Fall 2002 22.611J 6.651J, 8.613J
PS#6 Answer Key

1. The quick way:

Fourier transform Poisson's equation:

$$\nabla^2 \phi = 4\pi\delta(\vec{x}) \Rightarrow k^2\tilde{\phi} = 4\pi$$

$$\tilde{\phi} = \frac{4\pi}{k^2}$$

since $\phi = \frac{1}{|\vec{x}|}$ for a pt. charge,

$$\boxed{\int d^3x\, e^{-i\vec{k}\cdot\vec{x}} \frac{1}{|\vec{x}|} \text{ is simply the transform of } \phi, \text{ which is } \tilde{\phi} = \frac{4\pi}{k^2}}$$

Similarly,

$$\nabla\cdot E = 4\pi\delta(\vec{x}) \Rightarrow -i\vec{k}\cdot\tilde{E} = 4\pi \Rightarrow \tilde{E} = \frac{4\pi i\vec{k}}{k^2}$$

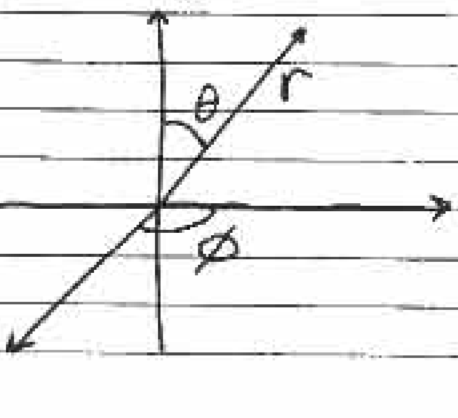
But $\vec{E} = -\nabla\phi = \frac{-\partial}{\partial\vec{x}}\frac{1}{|\vec{x}|}$

hence,

$$\boxed{\int d^3x\, e^{i\vec{k}\cdot\vec{x}} \frac{\partial}{\partial\vec{x}}\frac{1}{|\vec{x}|} = \int d^3x\, e^{i\vec{k}\cdot\vec{x}}\vec{E} = \frac{4\pi i\vec{k}}{k^2}}$$

The "manly" direct integral way

first, switch over to spherical coordinates



pick $\vec{k} = k\hat{z}$ (we can always rotate our system)

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$\vec{k}\cdot\vec{x} = kr\cos\theta$$

1. (Cont)

$$\int d\vec{x}^3 e^{-i\vec{k}\cdot\vec{x}} \frac{\partial}{\partial x}\left|\frac{1}{\bar{x}}\right| = \int_0^\infty\int_0^\pi r^2 \sin\theta \frac{e^{-ikr\cos\theta}}{r} \, dr \, d\theta \, 2\pi$$

Doing the θ integral :

$$= 2\pi \int_0^\infty \left[r \frac{e^{-ikr\cos\theta}}{+ikr} \right]_0^\pi dr$$

$$= 2\pi \int_0^\infty \frac{e^{ikr} - e^{-ikr}}{ik} dr = \frac{2\pi}{ik} \int_0^\infty e^{ikr} - e^{-ikr} \, dr.$$

→ Now, there's a few ways to resolve this integral. But let me try to do this the standard way first to illustrate the problem

$$\int_0^\infty e^{ikr} - e^{-ikr} dr = \left[\frac{e^{ikr} + e^{-ikr}}{ik} \right]_0^\infty = \frac{1}{ik} \Big[\cos(kr) \Big]_0^\infty$$

→ so our integral becomes undefined, since $\cos(kr \to \infty) = ?$

→ why did this happen? This is because this transform is really a ill-posed problem; it's well know that a $\frac{1}{r}$ potential is divergent as $r \to \infty$.

