_z)^2 = \mathbb{1}$. In general $e^{-i\phi\vec{n}\cdot\vec{\sigma}} = \mathbb{1}\cos(\phi) - i\vec{n}\cdot\vec{\sigma}\sin(\phi)$
Now we consider a system composed of two subsystems $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$, each being a two-level system. To describe operators on the two subsystems we normally use the shorthanded notations $A^1 \equiv A^1 \otimes \mathbb{1}^2$ and $B^2 \equiv \mathbb{1}^1 \otimes B^2$ and $A^1 B^2 \equiv A^1 \otimes B^2$.

a) What is $(\sigma_z^1\sigma_z^2)^2$?

b) Consider the operator $U = e^{-i\sigma_z^1\phi} \otimes e^{-i\sigma_z^2\phi}$: to which operator is this equivalent to, $U_+ = e^{-i(\sigma_z^1+\sigma_z^2)\phi}$ or $U_\times = e^{-i\sigma_z^1\sigma_z^2\phi}$?

Solution

This problem can be solved almost without any calculation (and without writing the explicit matrix representation)

a) $(\sigma_z^1\sigma_z^2)^2 = (\sigma_z^1 \otimes \sigma_z^2)^2(\sigma_z^1)^2 \otimes (\sigma_z^2)^2 = \mathbb{1}^1 \otimes \mathbb{1}^2 = \mathbb{1}$ (where the last identity is on the composite system).

b) Since σ_z^1 and σ_z^2 commute, we can do calculations as if they just were numbers. For exponential functions we know that the product of two exponentials is the exponential of the sum of the arguments. (Of course this is not valid when the operators do not commute). So we simply had: $e^{-i\sigma_z^1\phi} \otimes e^{-i\sigma_z^2\phi} = e^{-i\sigma_z^1\phi}e^{-i\sigma_z^2\phi} = e^{-i(\sigma_z^1+\sigma_z^2)\phi}$.

Notice that $U_\times = e^{-i\sigma_z^1\sigma_z^2\phi}$ is NOT an operator that acts separately on the two subsystems (i.e. it cannot be written for all phases as $A_1 \otimes B_2$, where $A_1 (B_2)$ acts on the first (second) space). Thus it is also an operator that can create entanglement (starting from a separable state).

With the formula above for the exponential and knowing that $(\sigma_z^1\sigma_z^2)^2 = \mathbb{1}$ you could have also calculated explicitly (and quickly!) all the exponentials and verified the result.

$$U_+ = e^{-i(\sigma_z^1+\sigma_z^2)\phi} = \cos(\phi)^2\mathbb{1} + \sin(\phi)^2\sigma_z^1\sigma_z^2 - i\sin(\phi)\cos(\phi)(\sigma_z^1 + \sigma_z^2)$$

and

$$U_\times = e^{-i\sigma_z^1\sigma_z^2\phi} = \cos(\phi)\mathbb{1} - i\sin(\phi)\sigma_z^1\sigma_z^2$$

For these operators it was also easy to calculate the exponentials from the matrix form, since the matrices are diagonal in the usual z-basis (and the exponential of a diagonal matrix is obtained by just exponentiating its diagonal elements). However, I hope that you understand also the arguments above, that simply the calculations in many other cases (e.g. if I had replace σ_z with σ_x).

Problem 3 Probing the number of photons in a cavity with Rydberg atoms   *30 points*

In the first lecture I presented a paper by Serge Haroche's group describing an experiment that follows the quantum life of a photon. We now have most of the tools to fully understand this paper. This will be discussed in problems 3–5.
[Note: you don't need to read the paper to solve the problems (you won't have the time!), just look at the Figures for reference.]