Hence, when we did Coulomb scattering earlier, we truncated the integral to λ_D from ∞. (when we were determining the transport cross sections and characteristic times)

→ A better way to do this problem is to use a shielded potential (i.e. problem 2) and take a limit ...

$$\phi = \lim_{k_0 \to 0} \frac{1}{|\bar{x}|} e^{\frac{-|\bar{x}| k_0}{1}}$$

(I think this is the most physical interpretation)

Then, we've

$$\lim_{k_0\to 0}\int_0^\infty \frac{(e^{ikr}-e^{-ikr})}{ik}\, e^{-k_0 r}\,dr$$

$$= \lim_{k_0\to 0} 2\pi \int_0^\infty \frac{e^{ikr-k_0 r}-e^{-ikr-k_0 r}}{ik}\,dr$$

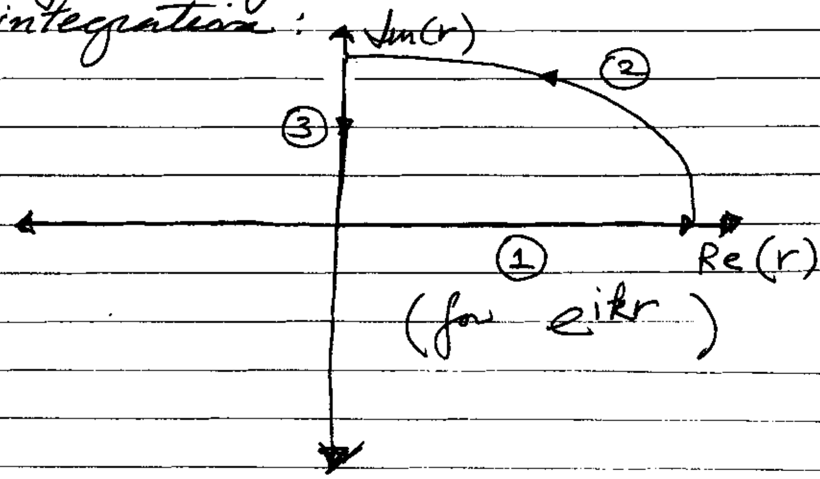
$$= \lim_{k_0\to 0} \frac{2\pi}{ik}\left[\frac{e^{i(k-k_0)r}-e^{-i(k+k_0)r}}{i(k-k_0)}\right]_0^\infty$$

$$= \lim_{k_0\to 0} \frac{2\pi}{ik}\left[\underbrace{\cancelto{0}{\frac{e^{ik\infty}e^{-k_0\infty}-e^{-ik\infty}e^{\mp k_0\infty}}{i(k-k_0)}}}_{\text{since } e^{-k_0\infty}\to 0 \text{ for any finite } k_0} - \frac{2}{i(k-k_0)}\right]$$

$$= \lim_{k_0\to 0} \frac{2\pi\cdot(-2)}{ik(i(k-k_0))} = \boxed{\frac{4\pi}{k^2} = \int d^3x\, \frac{e^{-i\vec{k}\cdot\vec{x}}}{|\vec{x}|}}$$

→ Another possible way of doing this is through contour integration:

$$\int_0^\infty e^{ikr}-e^{-ikr}\,dr \Rightarrow$$



So, for

$\int_0^\infty e^{ikr}\,dr$ we've

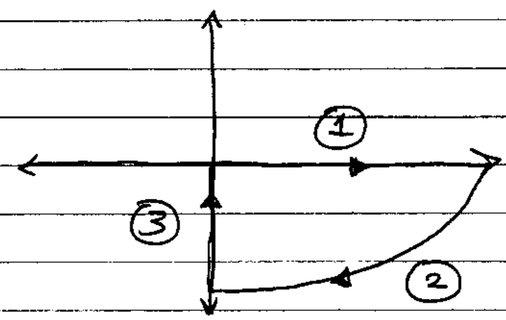
$$\int_{①} + \int_{②} + \int_{③} = 0$$

Now, if $\int_{②}$ equal zero (which is not completely obvious since Jordan's Lemma is for the full semi-circle) then, $\int_{①} = -\int_{③}$, so

$$\int_0^\infty e^{ikr}dr = -\int_{i\infty}^{0} e^{ikr}dr = +\left[\frac{e^{ikr}}{ik}\right]_0^{i\infty}$$

$$= +\cancel{\frac{e^{i^2k\infty}}{ik}}^{\to 0} + -\frac{1}{ik} = -\frac{1}{ik}$$

Similarly, for $\int e^{-ikr}dr$



$\int_{①} = -\int_{③}$ as before

$$\int_0^\infty e^{-ikr}dr = -\int_{-i\infty}^{0} e^{-ikr}dr$$

$$= +\left[\frac{e^{-ikr}}{+ik}\right]_{-i\infty}^{0} = +\left[\frac{1}{ik} + \cancel{\frac{-e^{-k\infty}}{ik}}^{\to 0}\right] = \frac{1}{ik}$$

hence,

$$\int e^{ikr} - e^{-ikr}dr \to -\frac{2}{ik}$$

which is what we needed to get the $\frac{4\pi}{k^2}$ answer.

→ Finally, a way that is related to the first method is to simply plug in a $e^{-\delta r}$:

$$\int_0^\infty (e^{ikr} - e^{-ikr}) \cdot \underbrace{\lim_{\delta\to 0} e^{-\delta r}}_{=1 \text{ at limit}} dr$$

, which is the same as using a shielded potential and taking the limit; however, starting w/ a shielded potential gives a more physical picture.

$$\Rightarrow \lim_{\delta\to 0}\int_0^\infty (e^{ikr} - e^{-ikr})e^{-\delta r}dr$$

1) The hard way for

$$\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{1}{|\vec{x}|}$$

using integration by parts,

$$\int dx\, e^{-i\vec{k}\cdot\vec{x}}\, \hat{i}\frac{\partial}{\partial x}\frac{1}{|\vec{x}|}$$ for the x-component.

(note: $\hat{i}$ is the unit x vector, i is $\sqrt{-1}$)

IBP: $\int u\,dv = uv - \int v\,du$

let $\frac{dv}{dx} = \frac{\partial}{\partial x}\frac{1}{|\vec{x}|}$, $v = \frac{1}{|\vec{x}|}$

$\frac{du}{dx} = -ik_x e^{-i\vec{k}\cdot\vec{x}}$ $u = e^{-i\vec{k}\cdot\vec{x}}$

then,

$$\Rightarrow \int dx\, e^{-i\vec{k}\cdot\vec{x}}\, \hat{i}\frac{\partial}{\partial x}\frac{1}{|\vec{x}|} = \hat{i}\cancelto{0}{\left.\frac{e^{-i\vec{k}\cdot\vec{x}}}{|\vec{x}|}\right|_{-\infty}^{\infty}} + \hat{i}\int_{-\infty}^{\infty} \frac{ik_x e^{-i\vec{k}\cdot\vec{x}}}{|\vec{x}|}\,dx$$

Doing the same w/ the y & z components, we get:

$$\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\,\vec{\nabla}\frac{1}{|\vec{x}|} = (\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)\, i\int_{-\infty}^{\infty}\frac{e^{-i\vec{k}\cdot\vec{x}}}{|\vec{x}|}\,dx$$

from above, $\int_{-\infty}^{\infty}\frac{e^{-i\vec{k}\cdot\vec{x}}}{|\vec{x}|}\,dx = \frac{4\pi}{k^2}$

$$\boxed{\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\frac{\partial}{\partial \vec{x}}\frac{1}{|\vec{x}|} = i\vec{k}\frac{4\pi}{k^2}}$$

2) First, we see that $f(\bar{x})$ is the inverse ~~the~~ fourier transform of $\frac{4\pi}{k^2+k_0^2}$

Now, from part one, we see that this is very similiar to Poisson's equation. Hence, try

$$\tilde{\phi} = \frac{4\pi}{k^2+k_0^2}$$

$$(k^2+k_0^2)\tilde{\phi} = 4\pi$$

Inverting,

$$(\nabla^2 + k_0^2)\phi = 4\pi\,\delta(\bar{x})$$

$$\nabla^2\phi + k_0^2\phi = 4\pi\,\delta(\bar{x})$$

which is the equation for debye shielding! (i.e. the potential of a shielded charge)

$$w/\ \lambda_D^2 = \frac{1}{k_0^2}$$

$$f(\bar{x}) = \phi = \phi_0\, e^{\left(\frac{-|\bar{x}|\,k_0}{1}\right)} = \frac{e^{(-|\bar{x}|k_0)}}{|\bar{x}|}$$

3) Maxwellian Distribution from Entropy Maximization:

from "Mathematical Methods for Physicist," Arfken & Weber 4th ed.