Fig. 1 describes the experimental apparatus, that consists of a first Ramsey cavity R_1, a cavity C, a second Ramsey cavity R_2 and the detector D. The atoms can be in two possible states $|g\rangle$ or $|e\rangle$ (the ground and excited states).
The Box B prepares a flux of atoms in their **ground state**. We will consider the evolution of a single atom from that initial state.
The Ramsey cavity R_1 operates on the atoms with the Hamiltonian

$$\mathcal{H}_{R1} = i\Omega_R(|e\rangle\langle g| - |g\rangle\langle e|)$$

a) What is the matrix representation of $\mathcal{H}_{R1}$ in the basis $|e\rangle, |g\rangle$?

b) The atoms take a time $t_R = \pi/(\Omega_R 4)$ to traverse the Ramsey cavity (during which time they experience the Hamiltonian above). What is the state of the atom at the exit of the Ramsey cavity?

In the cavity C, the atom interacts for a time t_C with the electro-magnetic field, with an Hamiltonian

$$\mathcal{H}_C = n\omega(|g\rangle\langle g| - |e\rangle\langle e|)$$

where n is the number of photons in the cavity itself. Finally, the second Ramsey cavity R_2 has an Hamiltonian that is similar to R_1:

$$\mathcal{H}_{R2} = -i\Omega_R(|e\rangle\langle g| - |g\rangle\langle e|)$$

and it takes the same time $t_R = \pi/(\Omega_R 4)$ for the atom to traverse this cavity.

c) What is the probability of finding the atom in the state $|g\rangle$ at the detector D?

Solution

a)
$$\mathcal{H}_{R1} = \Omega_R \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_R\sigma_y$$

Recognizing that this matrix is just the σ_y Pauli operator (and the other Hamiltonian as well could be expressed in terms of Pauli operators) and using the formula given above for the exponential of the Pauli operators should have simplified the calculations.

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b) With the formula provided $U_{R1} = e^{-i\sigma_y \pi/4} = \cos(\pi/4)\mathbb{1} - i\sin(\pi/4)\sigma_y = \frac{1}{\sqrt{2}}(\mathbb{1} - i\sigma_y)$. The atom before the cavity is in the ground state $|g\rangle$. Thus the atom after the cavity is in the state

$$U_{R1}|g\rangle = \frac{1}{\sqrt{2}}(\mathbb{1} - i\sigma_y)|g\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle)$$

c) Writing down the explicit matrix representation of $\mathcal{H}_C$ one recognizes that the hamiltonian is just $\mathcal{H}_C = n\omega\sigma_z$. In particular, if $n = 0$ the Hamiltonian is just 0.
Then the evolution in the cavity is $U_C = e^{-in\omega\sigma_z}$ giving a state

$$U_C \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle) = \frac{1}{\sqrt{2}}(e^{-in\omega t_C}|g\rangle + e^{in\omega t_C}|e\rangle)$$

The evolution in the second cavity is just $U_{R1} = e^{i\sigma_y \pi/4} = \frac{1}{\sqrt{2}}(\mathbb{1} + i\sigma_y)$ (as $\mathcal{H}_{R2} = -\Omega\sigma_y$). Finally we have:

$$|\psi\rangle_{fin} = \frac{1}{2}[e^{-in\omega t_C}(|g\rangle - |e\rangle) + e^{in\omega t_C}(|e\rangle + |g\rangle)] = \cos(n\omega t_C)|g\rangle + i\sin(n\omega t_C)|e\rangle$$

Thus the probability is $p(g) = |\langle g|\psi_{fin}\rangle|^2 = \cos(n\omega t_C)^2$.
Notice that if $n = 0$ $p(g) = 1$ always. This is also because the evolution is just given by the two Ramsey cavities (as the cavity Hamiltonian is zero). But the two cavities are just one the opposite of the other, so that $U_{R2}U_{R1} = \mathbb{1}$ so the initial state $|g\rangle$ does not evolve at all if there are 0 photons in the cavity.

Problem 4 Probing photon number states with Rydberg atoms   *30/40 points*

Now consider the e.m. field in the cavity C as a quantum mechanical system. Under the experiment conditions, the field is such that there's at most one photon in the cavity (that is, either 0 or 1 photons).

a) What is a basis for the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{e.m.}$? (where $\mathcal{H}_A$ is the atom's and $\mathcal{H}_{e.m.}$ the photon's Hilbert spaces. What is the Hamiltonian H_C for the interaction of the atom with the photon in the cavity? *[This should replace the operator $\mathcal{H}_C$ given above, for the cases $n = \{0, 1\}$]*

b) Assume that initially we do not know how many photons are in the cavity, but only that it is more probable to have zero photons (with 75% of probability). What is the state of the atom-photon system at the instant in which the atom enters the cavity C? (use the result found in Problem 3.b for the state of the atom).

c) The atom stays in the cavity for a time $t = \pi/(2\omega)$. We then detect the atom and find that it's in the state $|e\rangle$. What does this tell us about the number of photons in the cavity?
[Hint 1: $e^{-i(\sigma_z|1\rangle\langle 1|)\omega t} = e^{-i\sigma_z \omega t}|1\rangle\langle 1| + \mathbb{1}|0\rangle\langle 0|$. Hint 2: You might be able to give an answer even without doing any calculations.]

This is the method used in the paper to indirectly measure the photon number (either 0 or 1) by looking at the state of the atom. Even if we do not know a priori the number of photons in the cavity, we can infer it from the measurement of the atom's state. In the second experiment reported in the paper, the authors can initialize the photon number in a precise state. You should have found that the sequence of operations (First Ramsey cavity, Cavity, Second Ramsey cavity) gives the total propagator:

$$U = U_{R2}U_C U_{R1} = e^{-i\sigma_x \phi}|1\rangle\langle 1| + \mathbb{1}|0\rangle\langle 0| \rightarrow -i\sigma_x|1\rangle\langle 1| + \mathbb{1}|0\rangle\langle 0| \quad (\text{for } \phi = \pi/2)$$

d) If the field is initially in a superposition state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ (and the atom still $|g\rangle$ before the first Ramsey cavity), is the final state entangled? What would be the correlation between measuring the atom in the excited (ground) state and measuring one (zero) photons?

e) What is the probability of observing the atom in the $|g\rangle$ state at the detector D?
Now assume that the detector measures the atom's state, but the detector is unreliable, so that we do not know the outcome. What is the state of the photon after measuring the atom?

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Solution

a) We have a composite system of two two-level systems. The first system represents the atom, with basis $\{|g\rangle, |e\rangle\}$. The second system is the e.m. field, that we represent with the photon number. Here we restricted the photon number to either 0 or 1, so that the basis for the e.m. field is just $\{|0\rangle, |1\rangle\}$. A basis for the composite system is then $\{|g0\rangle, |g1\rangle |e0\rangle |e1\rangle\}$. Notation-wise, I set the first system to be the atom and the second to be the e.m. (or photon) system without further labeling the operators. This should be consistent with later notation.

The hamiltonian should represent the same evolution as before (given by $\mathcal{H}_C$) when we restrict to the possibility of having only 1 or 0 photons. Notice that for 0 photons $\mathcal{H}_C = 0$, while for 1 photon $\mathcal{H}_C = \omega \sigma_z^A$. Thus we can write the Hamiltonian for the composite system as

$$H_C = \omega \sigma_z^A \otimes |1\rangle\langle 1|^{e.m.}$$

b) The state of the photon is represented by the density operator: $\rho_{e.m.} = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$. The state of the system is thus:

$$\rho = \frac{1}{2}(|e\rangle + |g\rangle)(\langle e| + \langle g|) \otimes (\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|)$$

c) The propagator of the cavity is $e^{-i(\sigma_z|1\rangle\langle 1|)\omega t} = e^{-i\sigma_z \omega t}|1\rangle\langle 1| + \mathbb{1}|0\rangle\langle 0| = (-i\sigma_z|1\rangle\langle 1| + \mathbb{1}|0\rangle\langle 0|)$ thus:

$$U_C \rho U_C^\dagger = (-i\sigma_z|1\rangle\langle 1| + \mathbb{1}|0\rangle\langle 0|)\rho(+i\sigma_z|1\rangle\langle 1| + \mathbb{1}|0\rangle\langle 0|) = \frac{3}{8}(|e\rangle + |g\rangle)(\langle e| + \langle g|) \otimes |0\rangle\langle 0| + \frac{1}{8}(|e\rangle - |g\rangle)(\langle e| - \langle g|) \otimes |1\rangle\langle 1|$$

After the second Ramsey cavity this becomes:

$$\frac{3}{4}|g\rangle\langle g| \otimes |0\rangle\langle 0| + \frac{1}{4}|e\rangle\langle e| \otimes |1\rangle\langle 1|$$

Thus, if we detect the atom in the $|e\rangle$ state it must mean that we had a photon in the cavity. You could have argued this result from the Hamiltonian in 3.b or 4.a: if $n = 0$, $\mathcal{H}_C = 0$ so that it's not possible to find the atom in $|e\rangle$ since the two Ramsey cavity just cancel each other $U_{R2}U_{R1} = \mathbb{1}$. Physically, it means that if there is a photon, this photon can give some energy to the atom and induce an evolution to the excited state.

d) We apply the given U to the initial state

$$U|g\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = (-i\sigma_x|1\rangle\langle 1| + \mathbb{1}|0\rangle\langle 0|)[\frac{1}{\sqrt{2}}(|g0\rangle + |g1\rangle)] = \frac{1}{\sqrt{2}}(|g\rangle|0\rangle - i|e\rangle|1\rangle)$$

Thus the state is a maximally entangled state. e.g. the concurrence is 1, since $\alpha = \frac{1}{\sqrt{2}}$ and $\delta = \frac{-i}{\sqrt{2}}$, so that $C = 2|\alpha\delta - \beta\gamma| = 1$. There is a perfect correlation of finding 1 photon if the atom is found in the state $|e\rangle$:

$$\frac{1}{2}(\langle g|\langle 0| + i\langle e|\langle 1|)(\sigma_z^A \sigma_z^{e.m.})(|g\rangle|0\rangle - i|e\rangle|1\rangle) = 1$$

e) There is a 50% probability of finding the atom in the state $|g\rangle$: $|\langle g|\frac{1}{\sqrt{2}}(|g\rangle|0\rangle - i|e\rangle|1\rangle)|^2 = 1/2$

If the result of the measurement is unknown, then with 50% probability we measured $|g\rangle$ and there is no photon, and with 50% probability we measured $|e\rangle$ and the photon is in the state $|1\rangle$. Thus we have a mixed state for the photon equal to $\rho_{e.m.} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \mathbb{1}/2$ (maximally mixed state).

Problem 5   Quantum jumps of light recording the birth and death of a photon   *10/20 points*

In Fig. 3 of the paper, the authors show the decay of a one-photon state, as measured indirectly by the atom. Following their model [and guided by the results in (4.c) and (4.d)] we now consider the photon only and assume that we are directly observing its dynamics. Because of thermal fluctuations, the number of photons can jump from 1 to zero: at each instant of time, if there is a photon, it can be absorbed with probability p (or not, with probability $1 - p$). If there are already no photons, nothing happens.

a) Write Kraus operators describing this process.