If we've a function J, such that

$$J = \int f\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) dx_j ,$$

where x_j's are the independent variables, and y_i's the dependent variables, and we wish to find $\delta J = 0$ w/ the following constraints,

$$\int \varphi_k(y_i, \partial y_i/\partial x_j, x_j)\, dk_j = \text{constant},$$

we've to solve the Euler-Lagrange equations:

$$\frac{\partial g}{\partial y_i} - \sum_j \frac{\partial}{\partial x_j} \frac{\partial g}{\partial(\partial y_i/\partial x_j)} = 0 \qquad (A)$$

where $g\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) = f + \sum_k \lambda_k \varphi_k$
and λ_k are constants.

In our case, we've

$$g = -f \ln f + \alpha f + \beta \tfrac{1}{2} m v^2 f \qquad (B)$$

· where our f correspond to y_i above, and $v_x, v_y, v_z \leftrightarrow x_j$. We've no dependence on $\frac{\partial y_i}{\partial x_j}$ for g.

hence,

Plugging (B) → (A) we've

$$\frac{\partial}{\partial f}\left(-f\ln f + \alpha f + \beta \tfrac{1}{2} m v^2 f\right) = 0$$

$$-(\ln f + 1) + \alpha + \beta \tfrac{1}{2} m v^2 = 0$$

$$\ln f + 1 = \alpha + \beta \tfrac{1}{2} m v^2$$

$$\ln f = -1 + \alpha + \beta \tfrac{1}{2} m v^2$$

$$\boxed{f = e^{\alpha - 1} e^{\beta \frac{1}{2} m v^2}}$$

which looks very much like the maxwell distribution!

→ Now, use constraints to determine α and β

$$N = \int d^3v \; e^{\alpha-1} e^{\beta \frac{1}{2} m v^2}$$

$$= e^{\alpha-1} 4\pi \int_0^\infty v^2 e^{\beta \frac{1}{2} m v^2} dv \quad \text{(spherical coordinates)}$$

$$= e^{\alpha-1} 4\pi \left(\frac{\Gamma(3/2)}{2\left(-\frac{m\beta}{2}\right)^{3/2}} \right)$$

$$= e^{\alpha-1} 4\pi \frac{\sqrt{\pi}}{2} \left(\frac{1}{2}\right) \left(-\frac{2}{m\beta}\right)^{3/2}$$

$$N = e^{\alpha-1} \left(-\frac{2\pi}{m\beta}\right)^{3/2}$$

now, for E:

$$E = \int d^3v \, \tfrac{1}{2} m v^2 f = \int d^3v \, \frac{mv^2}{2}\left(e^{\alpha-1} e^{\frac{\beta m}{2} v^2}\right)$$

$$= 4\pi e^{\alpha-1} \frac{m}{2} \int_0^\infty v^4 e^{\frac{\beta m}{2} v^2} dv$$

$$= 2\pi e^{\alpha-1} m \left(\frac{\Gamma(5/2)}{2\left(-\frac{m\beta}{2}\right)^{5/2}} \right)$$

$$= \pi e^{\alpha-1} m \left(\frac{3}{4} (\pi)^{\frac{1}{2}} \left(\frac{-2}{m\beta}\right)^{5/2} \right)$$

$$E = \frac{3}{4} \pi^{3/2} e^{\alpha-1} m \left(-\frac{2}{m\beta}\right)^{5/2}$$

now, we've two eqn's for α & β.

Divide

$$\frac{N}{E} = \frac{\cancel{\pi^{3/2}}}{\frac{3}{\cancel{2}4}\cancel{\pi^{3/2}}\cancel{m}} \left(-\frac{\cancel{m}\beta}{\cancel{2}}\right) = -\frac{2\beta}{3}$$

$$\boxed{\beta = \frac{-3N}{2E}}$$

$$N = e^{\alpha-1}\left(\frac{-2\pi}{m\beta}\right)^{3/2}$$

$$N = e^{\alpha-1}\left(\frac{+2\pi \, 2E}{m \, 3N}\right)^{3/2}$$

$$N^{5/2} = e^{\alpha-1}\left(\frac{\pi 4E}{3m}\right)^{3/2}$$

$$\left(\frac{3m}{\pi 4E}\right)^{3/2} N^{5/2} = e^{\alpha - 1}$$

OR,

$$\boxed{\ln\left[\left(\frac{3m}{4E\pi}\right)^{3/2} N^{5/2}\right] + 1 = \alpha}$$

hence,

$$f = \left(\frac{3m}{4E\pi}\right)^{3/2} N^{5/2} e^{-\frac{3N}{2E}\cdot\frac{1}{2}mv^2}$$

defining $E = \frac{3}{2} N T$

$$\boxed{f = \left(\frac{m}{2T\pi}\right)^{3/2} N \, e^{-\frac{mv^2}{2T}}}$$

which is the maxwellian distribution.

→ FYI; normally f is derived through Boltzmann's H theorem, which is related to the collision operator.

4) The Vlasov equation is:

$$\frac{df}{dt} = 0,$$

$$\Rightarrow \frac{\partial f}{\partial t} + \vec{a} \cdot \vec{\nabla}_v f + \vec{v} \cdot \vec{\nabla} f = 0$$

$$\vec{a} = \frac{q}{m}(\vec{E} + \vec{v} \times \vec{B})$$

· Letting $q \to \frac{q}{N}$, $m \to \frac{m}{N}$ & $n \to Nn$

$$\frac{\cancel{N}\partial f}{\partial t} + \frac{\cancel{N}}{\cancel{N}}\frac{q}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla}_v \cancel{N} f + \vec{v} \cdot \vec{\nabla} \cancel{N} f = 0$$

→ all the N's divide out → hence, Vlasov remains invariant under transformation

→ furthermore, the parameter $\frac{1}{n\lambda_D^3}$ does not!

$$\Rightarrow \frac{1}{n\lambda_D^3} \propto \frac{1}{n\left(\frac{1}{nq^2}\right)^{3/2}} \propto n^{1/2} q^2$$

letting $n^{1/2} \to (Nn)^{1/2}$, $q^2 \to \left(\frac{q}{N}\right)^2$

$$\frac{1}{n\lambda_D^3} \Rightarrow (Nn)^{1/2}\frac{q^2}{N^2} = \frac{n^{1/2} q^2}{N^{3/2}} \Rightarrow 0 \text{ when } N \to \infty$$

Hence, $\frac{1}{n\lambda_D^3}$ is not invariant.

How does this discreteness parameter relate to the Vlasov equation?

In general, $\frac{df}{dt} = (C)$, where C is a collision operator.

$C = C\left(\frac{1}{n\lambda_D^3}\right)$, which means that as $N \to 0$, $C \Rightarrow C = 0$.

→ Hence, in the limit of 0 discreteness, the Vlasov equation, $\frac{df}{dt} = 0$, is exact!

→ In ~~and~~ other words, no particles/discreteness, no collisions!

5) Final Value theorem Proof:

We know

$$\mathcal{L}(g'(t)) = -i\omega\, \mathcal{L}(g(t)) - g(t=0)$$
$$= -i\omega\, g_\omega - g(t=0)$$

By definition,

$$\mathcal{L}(g'(t)) = \int_0^\infty dt\, e^{i\omega t} g'(t)$$

Letting $\omega \to 0$,

$$\mathcal{L}(g'(t)) = \int_0^\infty dt\, \frac{dg(t)}{dt}$$
$$= \Big|_0^\infty g(t)$$
$$= \lim_{t\to\infty} g(t) - g(t=0)$$

hence,

$$\lim_{\omega\to 0}(-i\omega g_\omega) - \cancel{g(t=0)} = \lim_{t\to\infty} g(t) - \cancel{g(t=0)}$$

$$\boxed{\lim_{\omega\to 0}(-i\omega g_\omega) = \lim_{t\to\infty} g(t)}$$

→ Stable in this case means no Poles above $\text{Im}(\omega) = 0$, and $\omega_r \Rightarrow 0$ as $t \to \infty$

→ the system ~~does~~ goes to 0 at $t \to \infty$

→ ja systems that oscillate indefinitely,

$$\lim_{t\to\infty} g(t) = \lim_{\omega_r \to \text{constant}} (-i\omega g_\omega)$$

(Note ~~that~~ ω_I still has to be ~~below~~ negative)