[Hint: you could consider the interaction of the photon e.m. field –the system S– with a bath –the environment E– giving the evolution:

$$U_{SE}|0\rangle_S|0\rangle_E \rightarrow |0\rangle_s|0\rangle_E$$
$$U_{SE}|1\rangle_S|0\rangle_E \rightarrow \sqrt{p}|0\rangle_S|1\rangle_E + \sqrt{1-p}|1\rangle_S|2\rangle_E$$

The same result can also be obtained by considering the different processes the photon can undergo].

b) What is the state ρ' after one step, if the initial state of the atom is $|1\rangle$? What is $\langle 1|\rho|1\rangle$ after n steps?

c) Assume nothing happens to the photon (initially in the state $|1\rangle$) until time t_0 where a jumps to the $|0\rangle$ state happens. Plot a figure similar to Fig. 3a.

d) Now we look instead at the average evolution of the photon, as described by the evolution of the density operator under the action of the superoperator $\rho(t) = S(t)[\rho]$. Take $p = \gamma \delta t$, with $\delta t = t/n$, in the limit of $n \rightarrow \infty$ (and fixed t). What is the time evolution of the density operator? How does this compare with Fig. 3d?

Solution

a) There are 3 Kraus operators:

$$M_0 = |0\rangle\langle 0| \quad M_1 = \sqrt{p}|0\rangle\langle 1| \quad M_2 = \sqrt{1-p}|1\rangle\langle 1|$$

The first operator describes the fact that if there is no photon nothing happens and the photon number stays $= 0$. The other two operators describe the processes in which with probability p one photon is absorbed (i.e. we go from 1 to 0 photons) and with probability $1 - p$ we still have 1 photon.

The Kraus operators can be found also from $M_k =_E \langle k|U_{SE}|0\rangle_E$, where the projection is taken on the environment state only. Here we have $M_0 = \langle 0|U_{SE}|0\rangle_E = |0\rangle\langle 0|_S$; $M_1 = \langle 1|U_{SE}|0\rangle_E = \sqrt{p}|0\rangle\langle 1|_S$; and $M_2 = \langle 2|U)SE|0\rangle_E = \sqrt{1-p}|1\rangle\langle 1|$.

b) The state after applying the Kraus operators is

$$\rho' = \sum_k M_k|1\rangle\langle 1|M_k^\dagger = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$$

At each step the matrix element $\rho_{11} = \langle 1|\rho|1\rangle$ is multiplied by a factor $(1 - p)$. The simplest way to calculate $\rho_{00} = \langle 0|\rho|0\rangle$ is just knowing that $\text{Tr}\{\rho\} = 1$ so that $\rho_{00} = 1 - \rho_{11}$. We thus have: $\rho(n) = [1 - (1 - p)^n]|0\rangle\langle 0| + (1 - p)^n|1\rangle\langle 1|$

c) The plot is exactly as in Fig.3a

d) With the approximation $(1 - \gamma t/n)^n \rightarrow e^{-\gamma t}$ for $n \rightarrow \infty$, and using the formula for $\rho(n)$ (after n steps) as above, the evolution of ρ_{11} is $\rho_{11}(t) = e^{-\gamma t}\rho_{11}(0) = e^{-\gamma t}$. So the plot would be the same as as in Fig. 3d (an exponential decay).
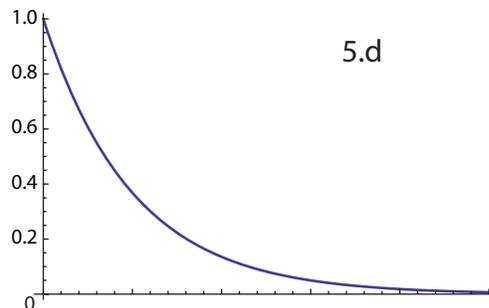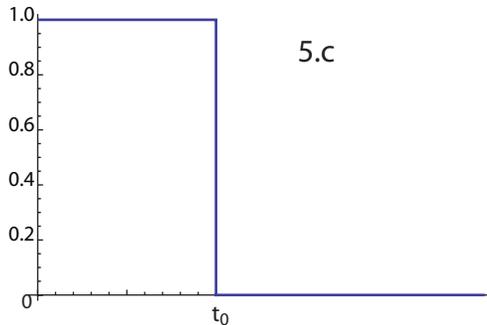
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